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PROPERTIES OF SOME SERIES WITH APPLICATIONS TO THE GAMMA AND BETA FUNCTIONS

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Abstract: A pair of series are studied which have applications to the theory of the gamma and beta functions. It is shown that these series can be used to define a related pair of differentiable functions on the positive real axis. One of these functions is directly related to the beta function. The second function is shown to be related to a beta function type integral. Some properties of these functions are developed as well.

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This is an introductory study of two related functions which are motivated in form by two different finite sums. The terms which appear in these closely related series depend on binomial coefficients and reciprocals of integers. These series in fact have some interesting mathematical properties in their own right. It is shown that the original series can be extended to define functions on a much larger domain, and that these functions have some very interesting properties. One of these functions is directly related to the beta function by means of the beta function integral. The second function is related to an integral which is much like that for the beta function, and a useful series representation of this function is formulated as well. Apart from this, these series have applications

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to the study of the beta and gamma functions, and some of these results may have relevance to the study of related special functions. Series which are related to the ones of interest here appear in **[1,2]**, and some of these results have been stated in **[1]** as well.

Let us begin by introducing the two functions which will be of interest here, and as well, a natural generalization of these functions. Consider here the pair of sums which will be defined first for positive integers n and m in the following way

$$u(m,n)=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{m+k},\qquad v(m,n)=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{m+k}. \tag{1}$$

In terms of the functions $u(m,n)$ and $v(m,n)$, an additional related function can be defined as follows,

$$w(m,n)=\frac{u(m,n)}{v(m,n)}. \tag{2}$$

Some properties of the functions which are defined by equations (1) and (2) will be established here. It will be shown that $w(m,n)$ is a positive integer for every pair of positive integers m and n.

In fact, the two functions u and v can be defined over a much larger domain in a way that makes a more direct correspondence with the theory of special functions. Suppose first that x is a nonnegative real number. Then the sums which define the functions u and v can be used to extend the domain of definition of these functions [3,4,5]. The functions u and v will be extended in the first argument and then written in the form of integrals as follows

$$u(x,n)=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+x}=\sum_{k=0}^{n}\binom{n}{k}\int_0^1 t^{k+x-1}\,dt=\int_0^1 t^{x-1}\sum_{k=0}^{n}\binom{n}{k}t^k=\int_0^1 t^{x-1}(1+t)^n\,dt, \tag{3}$$

and similarly

$$v(x,n)=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+x}=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^1 t^{k+x-1}\,dt=\int_0^1 t^{x-1}\sum_{k=0}^{n}\binom{n}{k}(-t)^k=\int_0^1 t^{x-1}(\tag{4}$$

The functions $u(x,n)$ and $v(x,n)$ can be extended in the second argument as well using the integrals obtained in (3) and (4)

$$u(x,y)=\int_0^1 t^{x-1}(1+t)^y\,dt,\qquad v(x,y)=\int_0^1 t^{x-1}(1-t)^y\,dt.$$

but only the case in which the first argument is a continuous variable will be treated here. The integral definition of these functions can be used to extend the definition of $w(m,n)$ presented in (2) in the following way,

$$w(x,n)=\frac{\int_0^1 t^{x-1}(1+t)^n\,dt}{\int_0^1 t^{x-1}(1-t)^n\,dt}. \tag{5}$$

For the case in which $n=0$ and $n=1$, these integrals can be calculated explicitly, and the following results are obtained

$$w(x,0)=1, \quad w(x,1)=2x+1, \quad x>0.$$

Moreover, when $x=1$, it is found that

$$w(1,n)=\frac{\frac{1}{n+1}(1+t)^{n+1}|_0^1}{-\frac{1}{n+1}(1-t)^{n+1}|_0^1}=2^{n+1}-1.$$

Neither $u(x,n)$ nor $v(x,n)$ as written in (3) and (4) is defined at $x=0$, however, the ratio of these two functions is defined at $x=0$. The following Lemma can be used to prove this claim.

