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MODELING PROPERTIES OF THE ENTROPY AS A LYAPUNOV-LIKE UTILITY FUNCTION IN DPPN

Julio Clempner

Centro de Investigación en Computación (CIC)
Instituto Politécnico Nacional
Apartado Postal 75-476, C.P. 07738, Mexico City, MEXICO
e-mail: julio@k-itech.com

Abstract: The theoretical foundations for the definition of the Entropy as a Lypunov-like utility function are laid out in this paper. It allows the introduction of a formal framework for game representation called game entropy decision process Petri nets (GEDPPN). In game sense, we introduce the entropy-Lyapunov equilibrium point as an alternative definition to the Nash equilibrium point for games. The advantage of this approach is that fixed-point conditions for games are given by the definition of the vector entropy function. As a result, new properties related with the characterization of the Lyapunov-like function by the entropy are introduced. A formal treatment is presented.

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1. Introduction

The notion of entropy arises in physics and statistical mechanics from the works of Maxwell, Boltzmann, Gibbs, etc.. In thermodynamics entropy is defined as the irreversible increase of non-disposable energy in the universe ([7]). This

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fact is expressed through the second law of thermodynamics: entropy can only increase in the universe. It is quite possible to decrease entropy in a system, but it must be balanced by an at last equal increase in entropy elsewhere.

Entropy becomes popular in computer science and information theory, particularly by the work of Shannon ([13]). He introduced the term $H(p) = \sum_{i=1}^{n} p_i \ln p_i$ as a measure of uncertainty about a discrete random variable having a probability mass function, p. Shannon's entropy term is also a measure of the increase of a probability distribution that achieves its maximum value when the distribution assigns equal probabilities to all outcomes. The entropy of a thing is the asymptotic average of the logarithm of the number of ways that the thing occurs. In entropy sense, equilibrium refers to the order state or minimum entropy. The order state is the opposite of entropy (disorder measure).

Non-cooperative game theory has been extensively used to analyze situations of interaction ([11], [12]). The most important results in non-cooperative games are related to the Nash equilibrium ([8], [9], [10]). Formally, a Nash equilibrium defines an equilibrium of a non-cooperative game with respect to a profile of strategies, one for each player in the game, such that each player's strategy attempts to maximize that player's expected utility, opposed to the set of strategies of the other players. Then, players are in *equilibrium* if each player's choice of strategy is a best response to the actions actually taken by his/her opponents.

This paper introduces a modeling paradigm for developing a game representation called Game entropy decision process Petri nets (GEDPPN). The main idea is to use the entropy as a vector Lyapunov-like utility function([6]) allowing the GEDPPN to join with game theory. The main idea is to use the entropy as a utility function that is non-negative and converges asymptotically to the equilibrium point. In this equilibrium each player chooses a strategy with a utility equal to the utility that this strategy is a best reply to a strategy profile chosen by the opponents. The advantage of this approach is that fixed-point conditions for the game are given by the definition of the entropy function. In addition, new properties related with the characterization of the entropy are introduced.

The paper is structured in the following manner. The next section presents the necessary mathematical background and terminology on Petri nets needed to understand the rest of the paper. Section 3 presents an isomorphism of the entropy to a probability space. Section 4 provides the necessary background in game decision process Petri nets. Section 5 discusses the main results of this paper, presenting the entropy-Nash equilibrium point and the related entropy properties. Finally, some concluding remarks and future work are provided in Section 6.

2. Petri Nets

Petri nets are a tool for the study of systems. Petri net theory allows a system to be modeled by a Petri net, a mathematical representation of the system. Analysis of the Petri net then can, hopefully, reveal important information about the structure and dynamic behavior of the modeled system. This information can then be used to evaluate the modeled system and suggest improvements or changes.

A Petri net is a 5-tuple, $PN = \{P, Q, F, W, M_0\}$, where: $P = \{p_1, p_2, ..., p_m\}$ is a finite set of places, $Q = \{q_1, q_2, ..., q_n\}$ is a finite set of transitions, $F \subseteq (P \times Q) \cup (Q \times P)$ is a set of arcs, $W : F \to \mathbb{N}_1^+$ is a weight function, M_0: $P \to \mathbb{N}$ is the initial marking, $P \cap Q = \emptyset$ and $P \cup Q \neq \emptyset$.

