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PARTIALLY RELAXED PSEUDOMONOTONE MAPPINGS IN APPROXIMATION-SOLVABILITY OF NONLINEAR VARIATIONAL INEQUALITIES

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Abstract: Let H be a real (finite-dimensional) Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \to H$ be $(\gamma) - \textit{partially}$ relaxed pseudomonotone, and let $L : K \to H$ be $(\delta, r) - \textit{partially}$ relaxed pseudomonotone. Then a class of nonlinear variational inequality (NVI) problems is described as: find an element $x^* \in K$ such that

$$\langle \rho(T(x^*) + L(x^*)), x - x^* \rangle \geq 0\,, \quad \forall\, x \in K \text{ and for } \rho > 0.$$

Let x^* be a solution to the NVI problem and a sequence $\{x^k\}$ be generated by a certain iterative algorithm. Suppose that mappings $T, L : K \to H$ satisfy the following assumptions:

(i) T is $(\gamma) - \textit{partially}$ relaxed pseudomonotone.

(ii) T is $(\beta) - \textit{Lipschitz}$ continuous.

(iii) L is $\delta, r) - \textit{partially}$ relaxed pseudomonotone.

(iv) L is $(s) - \textit{Lipschitz}$ continuous.

Then the sequence $\{x^k\}$ converges to x^* for $0 < \rho < \frac{1}{2r}$, $0 < \rho < \frac{1}{2(\gamma+\delta)}$, and the following estimates hold:

(a) $\|x^{k+1} - x^*\|^2 \leq (1 - 2\rho r)\|x^k - x^*\|^2 - [1 - 2\rho(\gamma + \delta)]\|x^k - x^{k+1}\|^2$.

(b) $\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - [1 - 2\rho(\gamma + \delta)]\|x^k - x^{k+1}\|^2$.

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1. Introduction

Verma [22, 24, 25], motivated by the work of Cohen [4, 5], extended the variational inequality type algorithms and applied them to the approximation solvability of a class of variational inequalities in a Hilbert space setting, which of course have applications to $\mathbf{R}^n$ space settings. We plan in this paper to present the approximation solvability of a class of variational inequalities involving the $(\gamma) -$ *partially* relaxed pseudomonotone and $(\delta, r) -$ *partially* relaxed pseudomonotone mappings with applications to the $\mathbf{R}^n$ space setting. For more details on the approximation solvability of variational inequalities and related developments on iterative algorithms, we recommend $[1 - 28]$.

Let H be a real Hilbert space with the inner product $\langle x, y\rangle$ and norm $\|x\|$ for all $x, y \in H$. Let $T : K \to H$ be $(\gamma) -$ *partially* relaxed pseudomonotone, and let $L : K \to H$ be $(\delta, r) -$ *partially* relaxed pseudomonotone. We consider a class of nonlinear variational inequalities (abbreviated as NVI): determine an element $x^* \in K$ such that

$$\langle \rho(T(x^*) + L(x^*)), x - x^*\rangle \geq 0\,, \quad \forall\, x \in K, \tag{1}$$

which is equivalent to the projection formula

$$x^* = P_K[x^* - \rho(T(x^*) + L(x^*))], \tag{2}$$

where P_K is the projection of H onto K, and ρ is a positive constant.

The projection P_K satisfies the following properties:

(i) For a given element $x \in H$, we have:

$$\langle P_K(x) - x, y - P_K(x)\rangle \geq\ 0\,, \quad \forall\, y \in K.$$

(ii) $\langle P_K(x) - P_K(y), x - y\rangle \geq\ \|P_K(x) - P_K(y)\|^2\,, \quad \forall\, x, y \in H.$

(iii) $\langle (I - P_K)(x) - (I - P_K)(y), x - y\rangle$

$$\geq\ \frac{1}{2}\|(I - P_K)(x) - (I - P_K)(y)\|^2, \quad \forall\, x, y \in H,$$

where I is the identity mapping.

Now we recall some auxiliary results, crucial to work on hand.

Lemma 1. *An element* $u \in K$ *is a solution to* (1) *if and only if*

$$u = P_K[u - \rho(T(u) + L(u))].$$

Lemma 2. *An element* $u \in K$ *is a solution to* (1) *if and only if*

$$\langle \rho(T(u) + L(u)), x - u \rangle \geq 0\,, \quad \forall\, x \in K.$$

Lemma 3. *For elements* $u, v \in H$*, we have*

$$\|u\|^2 + \langle u, v \rangle \geq -\frac{1}{4}\|v\|^2.$$

Lemma 4. *For* $u, v \in H$*, we have*

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u\|^2 - \|v\|^2}{2}.$$

2. Partial Relaxed Monotonicity/Pseudomonotonicity

In this section, we recall the notion of the partial relaxed monotonicity as well as partial relaxed pseudomonotonicity. The author [22] introduced partial monotonicity in the context of studying some nonlinear variational inequality problems. This notion is still less explored.

