International Journal of Pure and Applied Mathematics
Volume 21 No. 3 2005, 111-122

THE STABILITY OF SOLITARY WAVES FOR A GENERALIZED BOUSSINESQ EQUATION

Shaoyong Lai[1], Yong Hong Wu[2 §], Jibiao He[3], Lin Qun[4]

[1,2]Department of Mathematics and Statistics
Curtin University of Technology
GPO BOX U1987, Perth, WA6845, AUSTRALIA
[2]e-mail: yhwu@maths.curtin.edu.au

[3]Department of Mathematics
Mian Yang Teacher's College
Sichuan, P.R. CHINA

[4]Department of Mathematics
Sichuan Normal University, P.R. CHINA

Abstract: In this paper, we study an initial value problem for the following generalized Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} - \gamma^2 u + \beta(f(u))_{xx},$$

where $x \in R^1$, $t > 0$, $b > 0$, $\gamma \geq 0$, $\beta \in R^1$ and the function f is a polynomial with $f(0) = 0$. The conditions for the existence and uniqueness of a global solution to the problem in question are established in a Sobolev space. The results for sufficiently small β confirm Bony and Saches' suggestion[1] that initial data lying relatively close to a stabe solitary wave could evolve into a global solution for some equations.

AMS Subject Classification: 35K55, 35B40
Key Words: Boussinesq equation, solitary wave, nonlinear equation

1. Introduction

The typical model equations describing the propagation of small amplitude

Received: May 16, 2005

§Correspondence author

long waves on the surface of water were first established by Boussinesq in 1870s [2]. These equations have special, travelling wave solutions representing solitary waves. Thus, the phenomenon of solitary waves, discovered by Scott-Russell more than thirty years earlier, was first scientifically explained by the Boussinesq theory. In recent years, the study of various generalized Boussinesq equations has attracted the attention of many mathematicians, physicists and scientists. One of the classical Boussinesq equations attracting great attention is the following

$$u_{tt} = \alpha_1 u_{xxxx} + u_{xx} - \beta(u^2)_{xx}, \quad x \in R^1, t > 0, \tag{1}$$

where α_1 and β are real constants and the subscripts denote partial derivatives. Equation (1) has been studied by many researchers and the reader is referred to references [2-5] for more details . The conservation laws behind the equation and the soliton interaction of solutions to the equation were investigated in [6-11]. The conditions for nonexistence of global solutions in time to various boundary value problems associated with the equation were discussed in [12-13]. A number of exact solutions to the equation was also constructed by Clarkson [4]. More recently, to model dispersion in real processes, Varlamov [14-16] considered an initial boundary problem and a periodic Cauchy problem for the following damped boussinesq equation

$$u_{tt} - 2Bu_{txx} = -\alpha_1 u_{xxxx} + u_{xx} - \beta(u^2)_{xx} \tag{2}$$

with constant coefficients B, $\alpha_1 > B^2$ and $\beta \in R^1$. Biler [17,18] obtained a time estimate of the operator norm of the solution to the abstract Cauchy problem for the generalization of (1). Another generalized Boussinesq type equation, which arises in the modelling of nonlinear strings, was researched by Bony and Saches [1] and has the form of

$$u_{tt} - u_{xx} = -u_{xxxx} - (f(u))_{xx}. \tag{3}$$

Bony and Saches [1] deduced some properties leading to the inference that local smooth solutions may be extended uniquely to solutions defined globally in time under certain assumptions of initial data. The nonlinear stability of the solitary wave solutions to equation (3) was demonstrated for a range of propagation speed and pure power nonlinearities.

In this paper, we consider the existence of smooth global solutions for the following generalized Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} - \gamma^2 u + \beta\ (f(u))_{xx}, \tag{4}$$

where $b > 0$, $\gamma \geq 0, \alpha > b^2$, $\beta \in R^1$ and f is a $p-$order polynomial with $f(0) = 0$. Two initial value problems for equation (4) are considered in this paper. In the first problem, equation (4) is supplemented by the following auxiliary conditions

$$u(x,0) = \varepsilon\varphi(x), u_t(x,0) = \varepsilon\psi(x), \tag{5}$$

where ε is a small positive parameter. By using a method different from that in [1], in Section 3, the well-posedness of global solutions to problem (4)-(5) is established in a Sobolev space. The second problem is for the case where $\beta \to 0$ and in this case equation (4) is supplemented by

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \tag{6}$$

We use the contractive mapping to show that the existence and uniqueness of soliton solution associated to the problem are available. We also confirm that Bony and Saches'suggestion[1], that initial data lying relatively close to a stable solitary wave could evolve into a global solution for some equations, is valid for the problem considered in this paper.

