

NON-STABLE CLASSES OF ANALYTIC FUNCTIONS

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Abstract: If f and g are analytic functions in the unit disc \mathbb{D} , then f is said to be weakly subordinate to g , written $f \prec^w g$, if there exist analytic functions ϕ and $\omega : \mathbb{D} \rightarrow \mathbb{D}$, with ϕ an inner function, so that $f \circ \phi = g \circ \omega$. A class X of analytic functions in \mathbb{D} is said to be stable if it is closed under weak subordination, that is, if $f \in X$ whenever f and g are analytic functions in \mathbb{D} with $g \in X$ and $f \prec^w g$. For $0 < p < \infty$ and $\alpha > -1$, we let A_α^p denote the weighted Bergman space of all functions f , analytic in \mathbb{D} , such that $f \in L^p((1 - |z|^2)^\alpha dx dy)$ and the space of Dirichlet type \mathcal{D}_α^p consists of those f such that $f' \in A_\alpha^p$. Among other results, we prove that all the Bergman spaces A_α^p ($0 < p < \infty$, $\alpha > -1$) and all the \mathcal{D}_α^p -spaces except the space $\mathcal{D}_1^2 = H^2$ are non-stable classes of analytic functions in \mathbb{D} .

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1. Introduction and First Results

We denote by \mathbb{D} the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by $\mathcal{H}ol(\mathbb{D})$ the space of

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all analytic functions in \mathbb{D} . As usual, H^p ($0 < p \leq \infty$) are the classical Hardy spaces of analytic functions in \mathbb{D} (see [14] and [18]) and N is the Nevanlinna class (see [14] and [24]). A function I , analytic in \mathbb{D} , is said to be an inner function if $I \in H^\infty$ and I has a radial limit $I(e^{i\theta})$ of modulus one for almost every $e^{i\theta} \in \partial\mathbb{D}$.

We recall that an inner function I can be factored in the form $I = BS$ where B is a Blaschke product and S is a singular inner function, that is, S is of the form

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right), \quad z \in \mathbb{D},$$

where, μ is a finite positive Borel measure on $[0, 2\pi)$ that is singular with respect to Lebesgue measure.

K. Stephenson introduced in [26] the notion of weak subordination: if f and g are meromorphic functions in \mathbb{D} , then f is said to be *weakly subordinate* to g , written $f \prec^w g$, if there exist analytic functions ϕ and $\omega : \mathbb{D} \rightarrow \mathbb{D}$, with ϕ an inner function, so that $f \circ \phi = g \circ \omega$. If $\phi(z) \equiv z$ and $\omega(0) = 0$, we have the classical concept of subordination, and we shall simply write $f \prec g$. A class X of meromorphic functions in \mathbb{D} is said to be *stable* if it is closed under weak subordination, that is, if $f \in X$ whenever $f \prec^w g$ and $g \in X$.

Stephenson proved in [25] that the Hardy spaces H^p , $0 < p \leq \infty$, and the Nevanlinna class N are stable classes, although this terminology was not used there. Furthermore, Stephenson also proved in [26] that the space $BMOA$ is stable while the the Bloch space \mathcal{B} and the space $VMOA$ are not stable.

We recall that the space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$ and, $VMOA$ is the closed subspace of $BMOA$ which consists of those $f \in H^1$ whose boundary values have vanishing mean oscillation on $\partial\mathbb{D}$. Alternatively, $VMOA$ is the closure of the polynomials in $BMOA$. We mention [8], [18] and [19] for the theory of the spaces $BMOA$ and $VMOA$.

If f is an analytic function in \mathbb{D} , then f is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The space of all Bloch functions is denoted by \mathcal{B} . The little Bloch space \mathcal{B}_0 consists of those $f \in \mathcal{B}$ such that $\lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0$. Alternatively, \mathcal{B}_0 is the closure of the polynomials in the Bloch norm. A very good reference for the theory of Bloch functions is [4].

Stephenson's arguments can be used to obtain the following improvement of his results.

Theorem 1. *Let X be a space of analytic functions in \mathbb{D} which satisfies any of the two following conditions:*

(A) *X contains the polynomials and there is some inner function I such that $I \notin X$.*

(B) *$X \neq \mathcal{H}ol(\mathbb{D})$ and $\mathcal{B} \subset X$.*

Then X is a non-stable class of analytic functions in \mathbb{D} .

