International Journal of Pure and Applied Mathematics

Volume 21 No. 3 2005, 123-130

RANK TWO VECTOR BUNDLES AND SPECIAL RULED SURFACES

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Abstract: Let X be a smooth and connected projective curve and F a rank two vector bundle on X. Is there $L \in \mathrm{Pic}(X)$ such that $h^1(X, F \otimes L) > 0$ and $F \otimes L$ is spanned ? Describe all such pairs (F, L) (at least under certain assumptions on X and F). Here we give complete results when F is unstable and partial results when F is stable and " general " in its stratum for the stratification given by the order of stability.

AMS Subject Classification: 14H60, 14H50, 14J26
Key Words: rank two vector bundles on curves, ruled surface, projective bundles, projective bundles on curves, unstable vector bundle

1. Introduction

Stimulated from [4] we study the following question.

Question 1. Let X be a smooth and connected projective curve and F a rank two vector bundle on X. Is there $L \in \mathrm{Pic}(X)$ such that $h^1(X, F \otimes L) > 0$ and $F \otimes L$ is spanned ? Describe all such pairs (F, L) (at least under certain assumptions on X and F).

Of course, the same question may be raised also for higher rank vector bundles.

Notation 1. Let X be a smooth and connected projective curve and F a rank two vector bundle on X. Set $\tau(F) := \{L \in \mathrm{Pic}(X)$ such that $h^1(X, F \otimes$

Received: May 6, 2005

$L) > 0$ and $F \otimes L$ is spanned$\}$, $\tau(F, z) := \{L \in \mathrm{Pic}^z(X)$ such that $h^1(X, F \otimes L) > 0$ and $F \otimes L$ is spanned$\}$, $\eta(F) := \{z \in \mathbb{Z} : \tau(F, z) \neq \emptyset\}$, $\eta(F)_- := \inf\{z \in \eta(F)\}$ and $\eta(F)_+ := \sup\{z \in \eta\}$.

Obviously, there is no spanned vector bundle A on $\mathbf{P}^1$ such that $h^1(X, A) > 0$. For elliptic curves the answer to Question 1 easily follows from Atiyah's classification of vector bundles on elliptic curves (see Proposition 1). Here we may assume that X has genus $g \geq 2$. The case " F decomposable " is easy (see Remark 1). We have reasonably complete results when F is assumed to be unstable, but indecomposable (see Remarks 3, 4, 5). We have partial results when F is assumed to be stable and " general " in its stratum for the stratification given by the order of stability (see Propositions 2, 3 and 4).

In Section 3 we briefly consider the following question.

Question 2. Let X be a smooth curve of genus g. Fix an integer $r \geq 2$. What is the maximal integer t such that there is a chain of rank r vector bundles on X such that $E_0 \subset E_1 \subset \cdots \subset E_t$, $h^1(X, E_0) > 0$, $\deg(E_i) = \deg(E_0) + i$, and $h^0(X, E_i) = h^0(X, E_0) + i$ for all i. The latter condition is equivalent to $h^1(X, E_t) = h^1(X, E_0)$ (Riemann-Roch). Furthermore, we may add one of the following conditions:

(i) E_0 is spanned;

(ii) each E_i is semistable;

(iii) each E_i is stable.

Notice that if E_0 is spanned, then every E_i is spanned.

2. Question 1

Proposition 1. *Let X be an elliptic curve and F a rank 2 vector bundle on X. There is $L \in \mathrm{Pic}(X)$ such that $h^1(X, F \otimes L) > 0$ and $F \otimes L$ is spanned if and only if F is decomposable, say $A \oplus B$ with $\deg(A) \leq \deg(B)$, and either $A \cong B$ or $\deg(A) < \deg(B)$. In these cases L is unique and isomorphic to A^*.*

Proof. First assume F indecomposable. By [1], Lemma 15, we have $h^0(X, F \otimes L) = h^1(X, F \otimes L) \leq 1$ if $\deg(F \otimes L) = 0$, $h^0(X, F \otimes L) = 0$ if $\deg(F \otimes L) < 0$, and $h^1(X, F \otimes L) = 0$ if $\deg(F \otimes L) > 0$. If F is decomposable, then all the assertions follow from the known cohomology of line bundles on X. □

Remark 1. Let X be a smooth curve of genus $g \geq 2$, $A \in \mathrm{Pic}^a(X)$ and $B \in \mathrm{Pic}^b(X)$. Assume $a \leq b$ and set $F := A \oplus B$.

(i) If $z \geq 2g-1-a$, then $h^1(X, F \otimes L) = 0$ for every $L \in \mathrm{Pic}^z(X)$.

