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ON REDUCIBILITY OF ADJOINT SEMIGROUPS OF POSITIVE OPERATORS ON A NORMED RIESZ SPACE

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Abstract: We prove several results on the existence of invariant closed ideals for adjoint semigroups of positive operators on a normed Riesz space (of dimension greater than 1). If S is a multiplicative semigroup of positive operators on a such space that are quasinilpotent at an atom, then adjoint semigroup $S^{\odot}$ has a non-trivial invariant closed ideal. Furthermore, if T is a non-zero positive operator such that adjoint of T, T', is quasinilpotent at an atom and if S is a multiplicative semigroup of positive operators such that $RT \leq TR$ for all $R \in S$, then adjoint semigroup $S^{\odot}$ and adjoint T' have a common non-trivial invariant closed ideal.

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1. Introduction

Invariant closed subspaces of bounded operators are among the most studied topics in the operator theory. In the case of positive operators on Banach lattices the order structure enable us to obtain invariant subspaces. In the present dual normed Riesz space setting, we extend and strengthen the results of both papers, Abramovich et al [1], Drnovsek [3]. The proofs of our results

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mainly follow similar ideas that were introduced in [1], [4], [3]. In Jahandideh [4], he considered the positive commutant of a given positive operator and collections of positive operators dominated by a given operator. In this article, we show that the some results of Drnovsek [3] are valid in the dual normed Riesz spaces. For unexplained notions and terminology we refer to the books of Aliprantis et al [2], Meyer [5], Schaefer [7], and Zaanen [8].

A collection of linear transformations is called a semigroup if it is closed under multiplication as composition. A subset A of a semigroup S is said to be a semigroup ideal if RT and TR belong to A for all $R \in S$ and $T \in A$. It is well-known Radjavi [6] that a multiplicative semigroup S of operators on a Banach space has a non-trivial invariant closed subspace whenever a non-zero semigroup ideal of S has a non-trivial invariant closed subspace. Let X be a noormed Riesz space, and let $L(X)$ be the algebra of all bounded linear operators on X. X' is the norm dual of X and X_+ denotes the positive cone of X. A collection A of bounded operators on a normed Riesz space X is called reducible if there exists a closed ideal of X, other than $\{0\}$ and X, which is invariant under every member of A. The following result is a dual normed Riesz space analogue of this useful tool, Jahandideh [4], Drnovsek [3].

Theorem 1. (see [3]) *Let S be a multiplicative semigroup of positive operators on a normed Riesz space X. If a non-zero semigroup ideal τ of S has a non-trivial invariant closed ideal, then the adjoint semigroup $S^{\odot} = \{T' \in L(X') : T \in S\}$ has a non-trivial invariant closed ideal (band) as well.*

Proof. Let M be a non-trivial τ-invariant closed ideal. Then polar of M, M°, is a non-trivial $\tau^{\odot}$-invariant closed ideal, where $\tau^{\odot} = \{T' \in L(X') : T \in \tau\}$. Let us show that the order ideal N generated by the set

$$\{Tf : T \in \tau^{\odot}, f \in M^{\circ} \cap X'_+\}$$

is $S^{\odot}$-invariant. To end this, pick $g \in N$ and $S' \in S^{\odot}$. Since $g \in N$, there exist $0 \leq \lambda$, operators $T'_1, T'_2, ..., T'_n \in \tau^{\odot}$ and positive vectors $f_1, f_2, ..., f_n \in M^{\circ}$ such that $|g| \leq \lambda(T'_1 f_1 + ... + T'_n f_n)$.

Then $|S'g| \leq S'|g| \leq \lambda(S'T'_1 f_1 + ... + S'T'_n f_n)$. Since τ is a semigroup ideal, the operators $S'T'_1, ..., S'T'_n$ belong to $\tau^{\odot}$, so that $S'g \in N$. Thus the closure $\overline{N}$ of N is a closed $S^{\odot}$-invariant ideal. Since $N \subseteq M^{\circ}$, we have $\overline{N} \neq X'$. So, if $N \neq \{0\}$, we are done. Assume therefore that $N = \{0\}$. Then the closed ideal

$$K = \{f \in X' : T|f| = 0 \quad \forall T \in \tau^{\odot}\}$$

contains M°. Since $\tau^{\odot}$ is a nonzero ideal, K is not equal to X', and so it is a non-trivial closed ideal. To show that K is also $S^{\odot}$-invariant, fix $f\imath K$ and

$S \in S^{\odot}$. Then for any $T \in \tau^{\odot}$ we have $0 \leq T|Sf| \leq TS|f| = 0$, since $TS \in \tau^{\odot}$. This yields $Sf \in K$, and this completes the proof. □

Let X be a real normed Riesz space of dimension greater than 1, and let $f \in X'_+$ be an atom. Since X' is a Dedekind complete, the band generated by f in X', B_0, is a projection band in X',i.e., $X' = B_0 \oplus B_0^d$. Therefore, for each $h \in X'$ there exist unique real number λ and $g \in B_0^d$ such that $h = \lambda f + g$.