Proof. Suppose that X satisfies condition (A) and let I be an inner function which does not belong to X . Take $g(z) = \phi(z) = z$ and $\omega(z) = I(z)$ ($z \in \mathbb{D}$). Then ϕ and ω are inner functions and $I = I \circ \phi = g \circ \omega$. Hence, $I \prec^w g$, $g \in X$ and $I \notin X$. Thus X is not stable.

Suppose now that $X \neq \mathcal{H}ol(\mathbb{D})$ and $\mathcal{B} \subset X$. Let f be an analytic function in \mathbb{D} such that $f \notin X$. Let $E = \{m + ni : m, n \in \mathbb{Z}\}$ and $F = \{z \in \mathbb{D} : f(z) \in E\}$. We note that $F \neq \emptyset$ because $f \notin \mathcal{B}$. Since F is a countable subset of \mathbb{D} , it has capacity zero and therefore the universal covering map I from \mathbb{D} onto $\mathbb{D} \setminus F$ is an inner function (see, for instance, Chapter 2 of [11]). Set $g = f \circ I$. Then the image of g is contained in $\mathbb{C} \setminus E$ and, hence, it does not contain arbitrarily large discs. Consequently, see [4], g is a Bloch function. Since $\mathcal{B} \subset X$, we have that $g = f \circ I \in X$ even though $f \notin X$. Thus, X is not stable. \square

In addition to *VMOA*, among others, the following well known spaces satisfy condition (A) and, hence, are non-stable classes:

The little Bloch space \mathcal{B}_0 , [4]; the spaces Q_p and $Q_{p,0}$, $0 < p < 1$, ([6, 7, 16, 28]; the Besov spaces B^p , $1 \leq p < \infty$, [5, 12, 29]; the Dirichlet space \mathcal{D} of those f analytic in \mathbb{D} with finite Dirichlet integral.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $\int_{\mathbb{D}} (1 - |z|)^\alpha |f(z)|^p dA(z) < \infty$. Here, $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure in \mathbb{D} . We refer to [15] and [21] for the theory of these spaces.

If $\varphi : [0, 1) \rightarrow [0, \infty)$ is an increasing function with $\lim_{r \rightarrow 1} \varphi(r) = \infty$, we define

$$A(\varphi) = \{f \in \mathcal{H}ol(\mathbb{D}) : |f(z)| = O(\varphi(|z|)), \text{ as } |z| \rightarrow 1\}, \tag{1}$$

$$A_0(\varphi) = \{f \in \mathcal{H}ol(\mathbb{D}) : |f(z)| = o(\varphi(|z|)), \text{ as } |z| \rightarrow 1\}. \tag{2}$$

If $f \in \mathcal{B}$ then

$$|f(z)| = O\left(\log \frac{1}{1 - |z|}\right), \text{ as } |z| \rightarrow 1,$$

and then it follows that

$$\mathcal{B} \subset A_\alpha^p, \quad \alpha > -1, \quad 0 < p < \infty, \tag{3}$$

$$\mathcal{B} \subset A(\varphi), \text{ if } \log \frac{1}{1-r} = O(\varphi(r)), \text{ as } r \rightarrow 1, \quad (4)$$

and,

$$\mathcal{B} \subset A_0(\varphi), \text{ if } \log \frac{1}{1-r} = o(\varphi(r)), \text{ as } r \rightarrow 1. \quad (5)$$

Using (3), (4), (5) and Theorem 1, we obtain the following result.

Corollary 2. (i) If $\alpha > -1$ and $0 < p < \infty$ then the weighted Bergman space A_α^p is a non-stable class of analytic functions in \mathbb{D} .

(ii) If $\varphi : [0, 1) \rightarrow [0, \infty)$ is an increasing function such that $\log \frac{1}{1-r} = O(\varphi(r))$, as $r \rightarrow 1$, then the space $A(\varphi)$ is a non-stable class of analytic functions in \mathbb{D} .