(ii) If $a < b$, then $\omega_X \otimes A^* \in \tau(F)$.

(iii) If $a = b$, then $2g-2-a \in \eta(F)$ if and ond only either $A \cong B$ (and in this case $\tau(F, 2g-2-a) = \{\omega_X \otimes A^*)$ or $A \ncong B$ and $B \otimes \omega_X \otimes A^*$ is spanned (and in this case $\tau(F, 2g-2-a) = \{\omega_X \otimes A^*\})$ or $A \ncong B$ and $A \otimes \omega_X \otimes B^*$ is spanned (and in this case $\tau(F, 2g-2-a) = \{\omega_X \otimes B^*\}$).

Hence (as in the genus one case) $\tau(F) \neq \emptyset$ for every decomposable and not semistable vector bundle F.

Remark 2. Let X be a hyperelliptic curve of genus $g \geq 2$ and $R \in \mathrm{Pic}^2(X)$ the hyperelliptic line bundle. Fix $A, B, L \in \mathrm{Pic}(X)$ such that $\deg(A) = \deg(B)$ and set $F := A \oplus B$. If $h^1(X, A \otimes L) > 0$ and $A \otimes L$ is spanned, then $\deg(A \otimes L)$ is even and there is an integer t such that $0 \leq t \leq g-1$ and $A \otimes L \cong R^{\otimes t}$. Now assume $g = 2$. If $\deg(B \otimes L) \leq 2$ and $h^1(X, B \otimes L) = 0$, then $h^0(X, B \otimes L) \leq 1$, $B \otimes L \ncong \mathcal{O}_X$ (Riemann-Roch). Thus $h^0(X, B \otimes L) = 0$. Hence $B \otimes L$ is not spanned. Hence $\tau(F) = \emptyset$ if $g = 2$. Now assume $g \geq 3$, $\deg(A) = \deg(B)$, F spanned and $h^1(X, A \otimes L) > 0$. If $h^1(X, B \otimes L) > 0$, then we just saw that $A \cong B$. Now assume $A \ncong B$ and that there are no $P, Q \in X$, $P \neq Q$ such that $B \cong A(Q-P)$; since $g \geq 3$, this assumption is satisfied (for any fixed A) by a general $B \in \mathrm{Pic}^{\deg(A)}(X)$. Take $L := \omega_X \otimes A^*$. Hence $A \otimes L \cong \omega_X$ and $h^1(X, B \otimes L) = 0$. It is easy to check that our assumption is equivalent to the spannedness of $B \otimes L$ and hence of $F \otimes L$. Now assume $B \cong A(Q-P)$ for some $P, Q \in X$ such that $P \neq Q$. Take any L such that $h^1(X, A \otimes L) > 0$ and $A \otimes L$ spanned, i.e. such that $A \otimes L \cong R^{\otimes t}$ for some integer t such that $0 \leq t \leq g-1$. Using Riemann-Roch it is easy to see that Q is a base point of $R^{\otimes t}(Q-P)$ and hence $F \otimes L$ is not spanned.

Remark 3. Let X be a smooth curve of genus $g \geq 2$ and F an indecomposable rank two vector bundle on X. Assume that F is not semistable. Hence there is an exact sequence

$$0 \to A \to F \to B \to 0 \tag{1}$$

such that $A, B, \in \mathrm{Pic}(X)$ and $\deg(A) > \deg(B)$. Furthermore, (1) is uniquely determined by F. Set $L := \omega_X \otimes B^*$. Hence $F \otimes L$ is an extension of ω_X by a line bundle $A \otimes L$ of degree at least $2g-1$. Hence $h^1(X, A \otimes L) = 0$

and $h^1(X, F \otimes L) = 1$. Thus if $A \otimes L$ is spanned, then $F \otimes L$ is spanned. By Riemann-Roch $A \otimes L$ is spanned, unless $\deg(A) = \deg(B) + 1$ and there is $P \in X$ such that $A \cong B(P)$. Form now on we fix $P \in X$ and assume $A \cong B(P)$. Take again $L := \omega_X \otimes B^*$. Hence we have an exact sequence

$$0 \to \omega_X(P) \to F \otimes L \to \omega_X \to 0 \tag{2}$$

We know that $F \otimes L$ is spanned, except at most at P. $F \otimes L$ is spanned at P if and only if $h^0(X, F \otimes L(-P)) = h^0(X, F \otimes L) - 2 = 2g - 2$. Assume $h^0(X, F \otimes L(-P)) \neq 2g - 2$. From the exact sequence