2. Preliminary

In Bony and Sachs' work [1], the authors use Kato's theory for the initial value problem of (3). An equivalent form of (3), as a system of equations, is

$$\begin{aligned} u_t &= v_x, & (7)\\ v_t &= (u - u_{xx} - f(u))_x, x \in R,\ t > 0, & (8) \end{aligned}$$

where $f : R \to R$ is a C^∞ function with $f(0) = 0$. These equations are supplemented by the following initial data

$$u(x,0) = u_0^{\otimes}(x), v(x,0) = v_0^{\otimes}(x) \tag{9}$$

The stability theory established by Bony and Sachs [1] relies upon the Hamiltonian form of system (7)-(8). The solitary wave of speed c, if it exists, is a solution of (7)-(8) depending only on $\xi = x - ct$. Let

$$\begin{aligned} u(x,t) &= U_c(x - ct) = U_c(\xi), & (10)\\ v(x,t) &= V_c(x - ct) = V_c(\xi), & (11) \end{aligned}$$

then on substituting above into (7)-(8), we have the following system

$$-cU_c^{'} = V_c^{'}, \qquad (12)$$
$$-cV_c^{'} = (U_c - U_c^{''} - f(U_c))^{'}, \qquad (13)$$

where the prime denotes ∂_ξ, namely differentiation with respect to ξ . The proof of nonlinear stability follows from the abstract results of Grillakis, Shatah, and Straus[19](1987). Particular results of this type may be found in the reference due to Albert, Bona and Henry [20]. The basic idea, which goes back to Boussinesq, is to use the Lyapunov function to control the distance from (u,v) to the orbit of $(U_c,\ V_c)$. That is, provided that $(u(\cdot,0),v(\cdot,0))$ lies close to a given solitary wave, we shall only attempt to establish and estimate the distance $d((u,v),(U_c,V_c))$, where

$$d((u,v),(U_c,V_c)) = \inf_{x\in R} \|(u(\cdot,t),v(\cdot,t)) - (U_c(\cdot+y),V_c(\cdot+y))\|_{H^1(R)\times L^2(R)}. \quad (14)$$

Now we restate the main theorem presented in [1]

Theorem 2.1 *Let* $s \geq 1, 1 < p < 5,\ \frac{p-1}{4} < c^2 < 1$, *and let* $(U_c,\ V_c)$ *denote a solitary wave solution of (7)-(8) with* $f(u) = \mid u\mid^{p-1} u$. *If there exists a sufficiently small* $\lambda = \lambda(p,c) > 0$ *and there is a number* θ *such that*

$$\| u_0^{\otimes}(\cdot) - U_c(\cdot+\theta) \|_{H^1(R)} + \| v_0^{\otimes}(\cdot) - V_c(\cdot+\theta) \|_{L^2(R)} \leq \lambda, \qquad (15)$$

then the initial solution (u,v) *of (7)-(8) corresponding to the data* $(u_0^{\otimes}, v_0^{\otimes})$ *is global and lies in* $X_s(\infty)\times X_{s-1}(\infty)$. *Moreover, for all* $T>0$, *the mapping seeding* $(u_0^{\otimes}, v_0^{\otimes})$ *to the solution* (u,v) *of problem (7)-(8) is continuous from* $H^{s+2}(R)\times H^{s+1}(R)$ *into* $X_s(T)\times X_{s-1}(T)$, *where* $X_s(T) = C([0,T],H^{s+2}(R))\cap C^1([0,T],H^s(R))$ *denotes the Sobolev space with norm defined by*

$$|||u||| = \sup_{t\in[0,T]} \left(\|u\|_{s+2} + \|u_t\|_s\right), \qquad (16)$$

in which

$$\|u\|_s = \left(\int_{-\infty}^{+\infty} <\xi>^{2s} |\widehat{u}(\xi,t)|^2\, d\xi\right)^{\frac{1}{2}},$$
$$\widehat{u}(\xi,t) = F[u(x,t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\xi}u(x,t)dx,$$
$$<\xi> = \left(1+|\xi|^2\right)^{\frac{1}{2}}.$$

It should be addressed that solitary waves of problem (7)-(8) with $f(u) = | u |^{p-1} u$ do exist under certain conditions. In fact,this has been exemplified by Bona and Sachs [1]. In this paper, we study the solution of equation (4) supplemented respectively by initial conditions (5) and (6) in the Sobelev space $X_s(T) = C([0,T], H^{s+2}(R)) \cap C^1([0,T], H^s(R))$.