(iii) If $\varphi : [0, 1) \rightarrow [0, \infty)$ is an increasing function such that $\log \frac{1}{1-r} = o(\varphi(r))$, as $r \rightarrow 1$, then the space $A_0(\varphi)$ is a non-stable class of analytic functions in \mathbb{D} .

We remark that part (iii) of Corollary 2 improves Example II in pp. 575-576 of [26] which asserts that $A_0(\varphi)$ is not stable for $\varphi(r) = \exp\left(\frac{1+r}{1-r}\right)$, ($0 < r < 1$). The next two sections will be devoted to the main purpose of this paper which is finding other types of non-stable classes of analytic functions in \mathbb{D} . More precisely, we wish to study the possibility of finding a space X of analytic functions in \mathbb{D} which is not stable and does not satisfy either condition (A) or condition (B). In order to do so we shall consider spaces of Dirichlet type.

2. The Spaces \mathcal{D}_{p-1}^p

For $0 < p < \infty$, the space of Dirichlet type \mathcal{D}_{p-1}^p consists of all analytic functions in \mathbb{D} whose derivatives belong to A_{p-1}^p , that is,

$$\mathcal{D}_{p-1}^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} (1 - |z|^2)^{p-1} |f'(z)|^p dA(z) < \infty \right\}. \quad (6)$$

The spaces \mathcal{D}_{p-1}^p are closely related to the Hardy spaces. Indeed, a direct calculation with power series shows that $H^2 = \mathcal{D}_1^2$. A classical result of Littlewood and Paley [22] (see also [23]) asserts that

$$H^p \subset \mathcal{D}_{p-1}^p, \quad 2 \leq p < \infty. \quad (7)$$

On the other hand, we have

$$\mathcal{D}_{p-1}^p \subset H^p, \quad 0 < p \leq 2. \quad (8)$$

The inclusion (7) can be proved by Riesz-Thorin interpolation. The same method gives (8) for $1 \leq p \leq 2$, since the inclusion $\mathcal{D}_0^1 \subset H^1$ is trivial. Vinogradov ([27], Lemma 1.4) extended (8) to the range $0 < p \leq 2$. We refer to [27], [13], [9] and [20] for distinct aspects of the theory of the spaces \mathcal{D}_{p-1}^p . In particular, we mention that using Proposition 2.1 of [10] (see also Proposition A of [20]), we obtain the following result.

Proposition A. *If f is given by a power series with Hadamard gaps,*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \quad (z \in \mathbb{D}) \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k \ (\lambda > 1),$$

then, for every $p \in (0, \infty)$,

$$f \in \mathcal{D}_{p-1}^p \iff \sum_{k=1}^{\infty} |a_k|^p < \infty.$$

Since for Hadamard gap series we have, for $0 < p < \infty$,

$$f \in H^p \iff \sum_{k=1}^{\infty} |a_k|^2 < \infty,$$

we immediately deduce that $\mathcal{D}_{p-1}^p \neq H^p$ if $p \neq 2$.

We remark also that if $p \neq q$ then there is no relation of inclusion between \mathcal{D}_{p-1}^p and \mathcal{D}_{q-1}^q . However, it is easy to show (see the argument used in the proof of Lemma 2.5 of [27]) that

$$\mathcal{B} \cap \mathcal{D}_{p-1}^p \subset \mathcal{B} \cap \mathcal{D}_{q-1}^q, \quad 0 < p < q < \infty. \tag{9}$$

We can now prove the following result.

Theorem 3. *If $0 < p < \infty$ and $p \neq 2$, then \mathcal{D}_{p-1}^p is a non-stable class of analytic functions in \mathbb{D} .*

Remark 1. Using (7), we see that if $2 < p < \infty$, then \mathcal{D}_{p-1}^p contains all the inner functions. On the other hand, if we take the lacunary power series

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}, \quad z \in \mathbb{D},$$

we have that $f \in \mathcal{B}$, (see [4]), while Proposition (A) shows that $f \notin \mathcal{D}_{p-1}^p$. Consequently, if $2 < p < \infty$, the space \mathcal{D}_{p-1}^p is not stable and does not satisfy either condition (A) or condition (B).