A Petri net structure without any specific initial marking is denoted by N. A Petri net with the given initial marking is denoted by (N, M_0). Notice that if $W(p, q) = \alpha$ (or $W(q, p) = \beta$) then, this is often represented graphically by α, (β) arcs from p to q (q to p) each with no numeric label.

Let $M_k(p_i)$ denote the marking (i.e., the number of tokens) at place $p_i \in P$ at time k and let $M_k = [M_k(p_1), ..., M_k(p_m)]^T$ denote the marking (state) of PN at time k. A transition $q_j \in Q$ is said to be enabled at time k if $M_k(p_i) \geq W(p_i, q_j)$ for all $p_i \in P$ such that $(p_i, q_j) \in F$. It is assumed that at each time k there exists at least one transition to fire, i.e. it is not possible to block the net. If a transition is enabled, then it can fire. If an enabled transition $q_j \in Q$ fires at time k then, the next marking for $p_i \in P$ is given by

$$M_{k+1}(p_i) = M_k(p_i) + W(q_j, p_i) - W(p_i, q_j).$$

Let $A = [a_{ij}]$ denote an $n \times m$ matrix of integers (the incidence matrix), where $a_{ij} = a_{ij}^+ - a_{ij}^-$ with $a_{ij}^+ = W(q_i, p_j)$ and $a_{ij}^- = W(p_j, q_i)$. Let $u_k \in \{0, 1\}^n$ denote a firing vector, where if $q_j \in Q$ is fired, then its corresponding firing vector is $u_k = [0, ..., 0, 1, 0, ..., 0]^T$ with a "one" in the j-th position in the vector and zeros everywhere else. The matrix equation (nonlinear difference equation) describing the dynamical behavior represented by a Petri net is:

$$M_{k+1} = M_k + A^T u_k \,, \tag{1}$$

where if at step k, $a_{ij}^- < M_k(p_j)$ for all $p_j \in P$, then $q_i \in Q$ is enabled and if this $q_i \in Q$ fires, then its corresponding firing vector u_k is utilized in the difference equation (1) to generate the next step. Notice that if $M^{'}$ can be reached from some other marking M and, if we fire some sequence of d transitions with corresponding firing vectors $u_0, u_1, ..., u_{d-1}$ we obtain that

$$M^{'} = M + A^T u, \ u = \sum_{k=0}^{d-1} u_k. \tag{2}$$

Definition 2.1. The set of all the markings (states) reachable from some starting marking M is called the reachability set, and is denoted by $R(M)$.

Let $(\mathbb{N}_{n_0+}, d)$ be a metric space, where $d : \mathbb{N}_{n_0+} \times \mathbb{N}_{n_0+} \to \mathbb{R}_+$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^{m} \zeta_i \mid M_1(p_i) - M_2(p_i) \mid; \quad \zeta_i > 0, \ i = 1, ..., m.$$

3. Isomorphism to a Probability Space

The set of elements with respect to the preference graph is ordered by the utility function u_i. Each utility function has a linear transformation T to a function f bounded by the interval $[0, 1)$. The image transformation of T is given by T^*. This can also be expressed by requiring the identity result, i.e., $TT^* = I$. The function f has a *σ-algebra* and a measure bounded by a probability function. The space generated by f is a probability space $(\Omega, \Im, P)$.

In this probability space, the inner product of two random variables ξ and η will be defined by the relation $(\xi, \eta) = E\{\xi\overline{\eta}\}$, where E represents the expectation.

If ξ_1, ξ_2 and η_1, η_2 are two pairs of equivalent variables, this definition evidently gives $(\xi_1, \eta_2) = (\xi_2, \eta_2)$, considering that

$$E\{\xi\} = 0, \qquad E\{|\xi|^2\} < \infty\,, \tag{3}$$

with two variables $P\{\xi_1 = \xi_2\} = 1$ or $E\{|\xi_1 - \xi_2|^2\} = 0$ regarded as identical.

