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REAL ANALYTIC CONVEXITY ON DOMAINS OF TOPOLOGICAL VECTOR SPACES

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Abstract: Let V a locally convex and Hausdorff real topological vector space whose topology may be defined by a family of seminorms with finite-codimensional kernel, $\Omega \subset V$ a non-empty open subset, $S \subset \Omega$ a closed subset of Ω, and $P \in \Omega \backslash S$. Here we prove the existence a real analytic function $f : \Omega \to \mathbb{R}$ such that $f(P) > \sup_{Q \in S} |f(Q)|$.

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Key Words: real analytic function, real analytic function in infinite-dimensional topological vector spaces, topological vector space without a continuous norm, real analytic convex envelope

1. Real Analytic Convexity

In [1] we proved a result on the zero-loci of real analytic functions on open subsets of real Fréchet spaces without a continuos norm: no such zero-locus may have an isolated point. In particular, this result implies the non-existence 9 in such topological vector spaces) of "strict peak points" for the algebra of all real analytic functions. Nevertheless, we may easily adapt the finite-dimensional proof to show that any compact is "convex with respect to the set of all real analytic functions" in the following sense.

Theorem 1. *Let V a locally convex and Hausdorff real topological vector space whose topology may be defined by a family of seminorms with finite-*

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codimensional kernel, $\Omega \subset V$ *a non-empty open subset,* $S \subset \Omega$ *a closed subset of* Ω*, and* $P \in \Omega \backslash S$*. Then there exists a real analytic function* $f : \Omega \to \mathbb{R}$ *such that* $f(P) > \sup_{Q \in S} |f(Q)|$.

For instance, we may apply Theorem 1 to the Fréchet space $\mathbb{R}^{\mathbb{N}}$ or the the weak topology on any locally convex Hausdorff topological vector space.

Proof of Theorem 1. Since $P \in \Omega$ and S is closed in Ω, P is not contained in the closure $\bar{S}$ of S in V. Hence it is sufficient to prove find a real analytic $f : V \to \mathbb{R}$ such that $f(P) > \sup_{Q \in \bar{S}} |f(Q)|$. Up to a translation, we may assume $P = 0$. Since $\bar{S}$ is closed and $0 \notin \bar{S}$, there are $\epsilon > 0$ and a continuous seminorm $\| \ \|$ on V such that $\|Q| \geq \epsilon$ for all $Q \in \bar{S}$. Hence there is a finite-codimensional linear subspace W of V such that $W \cap \bar{S} = \emptyset$. Let n be the codimension of W in V. By Hahn-Banach W has a closed supplement, i.e. there is an n-dimensional linear subspace U of V such that $U + W = V$ and $U \cap W = \{0\}$. Let $g : V = U \oplus W \to U$ be the linear projection on the first factor. Choose linear coordinates $x_1, \dots, x_n$ on U and define $h : U \in \mathbb{R}$ by the formula $h(x_1, \dots, x_n) := 1/(x_1^2 + \cdots + x_n^2 + 1)$. Use the function $f := g \circ h$ to prove Theorem 1. $\square$

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References

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