Proof of Theorem 3. Suppose first that $0 < p < 2$. Then, using Theorem 3.11 of [27], we see that there exists a Blaschke product B such that $B \notin \mathcal{D}_{p-1}^p$, and then the fact that \mathcal{D}_{p-1}^p is not stable follows from Theorem 1.

Suppose now that $2 < p < \infty$. Take $f \in H^2 \setminus \mathcal{D}_{p-1}^p$, (for example, take $f(z) = \frac{1}{(1-z)^{1/p}}$, $z \in \mathbb{D}$). Now we argue as in the proof of Theorem 1: We set $E = \{m + ni : m, n \in \mathbb{Z}\}$ and $F = \{z \in \mathbb{D} : f(z) \in E\}$ and we let I be the universal covering map from \mathbb{D} onto $\mathbb{D} \setminus F$. Since the set F is countable, I is an inner function. The function $g = f \circ I$ is a Bloch function because its image does not contain arbitrarily large discs. Also, $g \in H^2$ because H^2 is closed under subordination. Thus, we have that $g = f \circ I \in \mathcal{B} \cap H^2 = \mathcal{B} \cap \mathcal{D}_1^2$. Using (9), we deduce that $g = f \circ I \in \mathcal{B} \cap \mathcal{D}_{p-1}^p$ even though $f \notin \mathcal{D}_{p-1}^p$. Consequently, \mathcal{D}_{p-1}^p is not stable. This finishes the proof. \square

Remark 2. If $2 < p < \infty$ and g is the function constructed in the proof of Theorem 3 then, bearing in mind that H^p is stable, we see that $g \in \mathcal{D}_{p-1}^p \setminus H^p$. For these values of p , most of the known examples of functions in $\mathcal{D}_{p-1}^p \setminus H^p$ are given by lacunary power series.

3. The Spaces \mathcal{D}_α^p , ($0 < p < \infty$, $\alpha > -1$)

For $0 < p < \infty$ and $\alpha > -1$ the space of Dirichlet type \mathcal{D}_α^p consists of all functions f which are analytic in \mathbb{D} and satisfy

$$\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|^p dA(z) < \infty.$$

We have proved in Theorem 3 that if $p \neq 2$ then the spaces \mathcal{D}_{p-1}^p are not stable. In this section we shall extend our work considering the spaces \mathcal{D}_α^p for all admissible values of p and α . We shall prove the following result.

Theorem 4. *If $0 < p < \infty$, $\alpha > -1$ and $(p, \alpha) \neq (2, 1)$ then \mathcal{D}_α^p is a non-stable class of analytic functions in \mathbb{D} .*

In other words, Theorem 4 asserts that all the \mathcal{D}_α^p -spaces ($0 < p < \infty$, $\alpha > -1$) except the space $\mathcal{D}_1^2 = H^2$ are non-stable classes of analytic functions in \mathbb{D} . *Proof of Theorem 4.* We shall distinguish several cases. (a) If $0 < p < \infty$, $p \neq 2$ and $\alpha = p - 1$ then the result follows from Theorem 3.

(b) If $0 < p < \infty$ and $\alpha > p - 1$ then it is well known that $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$ (see, e. g., Theorem 6 of [17]) and then the fact that \mathcal{D}_α^p is not stable follows using Corollary 2.

(c) Suppose now that $0 < p < 2$ and $-1 < \alpha < p - 1$. As noticed above, Theorem 3.11 of [27] implies that there exists an inner function I with $I \notin \mathcal{D}_{p-1}^p$. Since $\alpha < p - 1$, $\mathcal{D}_\alpha^p \subset \mathcal{D}_{p-1}^p$. Then we see that there is an inner function I which does not belong to \mathcal{D}_α^p . Now Theorem 1 yields that \mathcal{D}_α^p is not stable.

(d) Suppose now that $1 < p < \infty$ and $-1 < \alpha \leq p - 2$. Notice that the space \mathcal{D}_{p-2}^p is the Besov space B^p . Hence $\mathcal{D}_\alpha^p \subset B^p$. Now, there are inner functions which do not belong to B^p . Indeed, Theorem 3.1 of [12] asserts that the only inner functions in the space B^p are the finite Blaschke products. Since $\mathcal{D}_\alpha^p \subset B^p$, the same is true for the space \mathcal{D}_α^p . This implies that \mathcal{D}_α^p is not stable.