Finally the norm of the variable ξ is

$$\|\xi\| = (\xi, \xi)^{1/2} = \left[E\{|\xi|^2\}\right]^{1/2} \tag{4}$$

and will reduce to zero when and only when $P\{\xi = 0\} = 1$, that is, when the value of the variable ξ is equivalent to zero.

With this convention the set of random variables satisfying (3), will form a Hilbert space. Its elements are random variables, subject to the convention that equivalent random variables are regarded as identical. The Hilbert space is not separable, unless the probability space $(\Omega, \Im, P)$ is of particularly simple type.

From the previous definition (4) of the norm, it follows that the convergent in norm topology of the sequence of points in the Hilbert space coincides with convergent in quadratic mean of the corresponding sequence of random variables.

For the axioms of normative utility theory of von Neumann and Morgenstern (see [11]) we will consider a Hilbert space with random variables. Then, for the entropy the pay-off functions u_i will be defined up to positive affine transformations. For any given model of the game we will define u_i as a set-valued function defined for all events $\Xi \subset S$ by

$$u_i(O) = \sum_{s \in \Xi} u_i(s) \left(\prod_{j=1}^{n} \sigma_j(s_j|\Xi) \right) . \tag{5}$$

We will consider that the utility function is normalized so that $u_i(s) \geq 0$ for all s and $u_i(S)$, then the set function u defined by $u = u_i(s) \left(\prod_{j=1}^{n} \sigma_j(s_j|\Xi) \right)$ is a probability measure.

4. Game Entropy Decision Process Petri Net

The aim of this section is to associate to any game a game entropy decision process Petri net – GEDPPN – (see [1]). The GEDPPN structure will represent all the possible strategies existing within the game.

Definition 4.1. A non-cooperative game entropy decision process Petri net is a 8-tuple GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$, where:

- $\mathcal{N} = \{1, 2, ..., n\}$ denotes a finite set of players.
- $P = P_1 \times P_2 \times ... \times P_n$ is the set of places that represents the Cartesian product of states (each tuple is represented by a place).
- $Q = Q_1 \times Q_2 \times ... \times Q_n$ is the set of transitions that represents the Cartesian product of the conditions (each tuple is represented by a transition).

- $F \subset I \cup O$ is a set of arcs, where $I \subset (P \times Q)$ and $O \subset (Q \times P)$ such that $P \cap Q = \emptyset$ and $P \cup Q \neq \emptyset$,
- $W : F \to \mathbb{N}_1^+$ is a weight function,
- M_0: $P \to \mathbb{N}^n$ is the initial marking,
- $\pi : I \to \mathbb{R}_+^n$ is a routing policy representing the probability of choosing a particular transition (routing arc), such that for each $p_i \in P$, $\sum_{(p_i,q_j):q_j \text{ varying over } Q} \pi((p_{i\iota}, q_j)) = 1$, $\forall \iota \in \mathcal{N}$,
- $H : P \to \mathbb{R}_+^n$ is the Entropy function.

$H_k(.)$ denotes the utility at place $p_i \in P$ at time k and let $H_k = [H_k(.), ..., H_k(.)]^T$ denote the utility state of GEDPPN at time k. $FN : F \to \mathbb{R}_+$ is the number of arcs from place p to transition q (the number of arcs from transition q to place p). The rest of the GEDPPN functionality is as described in the PN preliminaries.

Consider an arbitrary $p_i \in P$ and for each fixed transition $q_j \in Q$ that forms an output arc $(q_j, p_i) \in O$, we look at all the previous places p_h of the place p_i denoted by the list (set) $p_{\eta_{ij}} = \{p_h : (p_h, q_j) \in I \ \& \ (q_j, p_i) \in O\}$ (η_{ij} is defined as equal to the index sequence of identifiers h of the previous places $p_h \in p_{\eta_{ij}}$), that materialize all the input arcs $(p_h, q_j) \in I$ and form the sum