Lemma: *For $n\geq 0$ and $x>0$*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+x}=\frac{n!}{(x+n)(x+n-1)\cdots(x+1)x}. \tag{6}$$

Proof: Beginning with the integral expression for the beta function, we can write

$$B(x,n+1)=\int_0^1 t^{x-1}(1-t)^n\,dt=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^1 t^{k+x-1}\,dt=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+x}.$$

Since the beta function can be written in terms of the gamma function as

$$B(x,n+1)=\frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)}=\frac{\Gamma(x)n!}{(x+n)(x+n-1)\cdots x\Gamma(x)},$$

the result (6) follows as required. •

Using the definition of $w(x,n)$ and applying the lemma, we can write explicitly

$$w(x,n)=\frac{u(x,n)}{v(x,n)}=\frac{x(x+1)\cdots(x+n)}{n!}\left(\frac{1}{x}+\binom{n}{1}\frac{1}{x+1}+\cdots+\binom{n}{n-1}\frac{1}{x+n-1}+\frac{1}{x+n}\right)$$

$$= \frac{(x+1)\cdots(x+n)}{n!}(1+\binom{n}{1}\frac{x}{x+1}+\cdots+\frac{x}{x+n}). \tag{7}$$

From (7), it is clear that $w(x,n)$ is a polynomial of degree n. From this form of w, not only can it be seen that the limit exists, the limit can be evaluated as well, and it is found to be finite

$$\lim_{x\to 0} w(x,n) = 1.$$

Theorem 1. *The function $w(x,n)$ increases monotonically for all $x > 0$.*

Proof. Let us begin with $w(x,n)$ in the specific form given by (7), that is,

$$w(x,n) = \frac{1}{n!}((x+1)\cdots(x+n)+\binom{n}{1}x(x+2)\cdots(x+n)+\cdots+x(x+1)\cdots(x+n-1)). \tag{8}$$

Since each individual term in (8) is of the form $\prod_{\substack{k=0\\k\neq j}}^{n}(x+k)$ for $j = 0,1,\cdots,n$, upon collecting terms after expanding, it is found that $w(x,n)$ is a polynomial of degree n, which has nonnegative coefficients. Therefore, we can write

$$n!w(x,n) = \sum_{k=0}^{n} a_k x^k,$$

with each a_j nonnegative. Therefore, the derivative of this expression with respect to x is given by

$$n!\partial_x w(x,n) = \sum_{k=1}^{n} k\, a_k x^{k-1}.$$

Clearly, the derivative with respect to x is positive for all $x > 0$ and since $w(0,n) = 1$, the function $w(x,n)$ is monotonically increasing for $x > 0$. Moreover, this implies that $w(x,n)$ is strictly positive for all $x \geq 0$ as well. ♠

The conclusion that $w(x,n)$ is positive follows as well by examining the integrals which define u and v. The integrands in both u and v are strictly positive over the interval of integration, hence the corresponding integrals are positive as well. This also proves that if $x = m$ is an integer as well as n, then $w(m,n)$ is a positive, nonzero real number.

When x is a positive integer, the integral which gives $v(m,n)$ is the beta function integral, and it can be written in the following form

$$v(m,n) = \int_0^1 t^{m-1}(1-t)^n\, dt = \frac{1}{m\binom{n+m}{m}}. \tag{9}$$

A related sum can be obtained by continuing (6) in x to negative values by replacing x by $-x$ in (6) to obtain

$$\sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{x-s} = \frac{(-1)^n n!}{(x-n)(x-n+1)\cdots(x-1)x}. \tag{10}$$

Introducing the harmonic numbers defined by the series $H_n = \sum_{k=1}^{n} 1/k$, we can develop the following result.

Theorem 2. *For positive integers n,*

$$\sum_{\substack{i=0\\ i\neq k}}^{n} \binom{n}{i} \frac{(-1)^i}{k-i} = (-1)^k \binom{n}{k} (H_{n-k} - H_k). \tag{11}$$

Proof. Using the sum given in (10), let us pull out the term with $s = k$ where $0 \leq k \leq n$, then we have

$$\sum_{\substack{i=0\\ i\neq k}}^{n} \binom{n}{i} \frac{(-1)^i}{x-i} = \frac{(-1)^n n!}{x(x-1)\cdots(x-n)} - (-1)^k \binom{n}{k} \frac{1}{x-k}.$$

$$= \binom{n}{k} [\frac{(-1)^n k!(n-k)! - (-1)^k x(x-1)\cdots(x-(k-1))(x-(k+1))\cdots(x-n)}{x(x-1)\cdots(x-n)}]. \tag{12}$$

The right hand side is clearly indeterminant as x tends to k. To evaluate the limit on the right-hand side, we apply l' Hospital's rule to the ratio in (12).