$$0 \to \omega_X \to F \otimes L(-P) \to \omega_X(-P) \to 0 \tag{3}$$

we obtain $h^0(X, F \otimes L(-P)) = 2g-1$ and that the map $\beta : H^0(X, F \otimes L(-P)) \to H^0(X, \omega_X(-P))$ is surjective. Hence $F \otimes L(-P)$ is spanned if $\omega_X(-P)$ is spanned. By Riemann-Roch it is obvious that $\omega_X(-P)$ is spanned if and only if X is not hyperelliptic. We give an independent proof that if β is surjective and $F \otimes L$ is not spanned, then (3) splits and hence F is decomposable, contradicting our assumption. Indeed, assuming $F \otimes L$ not spanned, call $E \subset F \otimes L$ the subsheaf of $F \otimes L$ spanned by $H^0(X, F \otimes L)$. We saw that E is a rank two vector bundle, that $\deg(E) = 4g - 4$, that $(F \otimes L)/E$ is supported by P and that the only subsheaf of $F \otimes L$ isomorphic to $\omega_X(P)$ has its proper subsheaf $\mathcal{I}_P \otimes \omega_X(P)$ as intersection with E. Hence E fits in an exact sequence

$$0 \to \omega_X \to E \to \omega_X \to 0 \tag{4}$$

Since $h^0(X, E) = h^0(X, F \otimes L) = 2g$, we easily see that (4) splits and hence $E \cong \omega_X^{\oplus 2}$. Hence $F \otimes L$ is obtained from $\omega_X^{\oplus 2}$ making a positive elementary transformation supported by P. It is easy to check that every positive elementary transformation, W, of $\mathcal{O}_X^{\oplus 2}$ supported by P is isomorphic to $\mathcal{O}_X(P) \oplus \mathcal{O}_X$ (hint: use $h^0(X, W) \geq 2$ and $g \geq 2$). Hence F is decomposable, contradiction.

Remark 4. Let X be a smooth curve of genus $g \geq 2$ and F an indecomposable rank two vector bundle on X. Later we will need to assume that X is not hyperelliptic. Assume that F is semistable, but not stable. Hence there is an exact sequence (1) such that $A, B, \in \mathrm{Pic}(X)$ and $\deg(A) = \deg(B)$. Furthermore, (1) is uniquely determined by F (hint: use that F is indecomposable). Here we assume $A \ncong B$. Set $L := \omega_X \otimes B^*$. Hence we have an exact sequence

$$0 \to M \to F \otimes L \to \omega_X \to 0 \tag{5}$$

such that $\deg(M) = 2g - 2$ and $h^1(X, M) = 0$. Hence $h^1(X, F \otimes L) = 1$, $h^0(X, F \otimes L) = 2g - 1$ and $F \otimes L$ is spanned if M is spanned. It is easy to check using Riemann-Roch that M is not spanned if and only if there are $P, Q \in X$ such that $P \neq Q$ and $M \cong \omega_X(P-Q)$. Hence from now on we assume $M \cong \omega_X(P-Q)$. M is spanned, except at P. Hence $H^0(X, F\otimes L)$ spans $F\otimes L$ outside P. $F\otimes L$ is spanned at P if and only if $h^0(X, F\otimes L(-P)) = 2g-3$. Assume that this in not the case. Twisting (5) with $\mathcal{O}_X(-P)$ we obtain the surjectivity of the map $\beta : H^0(X, F\otimes L(-P)) \to H^0(X, \omega_X(-P))$. Now we assume that X is not hyperelliptic. Then $\omega_X(-Q)$ and $\omega_X(-P)$ are spanned and special. Hence in this case we may take $F\otimes L(-P)$ as spanned and special twist of F.

Remark 5. Let X be a smooth curve of genus $g \geq 2$ and F an indecomposable rank two vector bundle on X which is an extension of a line bundle B by a line bundle isomorphic to B, i.e. we are in the set-up of Remark 4 with $A \cong B$. Set $L := \omega_X \otimes B^*$. Hence we have an extension

$$0 \to \omega_X \to F\otimes L \to \omega_X \to 0 \tag{6}$$

which does not split. By Serre duality the set of all such non-zero extensions, up to a non-zero multiplicative constant, is parametrized by the $(g-1)$-dimensional projective space $|\omega_X|$ of all hyperplanes of $H^0(X, \omega_X)$. More precisely, any non-zero extension ϵ induces a linear map $\beta_\epsilon : H^0(X, F\otimes L) \to H^0(X, \omega_X)$ whose image is the associated hyperplane H_ϵ. Thus $F\otimes L$ is not spanned if and only if there is $P \in X$ such that $H_\epsilon = H^0(X, \mathcal{I}_Q \otimes \omega_X)$. If $g \geq 3$, then a general extension ϵ corresponds to a spanned vector bundle $F\otimes L$.