Definition 2.1 For a constant $q > 0$, we say $u(x,t) \in \overline{Xs(T)}$ if $|||u||| < q$.

3. Existence and Uniqueness Theorem

Theorem 3.1 *Suppose $\varphi(x) \in H^{s+2}(R)$ and $\psi(x) \in H^s(R)$ with $s > \frac{1}{2}$ and ε is sufficiently small. If f is a pth order polynomial with $f(0) = 0$ and $\alpha > b^2$, $b > 0$, $\gamma \geq 0$, $\beta \in R^1$, then there exists a unique solution $u(x,t) \in X_s(\infty)$ to the problem (4) -(5).*

Proof. We divide the proof of the theorem into two parts. Firstly, we prove that there exists a solution $u(x,t) \in X_s(\infty)$ to the problem defined by (4) and (5). Then, we prove that the solution is unique. For simplicity, throughout the proof of this theorem, we denote by C any positive constant independent of ε, which may depend on p, s and etc.

(a) *Existence.* Taking the Fourier transform of equations (4) and (5), we get

$$\widehat{u}''(\xi,t) + 2b\xi^2\widehat{u}'(\xi,t) + \left(\alpha\xi^4 + \xi^2 + \gamma^2\right)\widehat{u}(\xi,t) = -\beta\xi^2\widehat{f}(u(\xi,t)), \tag{17}$$

$$\widehat{u}(\xi,0) = \varepsilon\widehat{\varphi}(\xi), \quad \widehat{u}_t(\xi,0) = \varepsilon\widehat{\psi}(\xi), \tag{18}$$

which give the following solution

$$\begin{aligned} \widehat{u}(\xi,t) &= \varepsilon^2 e^{-b\xi^2 t}\left\{\left[\cos(\sigma_\xi t) + b\xi^2\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\} \\ &- \frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]\sin[\sigma_\xi(t-\tau)]\widehat{f}(u(\xi,\tau))d\tau, \end{aligned}$$

where

$$\sigma_\xi = \sqrt{(\alpha - b^2)\xi^4 + \xi^2 + \gamma^2}, \quad \alpha - b^2 > 0.$$

Denoting u_0 by

$$u_0 = \varepsilon F^{-1}\left[e^{-b\xi^2 t}\left\{\left[\cos(\sigma_\xi t) + b\xi^2 \cdot \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) \ + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\}\right],$$

where F^{-1} represents the inverse Fourier transform, we have

$$\|u_0\|_{s+2} \leq C\varepsilon\left(\|\varphi\|_{s+2} + \|\psi\|_s\right). \tag{19}$$

Now, by letting $A = C(\|\varphi\|_{s+2} + \|\psi\|_s)$, we have, from (19), that

$$\|u_0\|_{s+2} \leq \varepsilon A. \tag{20}$$

The sequence $\{\widehat{u}_n\}$ can thus be constructed as follows

$$\begin{aligned} \widehat{u}_n(\xi,t) &= \widehat{u_0} - \frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]\sin\left[\sigma_\xi(t-\tau)\right] f(\widehat{u_{n-1}(\xi},t))d\tau, \\ n &= 1,2,3,\ldots. \end{aligned}$$

Since

$$\begin{aligned} \frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]d\tau &\leq \frac{\beta\xi^2}{\sqrt{(c-b^2)\xi^4+\xi^2+p^2}} \times \frac{1}{b\xi^2}\left(1-e^{-b\xi^2 t}\right) \\ &\leq \frac{2\beta/b}{\sqrt{(c-b^2)\xi^4+\xi^2+p^2}} \leq C, \end{aligned}$$

we get, from the Shaulder lemma, that

$$\begin{aligned} \|u_n\|_{s+2} &\leq \|u_0\|_{s+2} + C\|f(u_{n-1})\|_{s+2} \\ &\leq \|u_0\|_{s+2} + C\|u_{n-1}\|_{s+2}^p. \end{aligned}$$