Differentiating the numerator and denominator individually by using the product rule, the limit can then be evaluated in a straightforward way to obtain

$$\sum_{\substack{i=0\\ i\neq k}}^{n} \binom{n}{i} \frac{(-1)^i}{k-i} = (-1)^{k+1} \binom{n}{k} [\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{1} - \frac{1}{1} - \frac{1}{2} - \cdots - \frac{1}{n-k}]$$

$$= (-1)^{k+1} \binom{n}{k} (H_k - H_{n-k}) = (-1)^k \binom{n}{k} (H_{n-k} - H_k).$$

This is exactly the required result (11). ♠

The integrals which appear in the functions $u(x,n)$ and $v(x,n)$ can be used to derive a useful recursion relation for the function $w(x,n)$. To do so, integration by parts can be applied to reexpress the following integrals in the new form,

$$\begin{aligned}\textstyle\int_0^1 t^a(1+t)^b\,dt &= \frac{2^b}{a+1} - \frac{b}{a+1}\int_0^1 t^{a+1}(1+t)^{b-1}\,dt,\\ \textstyle\int_0^1 t^a(1-t)^b\,dt &= \frac{b}{a+1}\int_0^1 t^{a+1}(1-t)^{b-1}\,dt.\end{aligned} \tag{13}$$

Two more relations can be derived using integration by parts as well,

$$\begin{aligned}\textstyle\int_0^1 t^a(1+t)^b\,dt &= \frac{2^{b+1}}{b+1} - \frac{a}{b+1}\int_0^1 t^{a-1}(1+t)^{b+1}\,dt,\\ \textstyle\int_0^1 t^a(1-t)^b\,dt &= \frac{a}{b+1}\int_0^1 t^{a-1}(1-t)^{b+1}\,dt.\end{aligned} \tag{14}$$

Using the definition of $w(m,n)$ given in (5), we can write

$$w(x,n) = \frac{\frac{2^n}{x} - \frac{n}{x}\displaystyle\int_0^1 t^x(1+t)^{n-1}\,dt}{\int_0^1 t^{x-1}(1-t)^n\,dt} = \frac{2^n}{x}\frac{1}{\int_0^1 t^{x-1}(1-t)^n\,dt} - \frac{\frac{n}{x}\displaystyle\int_0^1 t^x(1+t)^{n-1}\,dt}{\frac{n}{x}\displaystyle\int_0^1 t^x(1-t)^{n-1}\,dt}$$

$$= \frac{2^n}{x}\frac{1}{\int_0^1 t^{x-1}(1-t)^n\,dt} - w(x+1,n-1).$$

Similarly, another relation for $w(x,n)$ can be obtained as follows

$$w(x,n) = \frac{\frac{2^{n+1}}{n+1} - \frac{x-1}{n+1}\displaystyle\int_0^1 t^{x-2}(1+t)^{n+1}\,dt}{\frac{x-1}{n+1}\displaystyle\int_0^1 t^{x-2}(1-t)^{n+1}\,dt} = \frac{2^{n+1}}{x-1}\frac{1}{\int_0^1 t^{x-2}(1-t)^{n+1}\,dt} - w(x-1,n+1).$$

Substituting the beta function integral into the expressions for $w(x,n)$, we obtain

$$w(x,n) = \frac{2^n\Gamma(x+n+1)}{x\Gamma(x)\Gamma(n+1)} - w(x+1,n-1) = 2^n\frac{\Gamma(x+n+1)}{\Gamma(x+1)\Gamma(n+1)} - w(x+1,n-1), \tag{15}$$

$$w(x,n) = \frac{2^{n+1}}{x-1}\frac{\Gamma(x+n+1)}{\Gamma(x-1)\Gamma(n+2)} - w(x-1,n+1) = 2^{n+1}\frac{\Gamma(x+n+1)}{\Gamma(x)\Gamma(n+2)} - w(x-1,n+1). \tag{16}$$

Equations (15) and (16) can also be written in terms of binomial coefficients as follows

$$w(x,n)=2^n\binom{x+n}{n}-w(x+1,n-1), \tag{17}$$

$$w(x,n)=2^{n+1}\binom{x+n}{n+1}-w(x-1,n+1). \tag{18}$$

Since $w(x,0)=1$ for all $x>0$, it follows from (17) that

$$w(x,1)=2\binom{x+1}{1}-1,$$

for all $x>0$. In fact, by repeatedly applying (17) as a type of lowering relation in the second argument, the following explicit formula for $w(x,n)$ can be developed in terms of binomial coefficients.