Definition 1. A mapping $T : H \to H$ is called:
(i) *monotone* if for each $x, y \in H$ we have

$$\langle T(x) - T(y), x - y \rangle \geq 0.$$

(ii) $(r) - \mathit{strongly}$ monotone if there exists a positive constant r such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2 \text{ for all } x, y \in H.$$

(iii) $(r) - \mathit{expansive}$ if

$$\|T(x) - T(y)\| \geq r\|x - y\|.$$

(iv) expansive if $r = 1$ in (iii).

(v) $(\gamma) -$ *cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \gamma \|T(x) - T(y)\|^2 \text{ for all } x, y \in H.$$

(vi) *pseudomonotone* if

$$\langle T(y), x - y)\rangle \geq 0 \Rightarrow \langle T(x), x - y)\rangle \geq 0.$$

(vii) $(b) -$ *strongly* pseudomonotone if

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq b\|x - y\|^2 \text{ for all } x, y \in H.$$

(viii) $(c) -$ *pseudococoercive* if there exists a constant $c > 0$ such that

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq c\|T(x) - T(y)\|^2 \text{ for all } x, y \in H.$$

(ix) *quasimonotone* if

$$\langle T(y), x - y\rangle > 0 \Rightarrow \langle T(x), x - y\rangle \geq 0 \text{ for all } x, y \in H.$$

(x) $(L)-$*relaxed* (also called weakly monotone) if there is a positive constant L such that

$$\langle T(x) - T(y), x - y\rangle \geq (-L)\|x - y\|^2 \text{ for all } x, y \in H.$$

(xi) *hemicontinuous* if for all $x, y, w \in H$ the function

$$t \in [0, 1] \to \langle T(y + t(x - y)), w\rangle$$

is continuous.

(xii) $(\beta) -$ *Lipschitz* continuous if there exists a constant $\beta \geq 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\|.$$

(xiii) $(\gamma) -$ *partially* relaxed monotone if there exists a positive constant γ such that

$$\langle T(x) - T(y), z - y\rangle \geq (-\gamma)\|z - x\|^2 \text{ for all } x, y, z \in H.$$

(xiv) $(\gamma) -$ *partially* relaxed pseudomonotone if there exists a positive constant γ such that

$$\langle T(y), z - y\rangle \geq 0 \Rightarrow \langle T(x), z - y\rangle \geq (-\gamma)\|z - x\|^2 \text{ for all } x, y, z \in H.$$

(xv) $(\gamma, r) -$ *partially* relaxed pseudomonotone if there exist positive constants γ, r such that

$$\langle T(y), z - y\rangle \geq 0 \Rightarrow \langle T(x), z - y\rangle \geq (-\gamma)\|z - x\|^2 + r\|x - y\|^2 \text{ for all } x, y, z \in H.$$

3. Algorithms and the NVI Problem

This section deals with an introduction to an iterative algorithmic procedure and its application to the approximation-solvability of NVI (1).

Algorithm 1. For an arbitrarily chosen initial point $x^0 \in K$, determine an iterate x^{k+1} such that $(k \geq 0)$

$$\langle \rho(T(x^0) + L(x^0)) + x^1 - x^0, x - x^1) \rangle \geq 0\,,$$

$$\vdots$$

$$\langle \rho(T(x^k) + L(x^k)) + x^{k+1} - x^k, x - x^{k+1} \rangle \geq 0.$$

Note that Algorithm 1 is equivalent to

$$x^{k+1} = P_K[x^k - \rho(T(x^k) + L(x^k))],$$

where P_K is the projection of H onto K.

We now present, based on Algorithm 1, the approximation solvability of the NVI (1) in a Hilbert space setting.

Theorem 1. *Let H be a real finite-dimensional Hilbert space and let K be a nonempty closed convex subset of H. Let x^* be a solution to the NVI (1) problem and a sequence $\{x^k\}$ be generated by Algorithm 1. Suppose that mappings $T, L : K \to H$ satisfy the following assumptions:*

(i) T *is* $(\gamma) -$ *partially relaxed pseudomonotone.*

(ii) T *is* $(\beta) -$ *Lipschitz continuous.*

(iii) L *is* $(\delta, r) -$ *partially relaxed pseudomonotone.*

(iv) L *is* $(s) -$ *Lipschitz continuous.*

Then following conclusions hold:

(a) $\|x^{k+1} - x^*\|^2 \leq (1 - 2\rho r)\|x^k - x^*\|^2 - [1 - 2\rho(\gamma + \delta)]\|x^k - x^{k+1}\|^2.$

(b) $\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - [1 - 2\rho(\gamma + \delta)]\|x^k - x^{k+1}\|^2.$

(c) The sequence $\{x^k\}$ *converges to* x^* *for* $0 < \rho < \frac{1}{2r}$, $0 < \rho < \frac{1}{2(\gamma+\delta)}$.

