International Journal of Pure and Applied Mathematics
Volume 22 No. 1 2005, 93-101

ON COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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Abstract: In this paper we prove some common fixed point theorems.

AMS Subject Classification: 47H10, 54H25
Key Words: common fixed point, fuzzy metric space

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [14]. Since then, it was developed extensively by many authors and used in various fields. Especially, [2, 4, 9] introduced the concept of fuzzy metric spaces in different ways. In [4, 5], George and Veeramani modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10], and obtained a Hausdorff topology for this kind of fuzzy metric spaces. Sessa [12] defined a generalization of commutativity, so called weak commutativity. Further Jungck [8] introduced more generalized commutativity, which is called compatibility in metric space and proved common fixed point theorems.

Recently, Singh and Chauhan [13] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems. Without the condition $\lim_{t\to\infty} M(x,y,t) = 1$ used in [13], Cho and Yang [1] proved some common fixed point theorems for compatible maps in fuzzy metric spaces. In this paper we have generalizations of results in [1].

Received: May 9, 2005

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2. Preliminaries

In this section, we give some definitions and lemmas.

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* [11] if $([0,1], *)$ is an Abelian topological monoid with 1 such that $a * b \le c * d$, whenever $a \le c, b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* [4] if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous, for all $x, y \in X$ and $t, s > 0$.

Example 1. Let (X, d) be a metric space and $a*b = ab$ or $a*b = \min\{a, b\}$, and let $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by d the standard fuzzy metric (see [4]).

Example 2. Let $X = [0, 1]$ and $a * b = \min\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \begin{cases} \frac{1}{2}, & (x \neq y), \\ 1, & (x = y). \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space (see [4]).

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *convergent* [6] to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called *Cauchy sequence* [6] if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, for all $t > 0$ and $p > 0$. A fuzzy metric space in which every Cauchy sequene is convergent is said to be *complete* [6]. Self mappings A and B of a fuzzy metric space $(X, M, *)$ is said to be *compatible* [13] if $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Lemma 2.1. (see [6]) *Let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing.*

Let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y, x_i, y_i \in X$ and $t > 0$ $(i = 1, 2, \cdots n)$, we denote $\frac{1}{M(x,y,t)} - 1$ by $N(x, y, t)$ and

$\frac{1}{\min\{M(x_1,y_1,t),M(x_2,y_2,t),\cdots,M(x_n,y_n,t)\}}-1$ by $Q(M(x_1,y_1,t),M(x_2,y_2,t),\cdots,M(x_n,y_n,t))$.

We know that $N(x,y,t)=Q(M(x,y,t))$.

From definitions, we get the next lemmas.

Lemma 2.2. *Let $(X,M,*)$ be a fuzzy metric space. Then we have the following:*

(1) A sequence $\{x_n\}$ in X converges to a point $x\in X$ if and only if $\lim_{n\to\infty}N(x_n,x,t)=0$.

(2) A sequence $\{x_n\}$ in X is Cauchy sequence if and only if $\lim_{n\to\infty}N(x_{n+p},x_n,t)=0$, for all $t>0$ and $p>0$.

(3) Self mappings A and B of X is compatible if and only if $\lim_{n\to\infty}N(ABx_n,BAx_n,t)=0$ for all $t>0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Bx_n=z$ for some $z\in X$.

Lemma 2.3. *Let $(X,M,*)$ be a fuzzy metric space and $\phi:[0,\infty)\to[0,\infty)$ be a continuous function with $\lim_{n\to\infty}\phi^n(t)=0$. If for all $x,y\in X$ and $t>0$, $N(x,y,t)\le\phi(N(x,y,t))$, then $x=y$.*

3. Common Fixed Point Theorems

In this section, we prove some common fixed point theorems for four mappings satisfying some conditions.

Theorem 3.1. *Let $(X,M,*)$ be a complete fuzzy metric space with continuous t-norm $*$ defined by $a*b=\min\{a,b\}$, $a,b\in[0,1]$. Let A,B,S and T be mappings from X into itself such that:*

(1) $A(X)\subset T(X)$ and $B(X)\subset S(X)$,

(2) S and T are continuous,

(3) A and S are compatible, and B and T are compatible,

(4) there exists a continuous function $\phi:[0,\infty)\to[0,\infty)$ with $\lim_{n\to\infty}\phi^n(t)=0$ such that

$$N(Ax,By,t)\le\phi(Q(M(Sx,Ty,t),M(Ax,Sx,t),M(By,Ty,t),\\ M(By,Sx,2t),M(Ax,Ty,t))),$$

for all $x,y\in X$ and $t>0$.

Then A,B,S and T have a unique fixed point in X.

Proof. Let $x_0\in X$. From (1), there exists $x_1\in X$ such that $Ax_0=Tx_1$. For this x_1, there exists $x_2\in X$ such that $Bx_1=Sx_2$.

Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$y_{2n} = Sx_{2n} = Bx_{2n-1}$ and $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2},$ for $n = 1, 2, \dots$.