Therefore, for $n = 1$, by using inequality (20), we have

$$\begin{aligned} \|u_1\|_{s+2} &\leq \|u_0\|_{s+2} + C\|u_0\|_{s+2}^p \\ &\leq \varepsilon A + C(\varepsilon A)^p, \\ &\leq \varepsilon A + C(2\varepsilon A)^p. \end{aligned}$$

Choosing ε sufficiently small such that

$$2C(2\varepsilon A)^{p-1} < 1, \tag{21}$$

we have that

$$\|u_1\|_{s+2} \leq 2\varepsilon A. \tag{22}$$

For $n = 2$, we have from (22) that

$$\|u_2\|_{s+2} \leq \varepsilon A + C\|u_1\|_{s+2} < \varepsilon A + C(2\varepsilon A)^p. \tag{23}$$

Further using (21), we have

$$\|u_2\|_{s+2} < 2\varepsilon A. \tag{24}$$

By induction, we get

$$\|u_n\|_{s+2} < 2\varepsilon A, \quad n = 1, 2, 3, \cdots . \tag{25}$$

Following the same procedure as that for deriving (25), we obtain

$$|||u_n||| < 2\varepsilon A, \quad n = 1, 2, 3, \cdots . \tag{26}$$

From the assumption of function f, we get

$$\|u_n - u_{n-1}||| \leq C|g(|||u_{n-1}|||, |||u_{n-2}|||)| \, |(|||u_{n-1}||| - |||u_{n-2}|||)|, \tag{27}$$

where $g(x, y)$ is a $(p-1)$th order homogeneous polynomial with respect to x and y. By using (26), it follows that

$$g(|||u_{n-1}|||, |||u_{n-2}|||) \leq C(2\varepsilon A)^{p-1}. \tag{28}$$

Thus from (27) and (28), we obtain

$$\begin{aligned} |||u_n - u_{n-1}||| &\leq C(2\varepsilon A)^{p-1} \, |(|||u_{n-1}||| - |||u_{n-2}|||)| \\ &\leq [C(2\varepsilon A)^{p-1}]^{n-1}(4\varepsilon A) \, . \end{aligned}$$

Furthermore

$$\begin{aligned} |||u_n||| &= |||u_0 + \sum_1^n (u_i - u_{i-1})|||, \\ &\leq |||u_0||| + \sum_1^n |||u_i - u_{i-1}||| \\ &\leq \varepsilon A + \sum_1^n [C(2\varepsilon A)^{p-1}]^{i-1}(4\varepsilon A) \end{aligned}$$

From above inequalities and noting that $C(2\varepsilon A)^{p-1} < 1$, it is clear that u_n is uniformly convergent in the space $X_s(\infty)$. Therefore there must exist a function $u(x,t) \in X_s(\infty)$ such that u_n uniformly converges to $u(x,t)$, i.e., a solution of the problem defined by (4) and (5).

(b) *Uniqueness.* Now we prove that the global solution $u(x,t) \in X_s(\infty)$ of the problem defined by(4)-(5) is unique. Suppose that there exist two solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ of the problem, then both $|||u^1|||$ and $|||u^2|||$ are bounded in the space $X_s(\infty)$. Letting

$$w(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t),$$

we have

$$\widehat{w}(\xi,t) = -\frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]\sin\left[\sigma_\xi(t-\tau)\right]\widehat{F}(\xi,\tau)d\tau, \tag{29}$$

where

$$F(x,t) = f(u^1(x,t)) - f(u^2(x,t)).$$

It follows from (29) that

$$\begin{aligned} |\widehat{w}(\xi,t)| &\le \frac{\beta\xi^2}{\sigma_\xi}\left[\int_0^t \exp\left[-2b\xi^2(t-\tau)d\tau\right]\right]^{\frac{1}{2}}\left[\int_0^t |\ \widehat{F}(\xi,\tau)\ |^2\ d\tau\right]^{\frac{1}{2}} \\ &\le C\left[\int_0^t |\ \widehat{F}(\xi,\tau)\ |^2\ d\tau\right]^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} <\xi>^{2(s+2)} |\widehat{w}(\xi,t)|^2 d\xi &\le C\int_{-\infty}^{+\infty}\int_0^t <\xi>^{2(s+2)} |g\widehat{(u^1,u^2)}w(\xi,t)|^2 d\tau d\xi \\ &\le C\int_0^t \|w\|_{s+2}^2 \mid \left(g(\|u^{(1)}\|_{s+2}, \|u^{(2)}\|_{s+2}\right) \mid d\tau \\ &\le C\int_0^t \|w\|_{s+2}^2 d\tau. \end{aligned}$$