Theorem 3. *For all $x\geq 0$, the $w(x,n)$ are given by the following finite sum*

$$w(x,n)=\sum_{j=0}^{n}2^{n-j}(-1)^j\binom{x+n}{n-j}. \tag{19}$$

Proof 1. The first proof is by induction on n and relies only on properties of the binomial coefficients. Consider equation (17) and write

$$w(x,n)=2^n\binom{x+n}{n}-w(x+1,n-1).$$

Suppose (19) holds for all $x\geq 0$ up to $n-1$, then we can obtain $w(x+1,n-1)$ from (19) with x replaced by $x+1$ and n replaced by $n-1$

$$w(x,n)=2^n\binom{x+n}{n}-\sum_{j=0}^{n-1}2^{n-1-j}(-1)^j\binom{x+n}{n-1-j}=2^n\binom{x+n}{n}+\sum_{j=0}^{n-1}2^{n-j-1}(-1)^{j-1}\binom{x+n}{n-j-1}.$$

Set $k=j+1$, so that k runs from 1 to n,

$$w(x,n)=2^n\binom{x+n}{n}+\sum_{k=1}^{n}2^{n-k}(-1)^k\binom{x+n}{n-k}=\sum_{k=0}^{n}2^{n-k}(-1)^k\binom{x+n}{n-k}.$$

Hence the case $n-1$ implies the case n, which is (19) as claimed. ♠

This result expresses $w(x,n)$ as a linear combination of binomial coefficients multiplied by powers of two and then combined under the operations of addition or subtraction. It immediately follows from (19) that when $x=m$ is a positive integer, $w(m,n)$ is itself an integer as well. This follows since the binomial coefficients are integers when both of their entries are positive integers and then multiplied by a power of two. Thus, $w(m,n)$ is given by (19) as a linear combination of integers. In fact, $w(m,n)$ is a positive integer, since it has been shown that $w(x,n)$ is a positive function.

There is another proof of this fact if we are allowed to use a nonelementary representation of $w(x,m)$ in terms of the hypergeometric function. This way has the advantage of conciseness, but it is not as elementary as Proof 1.

Proof 2. Writing $w(x,n)$ in terms of the hypergeometric function we have that

$$w(x,n)=\frac{\Gamma(n+x+1)}{\Gamma(x+1)\Gamma(n+1)}\,{}_2F_1\left(\begin{matrix}-n,x\\ \\ x+1\end{matrix};-1\right).$$

Pfaff's transformation is given by **[5]**,

$${}_2F_1\left(\begin{matrix}-n,a\\ \\ c\end{matrix};t\right)=\frac{(c-a)_n}{(c)_n}\,{}_2F_1\left(\begin{matrix}-n,a\\ \\ a+1-n-c\end{matrix};1-t\right).$$

Using this transformation on $w(x,n)$, one obtains that

$$w(x,n)=\frac{\Gamma(n+x+1)}{\Gamma(x+1)\Gamma(n+1)}\frac{(1)_n}{(x+1)_n}\,{}_2F_1\left(\begin{matrix}-n,x\\ \\ -n\end{matrix};2\right)=\sum_{k=0}^{n}\frac{(x)_k}{k!}2^k.$$

Setting $x=m$ in this result, we have the result,

$$w(m,n)=\sum_{k=0}^{n}\binom{k+m-1}{k}2^k.$$

♠

A straightforward application of Theorem 3 yields the following Theorem.