From (4), we have that

$$
\begin{aligned}
(y_{2n+1}, y_{2n+2}, t) &= N(Ax_{2n}, Bx_{2n+1}, t) \\
&\le \phi(Q(M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\
&\quad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Bx_{2n+1}, Sx_{2n}, 2t), M(Ax_{2n}, Tx_{2n+1}, t))) \\
&= \phi(Q(M(y_{2n}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+2}, y_{2n}, 2t))) \\
&\le \phi(Q(M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n}, t))), \\
&\qquad \text{and hence} N(y_{2n+1}, y_{2n+2}, t) \le \phi(N(y_{2n}, y_{2n+1}, t)\,.
\end{aligned}
$$

In general,

$$N(y_n, y_{n+1}, t) \le \phi(N(y_n, y_{n-1}, t)). \tag{3.1.1}$$

From (3.1.1),

$$
\begin{aligned}
N(y_n, y_{n+1}, t) &\le \phi(N(y_n, y_{n-1}, t)) \le \phi^2(N(y_{n-2}, y_{n-1}, t)) \\
&\le \cdots \le \phi^n(N(y_0, y_1, t)) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Therefore, $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ and so for $p \in \mathbb{N}$,

$$
\begin{aligned}
&N(y_{n+p}, y_n, t) \\
&\quad \le Q(M(y_{n+1}, y_n, t/p), M(y_{n+2}, y_{n+1}, t/p), \cdots M(y_{n+p-1}, y_{n+p}, t/p)) \to 0\,,
\end{aligned}
$$

as $n \to \infty$.

Hence $\{y_n\}$ is Cauchy. Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Thus $\{Ax_{2n}\}, \{Sx_{2n}\}$, $\{Bx_{2n}\}$ and $\{Tx_{2n-1}\}$ also converges to z. Since S is continuous, $SAx_{2n} \to Sz$, and so $N(SAx_{2n}, Sz, t/2) \to 0$ as $n \to \infty$.

From (3), $N(SAx_{2n}, ASx_{2n}, t/2) \to 0$ as $n \to \infty$. Thus

$$
\begin{aligned}
&N(ASx_{2n}, Sz, t) \\
&\qquad \le Q(M(SAx_{2n}, ASx_{2n}, t/2), M(SAx_{2n}, Sz, t/2)) \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

and so

$$ASx_{2n} \to Sz. \tag{3.1.2}$$

Similarly, we have

$$BTx_{2n-1} \to Tz. \tag{3.1.3}$$

From (4), we have that

$$N(ASx_{2n}, BTx_{2n-1}, t) \\ \leq \phi(Q(M(S^2x_{2n}, T^2x_{2n-1}, t), M(ASx_{2n}, S^2x_{2n}, t), \\ M(BTx_{2n-1}, T^2x_{2n-1}, t), M(BTx_{2n-1}, S^2x_{2n}, 2t), M(ASx_{2n}, T^2x_{2n-1}, t))).$$

Taking the limit as $n \to \infty$, and using (3.1.2) and (3.1.3), we have $N(Sz, Tz, t) \leq \phi(N(Sz, Tz, t))$, and hence

$$Sz = Tz. \tag{3.1.4}$$

From (4), we get

$$N(Az, BTx_{2n-1}, t) \leq \phi(Q(M(Sz, T^2x_{2n-1}, t), M(Az, Sz, t), \\ M(BTx_{2n-1}, T^2x_{2n-1}, t), M(BTx_{2n-1}, Sz, 2t), M(Az, T^2x_{2n-1}, t))).$$

Taking limit as $n \to \infty$, and using $(3.1.2), (3.1.3)$ and (3.1.4), we have $N(Az, Tz, t) \leq \phi(N(Az, Tz, t))$, and hence

$$Az = Tz. \tag{3.1.5}$$

From (4), (3.1.4) and (3.1.5), we have that

$$N(Az, Bz, t) \leq \phi(Q(M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), \\ M(Bz, Sz, 2t), M(Az, Tz, t))) = \phi(N(Az, Bz, t)),$$

and hence $Az = Bz$. Therefore,

$$Az = Bz = Sz = Tz. \tag{3.1.6}$$

Now, we show that z is a fixed point of B.

From (4),

$$N(Ax_{2n}, Bz, t) \leq \phi(Q(M(Sx_{2n}, Tz, t), M(Ax_{2n}, Sx_{2n}, t), \\ M(Bz, Tz, t), M(Bz, Sx_{2n}, 2t), M(Ax_{2n}, Tz, t))).$$

Taking limit as $n \to \infty$, and using (3.1.6), we get

$N(z, Bz, t) \leq \phi(N(z, Bz, t))$, and hence $z = Bz$. Thus, z is a common fixed point of A, B, S and T.