That is

$$\|w\|_{s+2}^2 \le C\int_0^t \|w\|_{s+2}^2 d\tau. \tag{30}$$

By the Growall inequality, we have $w(x,t) \equiv 0$ $(w(x,t) \in X_s(\infty))$. That is, $u^{(1)}(x,t) \equiv u^{(2)}(x,t)$ and hence there exists a unique solution $u(x,t) \in X_s(\infty)\left(s > \frac{1}{2}\right)$ of the problem defined by (4)-(5)

4. Stability of Solitary Waves

In this section, we assume that β is sufficiently small in equation (4) and the initial data are not sufficiently small. A result similar to theorem 3.1 can be obtained as follows.

Theorem 4.1 *Suppose that the initial data* $\varphi(x) \in H^{s+2}$, $\psi(x) \in H^s$ *with* $s > \frac{1}{2}$ *and* β *is sufficiently small. If* $\alpha > b^2$, $b > 0$, $\gamma \geq 0$ *and* f *is a* $p-$ *order polynomial with* $f(0) = 0$. *Then there exists a unique global solution* $u(x,t) \in \overline{X_s(\infty)}$ *to the problem defined by (4) and (6) .*

Proof. Taking Fourier transform of equations (4) and (6), we obtain

$$\begin{aligned} \widehat{u}(\xi,t) &= e^{-b\xi^2 t}\left\{\left[\cos(\sigma_\xi t) + b\xi^2\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\} \\ &- \frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]\sin[\sigma_\xi(t-\tau)]\widehat{f}(u(\xi,\tau))d\tau. \end{aligned}$$

Let the operator Λ be defined by

$$\begin{aligned} \widehat{\Lambda u}(\xi,t) &= e^{-b\xi^2 t}\left\{\left[\cos(\sigma_\xi t) + b\xi^2\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\} \\ &- \frac{\beta\xi^2}{\sigma_\xi}\int_0^t \exp\left[-b\xi^2(t-\tau)\right]\sin[\sigma_\xi(t-\tau)]\widehat{f}(u(\xi,\tau))d\tau. \end{aligned}$$

We shall show that the operator Λ is a contractive mapping from $\overline{X_s(\infty)}$ to $\overline{X_s(\infty)}$. For any $u(x)$,$v(x) \in \overline{X_s(\infty)}$, it follows from the definition of $\overline{X_s(\infty)}$ and Λ that

$$|||\Lambda u||| \leq (\|\varphi\|_{s+2} + \|\psi\|_s) + \beta C q^{p-1}|||u|||, \tag{31}$$

$$|||\Lambda u - \Lambda v||| \leq \beta C q^{p-1}|||u - v|||, \tag{32}$$

where C is independent of β and q. Let $(\|\varphi\|_{s+2} + \|\psi\|_s) < \frac{1}{2}q$, and β be sufficiently small such that $\beta C q^{p-1} < \frac{1}{2}$, we get from (31) and (32) that

$$|||\Lambda u||| \leq \frac{1}{2}q + \frac{1}{2}|||u||| \tag{33}$$

$$|||\Lambda u - \Lambda v||| \leq \frac{1}{2}|||u - v|||. \tag{34}$$

Up to now, we have shown that Λ is a contractive mapping from $\overline{X_s(\infty)}$ to $\overline{X_s(\infty)}$, henceforth there exists a unique global solution $u(x) \in \overline{X_s(\infty)}$ to the problem defined by (4) and (6).

As defined in Section 2, the solitary wave solutions for equation (4) exist if $u = U(x \pm ct)$ satisfies equation (4) for all wave speed $c > 0$. That is,

$$U_{tt} - 2bU_{txx} = -\alpha U_{xxxx} + U_{xx} - \gamma^2 U + \ \beta \ (f(U))_{xx} \ . \qquad (35)$$

For any wave speed c, as a direct use of theorem 4.1, we have the following theorem. At this point, this is quite different from that of the main result in [1], namely, theorem 7 in Section 2.