Notation 2. Let X be a smooth curve of genus g and E a rank two vector bundle on X. Fix any exact sequence

$$0 \to A \to E \to B \to 0 \tag{7}$$

with $A, B \in \mathrm{Pic}(X)$ and $\deg(A)$ maximal. Set $s(E) := \deg(B) - \deg(A)$.

The integer $s(E)$ is often called the stability degree of E because E is stable (resp. semistable) if and only if $s(E) > 0$ (resp. $s(E) \geq 0$). Notice that $s(E) \equiv \deg(E) \pmod 2$ and that $s(E\otimes L) = s(E)$ for all $L \in \mathrm{Pic}(X)$. By a theorem of C. Segre and M. Nagata we have $s(E) \leq g$ ([5]). Now assume $g \geq 2$. For all integers d, s let $M(X; 2, d)$ denote the moduli space of all rank two vector bundles on X with degree d. Set $M(X; 2, d; s) := \{E \in M(X; 2, d) : s(E) =\}$. Hence $M(X; 2, d; s) = \emptyset$ if either $s > g$ or $s < 0$ or $d - s$ is odd. Now assume $0 < s \leq g$ and $d \equiv s \pmod 2$. The set $M(X; 2, d; s)$ is non-empty, irreducible and of dimension $3g + s - 2$ (case $s \leq g-2$) or $4g - 3 = \dim(M(X; 2, d))$, (case $s \in \{g-1, g\}$) ([5], Prop. 3.1). In particular, $M(X; 2, d; g-1)$ (resp.

$M(X;2,d;g)$) is open and dense in $M(X;2,d)$ if $d \equiv g-1 \pmod 2$ (resp. $d \equiv g \pmod 2$). If $s \le g-2$, then a general $E \in M(X;2,d;s)$ has a unique line subbundle with maximal degree ([5], Prop. 3.3) and hence the associated exact sequence (7) is uniquely determined by E.

Proposition 2. *Let X be a smooth curve of genus $g \ge 4$. Fix integers d, s such that $0 < s \le g-3$ and $d \equiv s \pmod 2$, and a general $F \in M(X;2,d;s)$. Then $\tau(F) \neq \emptyset$ and $2g-2-(d+s)/2 \in \eta(F)$. More precisely for any extension F of a line bundle B by a line bundle A such that $h^0(X, Hom(A,B)) = 0$, the vector bundle $F \otimes \omega_X \otimes B^*$ is spanned and $h^1(X, F \otimes \omega_X \otimes B^*) = 0$.*

Proof. Take F fitting in a non-split exact sequence (1) with $\deg(A)$ maximal and take $L := \omega_X \otimes B^*$. Hence F fits in an exct sequence

$$0 \to M \to F \otimes L \to \omega_X \to 0 \tag{8}$$

with $\deg(M) = 2g-2-s \ge g+1$. For a general $F \in M(X;2,d;s)$ we may take M general in $\mathrm{Pic}^{2g-2-s}(X)$ because $\dim(\mathrm{Ext}^1(\omega_X, M)) = h^1(X, M \otimes \omega_X) = g+s-1$ is the same for all $M \in \mathrm{Pic}^{2g-2-s}(X)$. In particular, we may assume $h^1(X,M) = 0$ and that M is spanned. With these assumptions on M, any extension of ω_X by M is spanned. $\square$

Now (in the set-up of Proposition 2) we consider the case $h^0(X, Hom(A,B)) \neq 0$.

Proposition 3. *Let X be a smooth curve of genus $g \ge 5$. Fix integers d, s such that $0 < s \le g-4$ and $d \equiv s \pmod 2$. Fix $B \in \mathrm{Pic}(X)$ and a general effective degree s divisor D of X. Set $A := B(-D)$ and $L := \omega_X \otimes B^*$. Let F be a general bundle obtained from a general extension of B by A. Then $h^1(X, F \otimes L) > 0$ and $F \otimes L$ is spanned.*

Proof. We have an exact sequence (8) with $M \cong \omega_X(-D)$. Since $s \le g-4$, we have $h^1(X, \omega_X(-D)) = 1$ and $\omega_X(-D)$ is spanned. As in the proof of Proposition 2 it is sufficient to take F associated to an extension ϵ such that the associated map $\beta_\epsilon : H^0X, F \otimes L) \to H^0(X, \omega_X)$ has as its image a codimension one linear subspace of $H^0(X, \omega_X)$ spanning ω_X. $\square$

Now we will consider the case $s \in \{g-2, g-1, g\}$.