Theorem 4. *The function $w(x,n)$ can be expressed in terms of the gamma function and a finite sum of binomial coefficients*

$$u(x,m)=\frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)}w(x,n)=\frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)}\sum_{j=0}^{n}2^{n-j}(-1)^j\binom{x+n}{n-j}.$$

When $x=m$ is a positive integer, the result in Theorem 4 takes the specific form

$$\int_0^1 t^{m-1}(1+t)^n\,dt=\frac{1}{m\binom{m+n}{m}}\sum_{s=0}^{n}2^{n-s}(-1)^s\binom{n+m}{m+s}. \tag{20}$$

Theorem 5. *When $x = m$ is an integer, the integers $w(m,n)$ have the property that*

$$w(m, m+1) = w(m+2, m). \tag{21}$$

Proof. To prove (21), let us begin by writing (19) with (x, n) replaced by $(m, m+1)$

$$w(m, m+1) = \sum_{j=0}^{m+1} 2^{m+1-j}(-1)^j \binom{2m+1}{m+j} = \sum_{s=0}^{m+1} 2^s (-1)^{m+1-s} \binom{2m+1}{s}.$$

Similarly, replace (x, n) in (19) by $(m+2, m)$ to obtain

$$w(m+2, m) = \sum_{s=1}^{m+1} 2^{m-s+1}(-1)^{s-1} \binom{2m+2}{m+s+1} = \sum_{j=0}^{m} 2^{m-j}(-1)^j \binom{2m+2}{m-j}.$$

These expressions for $w(m, m+1)$ and $w(m+2, m)$ can be written in the equivalent form

$$w(m, m+1) = \sum_{j=0}^{m+1} 2^{m+1-j}(-1)^j \binom{2m+1}{m-j+1} = \sum_{s=0}^{m+1} 2^s (-1)^{m+1-s} \binom{2m+1}{s},$$

$$w(m+2, m) = \sum_{j=0}^{m} 2^{m-j}(-1)^j \binom{2m+2}{m-j} = \sum_{s=0}^{m} 2^s (-1)^{m-s} \binom{2m+2}{s}. \tag{22}$$

Using elementary properties of binomial coefficients, we can write this expression for $w(m+2, m)$ in the following ways,

$$w(m+2, m) = \sum_{s=0}^{m} 2^s (-1)^{m-s} \binom{2m+2}{s} = \sum_{s=0}^{m} 2^s (-1)^{m-s} \binom{2m+1}{s} + \sum_{s=1}^{m} 2^s (-1)^{m-s} \binom{2m+1}{s-1}$$

$$= 2^m \binom{2m+1}{m} + \sum_{s=0}^{m-1} 2^s (-1)^{m-s} \binom{2m+1}{s} - 2 \sum_{s=0}^{m-1} 2^s (-1)^{m-s} \binom{2m+1}{s}$$

$$= -2^m \binom{2m+1}{m} + 2^{m+1} \binom{2m+1}{m+1} + \sum_{s=0}^{m-1} 2^s (-1)^{m-s+1} \binom{2m+1}{s}$$

$$= \sum_{s=0}^{m+1} 2^s (-1)^{m+1-s} \binom{2m+1}{s} = w(m, m+1).$$

The results given in (22) have been used to complete the proof. ♠

In the case in which $m = n$, we can write $w(n,n)$ in the form

$$w(n,n) = n\binom{2n}{n}\int_0^1 t^{n-1}(1+t)^n\,dt. \tag{23}$$

If we divide both sides of this by $2^n\binom{2n}{n}$, we obtain a new function for which the limit as n becomes large exists. An asymptotic expansion in n can be obtained by means of Laplace's method **[6]** for $w(n,n)$, and to this end we define

$$\Pi_n = \frac{w(n,n)}{2^n\binom{2n}{n}} = \frac{n}{2^n}\int_0^1 t^{n-1}(1+t)^n\,dt. \tag{24}$$

Applying Laplace's method to the integral and keeping terms to second order in t, it follows from (24) that

$$\Pi_n = \frac{2}{3} + \frac{2}{27n} + \frac{38}{81n^2} + O(n^{-3}).$$

Clearly, this implies that the limit of Π_n as $n \to \infty$ exists, and equals 2/3. Hence, the function $w(n,n)$ can be approximated for large n by

$$w(n,n) = 2^n\binom{2n}{n}(\frac{2}{3} + \frac{2}{27n} + \frac{38}{81n^2} + O(n^{-3})). \tag{25}$$

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