For uniqueness, let y be another common fixed point of A, B, S and T. Then we have

$$N(Az, By, t) \le \phi(Q(M(Sz, Ty, t), M(Az, Sz, t), M(By, Ty, t), M(By, Sy, 2t), M(Az, Ty, t))),$$

and $N(Az, By, t) \le \phi(N(Az, By, t))$. Hence, $N(z, y, t) \le \phi(N(z, y, t))$.

Thus we have $z = y$. This completes the proof. □

Let $\phi(t) = kt$ for $k \in (0, 1)$. Then we have the following result.

Corollary 3.2. (see [1]) *Let $(X, M, *)$ be a complete fuzzy metric space with continuous t-norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that:*

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) S and T are continuous,

(3) A and S are compatible, and B and T are compatible,

(4) there exists $k \in (0, 1)$ such that $N(Ax, By, t) \le kQ(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t))$, for all $x, y \in X$ and $t > 0$.

Then A, B, S and T have a unique fixed point in X.

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space with continuous t-norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that:*

(1) S and T are continuous,

(2) $AS = SA$ and $TB = BT$,

(3) $A^a(X) \subset T^c(X)$ and $B^b(X) \subset S^s(X)$,

(4) there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that

$$N(A^a x, B^b y, t) \le \phi(Q(M(S^s x, T^c y, t), M(A^a x, S^s x, t), M(B^b y, T^c y, t), M(B^b y, S^s x, 2t), M(A^a x, T^c y, t))) \text{ for all } x, y \in X \text{ and } t > 0.$$

Then A, B, S and T have a unique fixed point in X.

Proof. From (2), $A^a S^s = S^s A^a$ and $T^c B^b = B^b T^c$. We know that commutativity implies compatibility, and hence from Theorem 3.1, there exists unique $z \in X$ such that

$$z = A^a z = B^b z = S^s z = T^c z. \tag{3.2.1}$$

Then we have

$$\begin{aligned} &Az = A(A^a z) = A^a(Az), \quad Az = A(S^s z) = S^s(Az), \\ &Bz = B(B^a z) = B^a(Bz) \text{ and } Bz = B(T^c z) = T^c(Bz). \end{aligned} \tag{3.2.2}$$

Similarly,

$$\begin{gathered} Sz = A^a(Sz), \quad Tz = B^b(Tz), \\ Sz = S^s(Sz) \text{ and } Tz = T^c(Tz). \end{gathered} \tag{3.2.3}$$

From $(4), (3.2.1)$ and $(3.2.2)$, we get

$$\begin{aligned} N(Az, Bz, t) &= N(A^a Az, B^b Bz, t) \\ &\le \phi(Q(M(S^s Az, T^c Bz, t), M(A^a Az, S^s Az, t), M(B^b Bz, T^c Bz, t), \\ &\qquad M(B^b Bz, S^s Az, 2t), M(A^a Az, T^c Bz, t))) = \phi(N(Az, Bz, t)), \end{aligned}$$

and hence $Az = Bz$.

Similarly, $Sz = Tz$ and $Az = Tz$.

Thus we have

$$Az = Bz = Sz = Tz. \tag{3.2.4}$$

From $(4), (3.2.1)$ and $(3.2.2)$, we have

$$\begin{aligned} N(z, Bz, t) &= N(A^a z, B^b Bz, t) \\ &\le \phi(Q(M(S^s z, T^c Bz, t), M(A^a z, S^s z, t), M(B^b Bz, T^c Bz, t), \\ &\qquad M(B^b Bz, S^s z, 2t), M(A^a z, T^c Bz, t))) = \phi(N(z, Bz, t)), \end{aligned}$$

and hence $z = Bz$.

From (3.2.4), $z = Az = Bz = Sz = Tz$. This completes the proof. □

From Theorem 3.2, we get the next result.

Corollary 3.4. (see [1]) *Let $(X, M, *)$ be a complete fuzzy metric space with continuous t-norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that:*

(1) *S and T are continuous,*

(2) *$AS = SA$ and $TB = BT$,*

(3) *$A^a(X) \subset T^c(X)$ and $B^b(X) \subset S^s(X)$,*

(4) *there exists $k \in (0, 1)$ such that*

$$\begin{aligned} N(A^a x, B^b y, t) &\le kQ(M(S^s x, T^c y, t), M(A^a x, S^s x, t), M(B^b y, T^c y, t), \\ &\qquad M(B^b y, S^s x, 2t), M(A^a x, T^c y, t))) \text{ for all } x, y \in X \text{ and } t > 0. \end{aligned}$$

Then A, B, S and T have a unique fixed point in X.

Example 3. For fuzzy metric space $(X, M, *)$ given in Example 2, let $Ax = \frac{x}{16}, Tx = \frac{x}{2}, Bx = \frac{x}{8}$ and $Sx = \frac{x}{4}$. Then conditions (1), (2) and (3) of Theorem 3.1 are satisfied. Let $\phi(t) = kt$ for $k \in [\frac{1}{2}, 1)$. Then the condition (4) of Theorem 3.1 is satisfied and zero is the unique common fixed point of A, B, S and T.

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