Theorem 4.2 *Suppose that a globally solitary wave solution $U(x \pm ct)$ of equation(4) exists, $\beta \to 0$ and all the assumptions in theorem 4.1 are satisfied. If there exists a sufficiently small $\varepsilon = \varepsilon(p, c) > 0$ and there is a number θ such that*

$$|||u_0(x) - U(x+\theta)\ |||\ _{s+2} + \ |||u_1(x) - U_t(x+\theta)\ |||\ _s \ \ < \ \varepsilon, \qquad (36)$$

then the solution u of problem (4) subject to initial condition (6) is global in time and unique in the space $\overline{X_s(\infty)}$.

Remark 1: Theorem 4.1 states that the initial data lying sufficiently close to a globally solitary wave could develop into a global solution of equation (4) if β is sufficiently small.

Remark 2: The result of theorem 4.2 suggests that we do not need to control the wave speed c of the solitary solution, although we actually do not know what assumptions we should give to the existence of solitary solutions. We also can get the conclusion that if the soliton solution to equations (4) and (6) exists for β sufficiently small, then the solitary wave is stable.

References

[1] J. Bona and R. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Commun. Math. Phys. 118 (1988), 12–29.

[2] J.Boussinesq, Theorie des ondes et de remous qui se propagent le long d un canal recangulaier horizontal, et communiquant au liquide contene dans ce cannal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl. Ser 2, 17 (1872), 55–108.

[3] F. Calogero and A. Degasperis, Spectral Transform and Solitons, Vol. 1, North Holland Publ. Amsterdam, 1982.

[4] P. Clarkson, New exact solutions of the Boussinesq equation, Eur. J. Appl. Maths. 1 (1990), 279–300.

[5] V.M. Galkin, D.E. Pelinovksy, and Yu. A. Stepanyants, The structure of the rational solution to the Boussinesq equation, Physica D. 80 (1995), 246–255.

[6] R. Hirota, Solutions of the classical Boussinesq equation and the spherical Boussinesq equation : The Wronskian Technique, J.Phys. Soc. Japan. 55 (1986), 2137–2150.

[7] J.W. Miles, Solitary waves, Ann. Rev. Fluid Mech. 12 (1980), 11–43.

[8] P. A. Milewski and J. B. Keller, Three-dimensional water waves, Studies in Applied Math. 37 (1996), 149–166.

[9] A. Nakamura, Exact solitary wave solutions of the spherical Boussinesq equation, J. Phys. Soc. Japan, 54 (1985), 4111–4114.

[10] P.I. Naumkin and I.A. Shishmarev, Nonlinear nonlocal equations in the theory of waves, Transl. Math. Monog. 133, AMS, (1994).

[11] M. Tajiri and T. Nishitani, Two-soliton resonant interaction in one spatial dimension of Boussinesq-type equation, J. Phys. Soc. Japan, 51(1982), 3720–3723.

[12] H.A. Levine, Instability and nonexistence of global solution to nonlinear wave equations of the form, Trans. Amer. Math. Soc. 192 (1974), 1–21.

[13] B. Straughan, Global nonexistence of solutions to some Boussinesq-type equation, J. Math.Phys. Sci. 26(1992), 155–164.

[14] V. V. Varlamov, On the Cauchy problem for the damped Boussinesq equation, Differential and Integral Equations, 9 (1996), 619–634.

[15] V. V. Varlamov, On the Initial boundary Value Problem for the damped Boussinesq equation, Discrete and Continuous Dynamical Systems, 3 (1998), 431–444.

[16] V. V. Varlamov, Asymptotics as $t \to \infty$ of solution to the Periodic Cauchy Problem for the Damped Boussinesq Equation, Mathematical Methods in Applied Sciences, 20(1977), 805-812.

[17] P. Biler, Time decay of solutions of semilinear strongly damped generalized wave equations, Math. Meth. Appl. Sci. 14 (1991), 427–443.

[18] P. Biler, Regular decay of solutions of strongly damped nonlinear hyperbolic equations, Appl. Anal. 32 (1989), 277–285.

[19] Grillakis J.,Shatah J., Strauss W.A., Stability theory of solitary waves in the presence of symmetry I.J. Funct.Anal. 74(1987),160-197.

[20] Albert J., Bona J.L. and Henry D., Sufficient conditions for stability of solitary-wave solutions of model equations for long waves, Physica, 24D(1987), 343-366.