Proof. First we compute the estimates, and then we show the convergence of the sequence $\{x^k\}$ to x^*, a solution to NVI (1) problem. Since x^k satisfies Algorithm 1, we have (for $\rho > 0$) that

$$\langle \rho(T(x^k) + L(x^k)) + x^{k+1} - x^k, x - x^{k+1}\rangle \geq 0\,, \quad \forall\, x \in K. \tag{3}$$

On the top of that, x^* is a solution to (1), that is, we have for $x = x^{k+1}$ that

$$\langle \rho(T(x^*) + L(x^*)), x^{k+1} - x^*\rangle \geq 0. \tag{4}$$

Replacing x by x^* in (3), we obtain

$$0 \leq -\langle \rho(T(x^k) + L(x^k)), x^{k+1} - x^*\rangle + \langle x^{k+1} - x^k, x^* - x^{k+1}\rangle.$$

Since T is $(\gamma) - \mathit{partially}$ relaxed pseudomonotone and L is $(\delta, r) - \mathit{partially}$ relaxed pseudomonotone, it implies that

$$0 \leq \rho(\gamma+\delta)\|x^{k+1} - x^k\|^2 - \rho r\|x^k - x^*\|^2 + \langle x^{k+1} - x^k, x^* - x^{k+1}\rangle. \tag{5}$$

Taking $u = x^{k+1} - x^k$ and $v = x^* - x^{k+1}$ in Lemma 4, and applying to (5), we have

$$\begin{aligned} 0 \leq \frac{1}{2}[\|x^* - x^k\|^2 - \|x^{k+1} - x^k\|^2 - \|x^* - x^{k+1}\|^2] \\ + \rho(\gamma+\delta)\|x^{k+1} - x^k\|^2 - \rho r\|x^k - x^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 \\ + 2\rho(\gamma+\delta)\|x^{k+1} - x^k\|^2 - 2\rho r\|x^k - x^*\|^2. \end{aligned} \tag{6}$$

That means, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq (1 - 2\rho r)\|x^k - x^*\|^2 - [1 - 2\rho(\gamma+\delta)]\|x^{k+1} - x^k\|^2 \\ &\leq \|x^k - x^*\|^2 - [1 - 2\rho(\gamma+\delta)]\|x^{k+1} - x^k\|^2, \end{aligned} \tag{7}$$

where $1 - 2\rho r > 0$. Therefore, (7) implies that

$$\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \geq [1 - 2\rho(\gamma+\delta)]\|x^{k+1} - x^k\|^2. \tag{8}$$

Under the assumptions of the theorem, we infer from (8) that $\{\|x^k - x^*\|^2\}$ is a strictly decreasing sequence, and the difference of two consecutive terms tends to zero. It follows that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0.$$

Let x' be a limit of the subsequence $\{x^k\}'$ of the sequence $\{x^k\}$. Since the left hand side of (7) is bounded, it ensures the existence of such a convergent subsequence. Furthermore, the projection- an equivalent form of Algorithm 1 – defined by

$$x^{k+1} = P_K[x^k - \rho(T(x^k) + L(x^k))],$$

where P_K is the projection of H onto K, in light of the $(\beta)-$*Lipschitz* continuity of the mapping T and $(s)-$*Lipschitz* continuity of the mapping L, is continuous, and so x' is a fixed point of this projection mapping. Hence, x' is a solution to the NVI problem (1).

This completes the proof. □

4. Applications

In this section, the results from Section 3 are applied to the $\mathbf{R^n}$ space setting. Let $F, L : X \to \mathbf{R^n}$ be mappings from a closed convex X of $\mathbf{R^n}$ into $\mathbf{R^n}$. Let P_{D_ρ} denote the projection on X, and let $\|.\|_{D_\rho}$ denote the norm induced by the symmetric positive-definite matrix D_ρ. We consider the nonlinear variational inequality problem: find an element $u \in X$ such that

$$[F(u) + L(u)]^t(x - u) \geq 0\,, \quad \forall\ x \in X, \tag{9}$$

where $[F(u)]^t$ denotes the transpose of the vector $F(u)$.