It only remains to prove that \mathcal{D}_α^p is not stable for $2 \leq p < \infty$ and $p - 2 < \alpha < p - 1$. Using again Theorem 1, this will follow from the following result.

Theorem 5. *If $1 \leq p < \infty$ and $p - 2 < \alpha < p - 1$, then there exists an inner function I such that $I \notin \mathcal{D}_\alpha^p$.*

We need to introduce some notation and several results to prove Theorem 5. If I is an inner function, we shall write

$$\Delta(r, I) = \frac{1}{2\pi} \int_0^{2\pi} (1 - |I(re^{it})|^2) dt. \tag{10}$$

The quantity $\Delta(r, I)$ plays a very important role to study the membership of the derivative of an inner function in classical spaces of analytic functions in \mathbb{D} , (see [1], [2], [3]).

The following result is a special case of Theorem 6 of [2].

Theorem A. *Suppose that $\alpha > -1$, $p > 1 + \alpha$ and I is an inner function. Then $I \in \mathcal{D}_\alpha^p$ if and only if*

$$\int_0^1 (1 - r)^{\alpha-p} \frac{1}{2\pi} \int_0^{2\pi} (1 - |I(re^{it})|^2)^p dt dr < \infty.$$

Now, if μ is a finite positive Borel measure on $[0, 2\pi]$, the modulus of continuity ω_μ of μ is defined by

$$\omega_\mu(\delta) = \sup\{\mu([\theta, \theta + \delta]) : \theta \in [0, 2\pi]\}, \quad 0 \leq \delta.$$

If $0 < \beta < 1$ we let S_β denote the class of all inner functions I whose singular inner factor S is the singular inner function associated to a singular measure μ whose modulus of continuity ω_μ satisfies that $\omega_\mu(\delta) = O(\delta^\beta)$, as $\delta \rightarrow 0$. Ahern proved in Theorem 2.5 of [1] the following result.

Theorem B. *Suppose that $0 < \beta < 1$ and $I \in S_\beta$. Then there exist a constant $\varepsilon > 0$ and $r_0 \in (0, 1)$ such that*

$$\Delta(r, I) \geq \varepsilon(1 - r)^{\frac{1-\beta}{2-\beta}}, \quad r \in (r_0, 1). \tag{11}$$

Proof of Theorem 5. Take and fix p with $1 \leq p < \infty$. Since the spaces \mathcal{D}_α^p grow with α , it suffices to prove the theorem for those α such that $\max\{p-2, \frac{p-2}{2}\} < \alpha < p-1$. So, take such an α and set

$$\beta = 2 - \frac{p}{1+a}. \quad (12)$$

Observe that $0 < \beta < 1$. Let I be an inner function with $I \in S_\beta$. Using Theorem B we see that there exist $\varepsilon > 0$ and $r_0 \in (0, 1)$ such that

$$\Delta(r, I) \geq \varepsilon(1-r)^{\frac{1-\beta}{2-\beta}}, \quad r \in (r_0, 1). \quad (13)$$

Since $p \geq 1$, we have that

$$\frac{1}{2\pi} \int_0^{2\pi} (1 - |I_\alpha(re^{it})|^2)^p dt \geq \left(\frac{1}{2\pi} \int_0^{2\pi} (1 - |I_\alpha(re^{it})|^2) dt \right)^p = \Delta^p(r, I),$$

$0 < r < 1$, which, together with (13), implies

$$\begin{aligned} \int_{r_0}^1 (1-r)^{\alpha-p} \frac{1}{2\pi} \int_0^{2\pi} (1 - |I(re^{it})|^2)^p dt dr &\geq \int_{r_0}^1 (1-r)^{\alpha-p} \Delta^p(r, I_\alpha) dr \\ &\geq \varepsilon^p \int_{r_0}^1 (1-r)^{\alpha-p} (1-r)^{\frac{p(1-\beta)}{2-\beta}} dr = \varepsilon^p \int_{r_0}^1 (1-r)^{-1} dr = \infty. \end{aligned} \quad (14)$$

Using Theorem A we deduce that $I \notin \mathcal{D}_\alpha^p$. This finishes the proof. \square

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