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ON THE EXISTENCE OF CURVES WITH ONE OR MORE WEIERSTRASS POINTS WITH A CERTAIN SEMIGROUP

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Abstract: Let X be a smooth projective curve of genus $g \geq 4$ and $Q \in X$. Here we will find (if $g \gg z \geq 1$ and $g \gg k \geq 4$) a smooth genus g curve X and $Q_1, \dots, Q_z \in X$, $Q_i \neq Q_j$ for all $i \neq j$, with the same Weierstrass semigroup whose first non-gap is k and the other generators "prescribed" (freely depending on one integer parameter).

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1. The Existence of Certain Weierstrass Semigroup

Let X be a smooth projective curve of genus $g \geq 4$ and $Q \in X$. Set $h_{X,Q}(i) := h^0(X, \mathcal{O}_X(iQ))$. Q is a Weierstrass point of X if and only if $h_{Q,X}(g) \geq 2$. The knowledge of the non-decreasing function $h_{X,Q} : \mathbb{N} \to \mathbb{N}$ is equivalent to the knowledge of the semigroup of non-gaps of Q. Here we will adapt the proofs in [2] to find (if $g \gg z \geq 1$) a smooth curve X of genus g and $Q_1, \dots, Q_z \in X$, $Q_i \neq Q_j$ for all $i \neq j$, such that $h_{X,Q_i}(t) = h_{X,Q_i}(t)$ for all i, t, and the function h_{X,Q_1} is "prescribed". More precisely, we will show that the proofs in [2] easily give the following result.

Theorem 1. *Fix integers z, k, c, y such that $z \geq 1$, $k \geq 4$, $c \geq 4$, $y \geq kz+kc+1$. Set $G(c,y) := 1+k(kc-c-2)+(k-1)y$ and take any integer g such that $G(c,y-1) < g \leq G(c,y)$. Then there exist a smooth curve X of genus g*

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and $Q_1, \dots, Q_z \in X$*,* $Q_i \neq Q_j$ *for all* $i \neq j$*, such that* $h_{X,Q_i}(t) = h_{X,Q_i}(t)$ *for all* i, t *and the Weierstrass semigroup of each* Q_i *is generated by the integers* $k + mc$ *for all* $m \geq 0$*, and all integers* $x \geq 2g - 1$*.*

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Sketch of the proof of Theorem 1. Set $t := G(c, y) - g$. Fix a general subset S_t of the Hirzebruch surface F_c such that $\sharp(S_t) = t$. Let h, f be a base of $\mathrm{Pic}(F_c)$,with f a fiber of the ruling $\pi : F_c \to \mathbf{P}^1$ and h a section of π with minimal self-intersection. In the proofs of [2] we proved the existence of an integral nodal curve $C_{y,t} \subset F_c$ such that $C_{y,t} \in |kh + (kc + y)f|$, $\mathrm{Sing}(C_{y,t}) = S_t$ and $C_{y,t}$ has an ordinary node at each point of S_t. Let $f : X \to C_{y,t}$ be the normalization map. Hence $p_a(C_{y,t}) = G(c, y)$ and $p_a(X) = g$. Here we need only to check (omitted) that (with our new restriction $y \geq kz + kc + 1$) there is such a curve $C_{y,t}$ with the additional condition that there are $P_i \in (C_{y,t})_{reg}$, $1 \leq i \leq z$, such that $P_i \neq P_j$ for all $i \neq j$ and $\pi|C_{y,t}$ has a total ramification point at each P_i: set $Q_i := f^{-1}(P_i)$. □

It is easy (and omitted) to adapt the sketch of proof of Theorem 1 to the case $c = 0$, i.e. to the Hirzebruch surface $F_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$, and get the following result.

Theorem 2. *Fix integers* $z \geq 1$*,* $k \geq 4$ *and* $u > 0$*. Then there exists an integer* $g(z, k, u)$ *such that for all integers* $g \geq g(z, k, u)$ *there exist a smooth curve* X *of genus* g *and* $Q_1, \dots, Q_z \in X$*,* $Q_i \neq Q_j$ *for all* $i \neq j$*, such that* $h_{X,Q_i}(t) = h_{X,Q_i}(t)$ *for all* i, t*, and for any choice of* u *integers* $2 \leq a_1 < \cdots < a_t \leq g/k$ *the Weierstrass semigroup of each* Q_i *is generated by* k*,* $a_j k + j$*,* $1 \leq j \leq t$*, and all integers* $x \geq 2g - 1$*.*

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References

[1] E. Ballico, Scrollar invariants of smooth projective curves, *J. Pure Appl. Algebra*, **166**, No. 3 (2002), 239-246.

[2] E. Ballico, Smooth curves with a pencil with certain scrollar invariants, *Kyushu J. Math.*, **57**, No. 2 (2003), 217-226.