When $L = 0$, we find the nonlinear variational inequality problem: determine an element $u \in X$ such that

$$[F(u)]^t(x - u) \geq 0\,, \quad \forall\, x \in X, \tag{10}$$

Based on Algorithm 1, we have

Algorithm 2. For an arbitrarily chosen initial point $x^0 \in X$, compute an iterate x^{k+1} such that $(k \geq 0)$

$$[\rho(F(x^0) + L(x^0)) + D_\rho(x^1 - x^0)]^t(x - x^1) \geq 0\,,$$

$$\vdots$$

$$[\rho(F(x^k) + L(x^k)) + D_\rho(x^{k+1} - x^k)]^t(x - x^{k+1}) \geq 0,$$

where D_ρ is a fixed positive-definite matrix.

In what follows, D_ρ shall denote a symmetric matrix in Algorithm 2 regarding the convergence of the projection method. The symbol $\lambda_{min}(S)$ shall denote the minimum eigenvalue of a symmetric matrix S.

Since D_ρ is symmetric, it implies that Algorithm 2 is equivalent to

$$x^{k+1} = P_{D_\rho}[x^k - D_\rho^{-1}(\rho(F(x^k) + L(x^k)))], \tag{11}$$

where P_{D_ρ} is the projection of $\mathbf{R^n}$ onto X with the norm $\|.\|_{D_\rho}$ induced by the positive-definite symmetric matrix D_ρ.

For $L = 0$, Algorithm 2 reduces to the following one.

Algorithm 3. *For an arbitrarily chosen initial point $x^0 \in K$, compute an iterate x^{k+1} such that $(k \geq 0)$*

$$[\rho F(x^0) + D_\rho(x^1 - x^0]^t(x - x^1) \geq 0\,,$$

$$\vdots$$

$$[\rho F(x^k) + D_\rho(x^{k+1} - x^k)]^t(x - x^{k+1}) \geq 0,$$

where D_ρ is a fixed positive-definite matrix.

Note that Algorithm 3 is equivalent to

$$x^{k+1} = P_{D_\rho}[x^k - D_\rho^{-1}(\rho F(x^k))]. \tag{12}$$

Theorem 2. *Let X be a nonempty closed convex subset of $\mathbf{R^n}$, and let x^* be a solution to the NVI problem (9). Let a sequence $\{x^k\}$ be generated by Algorithm 2. Suppose that mappings $F, L : X \to \mathbf{R^n}$ satisfy the following assumptions:*

(i) F is $(\gamma) -$ partially relaxed pseudomonotone.

(ii) F is $(\beta) -$ Lipschitz continuous.

(iii) L is $\delta, r) -$ partially relaxed pseudomonotone.

(iv) L is $(s) -$ Lipschitz continuous.

Then following conclusions hold:

(a) $\|x^{k+1} - x^*\|_D^2 \leq (1 - \frac{2\rho r}{\lambda_{min(D)}})\|x^k - x^*\|_D^2 - [1 - \frac{2\rho(\gamma+\delta)}{\lambda_{min(D)}}]\|x^k - x^{k+1}\|_D^2$.

(b) $\|x^{k+1} - x^*\|_D^2 \leq \|x^k - x^*\|_D^2 - [1 - \frac{2\rho(\gamma+\delta)}{\lambda_{min(D)}}]\|x^k - x^{k+1}\|_D^2$.

(c) The sequence $\{x^k\}$ converges to x^* for $0 < \rho < \frac{\lambda_{min(D)}}{2r}$, $0 < \rho < \frac{\lambda_{min(D)}}{2(\gamma+\delta)}$, where D is a symmetric positive-definite matrix.

Proof. The proof is similar to that of Theorem 1. □

For $L = 0$, we arrive at

Theorem 3. *Let X be a nonempty closed convex subset of $\mathbf{R^n}$, and let x^* be a solution to the NVI problem (10). Let a sequence $\{x^k\}$ be generated by Algorithm 3. Suppose that mappings $F, L : X \to \mathbf{R^n}$ satisfy the following assumptions:*

(i) F is $(\gamma) - partially$ relaxed pseudomonotone.

(ii) F is $(\beta) - Lipschitz$ continuous.

Then following conclusions hold:

(a) $\|x^{k+1} - x^*\|_D^2 \leq -[1 - \frac{2\rho(\gamma)}{\lambda_{min(D)}}]\|x^k - x^{k+1}\|_D^2$.

(b) The sequence $\{x^k\}$ converges to x^* for $0 < \rho < \frac{\lambda_{min(D)}}{2\gamma}$, where D is a symmetric positive-definite matrix.

Remark 1. If we replace the $(\gamma) - partial$ relaxed pseudomonotonicity and $(\beta) - Lipschutz$ continuity of F in Theorem 3 by the $(\gamma) - cocoercivity$, we arrive at Marcotte and Wu [15, Theorem 2.1].

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