Theorem 2. (see [3]) *Let $\phi_0 \in X'_+$ be an atom and let S be a multiplicative semigroup of positive operators on a normed Riesz space X and let $x \in X_+$ be such that $\phi(Sx) = 0$ for all $S \in S$. Then adjoint semigroup $S^{\odot}$ has a non-trivial invariant closed ideal. In addition, since each member of $S^{\odot}$ is order continuous, $S^{\odot}$ has a nontrivial invariant closed band.*

Proof. If $S\phi_0 = 0$ for all $S \in S^{\odot}$, then B_0 is a non-trivial $S^{\odot}$-invariant band. Therefore, we have to consider the case when $S\phi_0 \neq 0$ for some $S \in S^{\odot}$.

Let I be the order ideal generated in X' by the set $\{S\phi_0 : S \in S^{\odot}\}$, i.e., I is the ideal of all $f \in X'$ such that there exist $0 \leq \lambda$ and $S_1, ..., S_n \in S^{\odot}$ such that $|f| \leq \lambda(S_1 + ... + S_n)\phi_0$. We claim that I is invariant under arbitrary $S \in S^{\odot}$. To show this, pick $f \in I$. Then $|f| \leq \lambda(S_1 + ... + S_n)\phi_0$ for some $0 \leq \lambda$ and $S_1, S_2, ..., S_n \in S^{\odot}$, and hence

$$|Sf| \leq S|f| \leq \lambda(SS_1 + SS_2 + ... + SS_n)\phi_0.$$

Since $S^{\odot}$ is a semigroup, we conclude that $Sf \in I$. Now, the closure $\overline{I}$ of I is invariant under S as well, because the operator S is continuous. Since $S\phi_0 = 0$, we have $S\phi_0 \in B_0^d$ for all $S \in S^{\odot}$, so that $\{0\} \neq \overline{I} \subseteq B_0^d \neq X'$. Thus we have shown that $\overline{I}$ is a non-trivial $S^{\odot}$-invariant closed ideal. □

Since adjoint of a positive operator is order continuous, each member of $S^{\odot}$ is order continuous. The band B generated by I is non-trivial, since $B \subseteq B_0^d \neq X'$. It is also invariant under each $S \in S^{\odot}$. Indeed, for each $f \in B \cap X'_+$ there exists an upwards net $\{f_\alpha\}$ of vectors in I such that $0 \leq f_\alpha \uparrow f$. Hence $0 \leq Sf_\alpha \uparrow Sf$, since S is order continuous. This implies that $Sf \in B_0$. Therefore, B is $S^{\odot}$-invariant band.

Theorem 3. (see [3]) *Let S be a multiplicative semigroup of positive operators on a normed Riesz space X such that each of them is quasinilpotent at an atom $f \in X'_+$. Then adjoint semigroup $S^{\odot}$ has a non-trivial invariant closed ideal (band).*

Proof. Fix $T \in S$ and put $\lambda = (fT)(x)$ for some $0 \leq x \in X$. From $\lambda f \leq T'f$ we obtain that $\lambda^n f \leq {T'}^n f$ for all $n \in N$, so that $\lambda \|f\|^{\frac{1}{n}} \leq \|{T'}^n f\|^{\frac{1}{n}}$. Since T' is quasinilpotent at f, this implies that $\lambda = 0$, and by the previous result we finish the proof. □

Theorem 4. (see [3]) *Let T be a non-zero positive operator on a normed Riesz space X such that adjoint of T is quasinilpotent at an atom $f \in X'_+$. Let S be a multiplicative semigroup of operators on X. If $T'|S'| \leq |S'|T'$ for all $S \in S$, then T' has a non-trivial invariant closed ideal (band) that is also $S^{\odot}$-invariant.*

Proof. Let τ be a semigroup generated by T' and the set $\{|S'| : S \in S\} = S^{\odot}$. Denote by I the semigroup ideal of τ generated by T'. We claim that each member of I is quasinilpotent. Indeed, if $U \in I$, then $U \leq |S'_1|...|S'_p|T'^k$ for some $k \in N$, $p \in N \cup \{0\}$, and $S_1, ..., S_p \in S$. Denoting $V = |S'_1|...|S'_p|T'^{(k-1)}$ we have $U \leq VT'$. Since $T'|S'| \leq |S'|T'$ by the assumption on S , an easy induction shows that $U^n \leq V^n T'^n$ for all $n \in N$. We therefore have

$$\|U^n f\| \leq \|V^n T'^n f\| \leq \|V\|^n \|T'^n f\|$$

for all $n \in N$, which implies that U is quasinilpotent at f. By Theorem 3 the semigroup ideal I has a non-trivial invariant closed ideal. Now Theorem 1 gives a non-trivial τ-invariant closed ideal J. Let us show that J is $S^{\odot}$-invariant as well. To end this, let $f \in J$ and $P \in S$. Since $|f| \in J$ and $|P'| \in \tau$, we conclude that $|P'||f| \in J$, and finally the inequality $|P'f| \leq |P'||f|$ implies that $P'f \in J$ (as J is an order ideal). □

Corollary 5. *Let $0 \leq S$ be a linear operator on a normed Riesz space X and let T be a non-zero positive operator on X such that:*

(i) $T'|S'| \leq |S'|T'$, and

(ii) T' is quasinilpotent at an atom f of X'_+,

then S' and T' have a common non-trivial invariant closed ideal.

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