Proposition 4. *Let X be a smooth curve of genus $g \ge 5$. Fix integers d, s such that $s \in \{g-2, g-1, g\}$ and $d \equiv s \pmod 2$ and a general $F \in M(X;2,d;s)$. If $s = g-2$, assume that X is not hyperelliptic. If $s = g-1$, assume that X is neither hyperelliptic, nor bielliptic nor isomorphic to a plane quintic. If $s = g$,* *assume $g \geq 11$ and that X has Clifford index at least 3 Then $\tau(F) \neq \emptyset$ and $1 - g + (d - s)/2 \in \eta(F)$.*

Proof. Our assumptions on s and X imply the existence of $M \in \mathrm{Pic}^{2g-2-s}(X)$ such that $h^0(X, M) = 2$ (and hence $h^1(X, M) > 0$ by Riemann-Roch) and M is spanned; for the case $s = g - 1$, see [3], Lemma 2.1.1; the case $s = g - 2$ is easier and well-known; if $s = g$, the reselt is checked in the middle of the proof of [3], Prop. 2.3.1. Take as $F \otimes L$ a general extension (8) of ω_X by M. We may find such an extension so that the image of the associated map $H^0(X, F \otimes L) \to H^0(X, \omega_X)$ contains a $h^1(X, M)$-codimensional linear subspace of $H^0(X, \omega_X)$ spanning ω_X, concluding the proof. □

3. Question 2

Notation 3. Let X be a smooth curve of genus g and $L \in \mathrm{Pic}(X)$. For any integer $x \geq 0$ consider the following condition (β, x) that L may satifies:

(β, x): For a general $S \subset X$ such that $\sharp(S) = x$ the line bundle $R(-S)$ is spanned.

Notice that for any fixed $R \in \mathrm{Pic}(X)$ if we take a general $S \subset X$ such that $\sharp(S) = x$, then $h^0(X, R(-S)) = \max\{0, h^0(X, R) - x\}$ and $h^1(X, R(-S)) = \max\{h^1(X, R), x - h^0(X, R)\}$. L satisfies $(\beta, 0)$ if and only if it is spanned. If L satisfies (β, x), then it satisfies (β, y) for all $0 \leq y \leq x$.

For simplicity we will just state the next result for $r = 2$. The interested reader may easily extend it to the case $r \geq 3$ using [2].

Proposition 5. *Let X be a smooth curve of genus $g \geq 2$, $L_i \in \mathrm{Pic}(X)$, $i = 1, 2$, and an integer $t > 0$. Assume that each L_i satisfies $(\beta, t + 1)$. Set $d_i := \deg(L_i)$ and $E_t := L_1 \oplus L_2$. Assume define the rank two vector bundles E_y for all $j < y \leq x$. Let E_j be any rank two vector bundle obtained from E_{j+1} making a general negative elementary transformation. Then $\deg(E_j) = \deg(E_{j+1} - 1)$, $h^0(X, E_j) = \max\{0, h^0(X, L_1) + h^0(X, L_2) - t + j\}$ and $h^1(X, E_j) = \max\{h^1(L_1) + h^1(L_2), t - j - h^0(X, L_1) + h^0(X, L_2)\}$. For all $j \geq 0$ the vector bundle E_j is spanned. If $d_1 \neq d_2$, then E_j is semistable (resp. stable) if and only if $t - j > |d_1 - d_2|$ (resp. $t - j \geq |d_1 - d_2|$). If $d_1 = d_2$, then E_t is semistable. If $d_1 = d_2$ and $L_1 \ncong L_2$, then E_j is stable for all $j < t$. If $L_1 \cong L_2$, then E_{t-1} is not semistable, E_{t-2} is semistable, indecomposable, but not stable, while all E_j, $j \leq t - 3$, are stable.*

Proof. The cohomology groups of each E_j were computed in [2], Lemma 2.2. For the other assumptions, use [2], Prop. 2.3 and Cor. 2.4. $\square$

It is easy (e.g. using plane nodal curves or nodal curves in $\mathbf{P}^1 \times \mathbf{P}^1$) to give example of curves X equipped with line bundles L_i, $i = 1, 2$, satisfying (β, t) for certain t and hence curves for which we may apply Proposition 5 to get examples related to Proposition 5.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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