$$\sum_{h \in \eta_{ij}} \left(\langle \Psi(p_h, q_j, p_i) * H_k(p_h) \rangle \right)_\iota , \tag{6}$$

where $\Psi(p_h, q_j, p_i) = (\pi(p_{h_1}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)}, \pi(p_{h_2}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)}, ..., \pi(p_{h_n}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)})$, p_{h_ι} is the $\iota \in \mathcal{N}$ element of the tuple routing policy π, $(\langle * \rangle)_\iota$ represent the product of the vector element by element, i.e. $(\langle (a_1, a_2, ..., a_n) * (b_1, b_2, ..., b_n) \rangle)_\iota = (a_1 b_1, a_2 b_2, ..., a_n b_n)$ and the index sequence j is the set $\{j : q_j \in (p_h, q_j) \cap (q_j, p_i) \ \& \ p_h \text{ running over the set } p_{\eta_{ij}}\}$.

Proceeding with all the q_js we form the vector indexed by the sequence j identified by $(j_0, j_1, ..., j_f)$ as follows:

$$\left[\sum_{h \in \eta_{ij_0}} \left(\langle \Psi(p_h, q_{j_0}, p_i) * H_k(p_h) \rangle \right)_\iota , \sum_{h \in \eta_{ij_1}} \left(\langle \Psi(p_h, q_{j_1}, p_i) * H_k(p_h) \rangle \right)_\iota , ..., \sum_{h \in \eta_{ij_f}} \left(\langle \Psi(p_h, q_{j_f}, p_i) * H_k(p_h) \rangle \right)_\iota \right] \tag{7}$$

Intuitively, the vector (7) represents all the possible trajectories through the transitions q_j, where $(j_1, j_2, ..., j_f)$ to a place p_i for a fixed i.

Definition 4.2. A final decision point $p_f \in P$ with respect to a game entropy decision process Petri net GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ is a place $p \in P$, where the infimum is asymptotically approached (or the minimum is attained), i.e. $H(p) = \mathbf{0}$ or $H(p) = \mathbf{C}$.

Definition 4.3. An optimum point $p^{\triangle} \in P$ with respect a game entropy decision process Petri net GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ is a final decision point $p_f \in P$, where the best choice is selected 'according to some criteria'.

Property 4.1. Every game entropy decision process Petri net GEDPPN $= (\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ has a final decision point.

Definition 4.4. A strategy with respect a game entropy decision process Petri net GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ is identified by σ and consists of the routing policy transition sequence represented in the GEDPPN graph model such that some point $p \in P$ is reached.

Definition 4.5. An optimum strategy with respect a game entropy decision process Petri net GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ is identified by $\sigma^{\triangle}$ and consists of the routing policy transition sequence represented in the GEDPPN graph model such that an optimum point $p^{\triangle} \in P$ is reached.

We denote $S_\iota = \{s_i\}$ the set of pure strategies for player ι in the GEDPPN. For notational convenience we write $S = \prod_{\iota \in \mathcal{N}} S_\iota$ (the pure strategies profile), and $S_{-\iota} = \prod_{j \in \mathcal{N}|\{\iota\}} S_j$ (the pure strategies profile of all the players but for player ι). For an action tuple $s = (s_1, ..., s_n) \in S$ we denote $s_{-\iota} = (s_1, ..., s_{\iota-1}, s_{\iota+1}, ..., s_n)$ and, with an abuse of notation, $s = (s_\iota, s_{-\iota})$.

Similarly, we denote $\Gamma_\iota = \{\sigma_i\}$ the set of mixed strategies for player ι in the GEDPPN, identified with the routing policy representing the probability of choosing a particular transition. Analogously, we use notations $\Gamma = \prod_{\iota \in \mathcal{N}} \Gamma_\iota$ to denote the mixed strategies profile that combine strategies one for each player and $\Gamma_{-\iota} = \prod_{j \in \mathcal{N}|\{\iota\}} \Gamma_j$ to denote the mixed strategies profile of all the players except for player ι. For a strategy tuple $\sigma = (\sigma_1, ..., \sigma_n) \in \Gamma$ we denote $\sigma_{-\iota} = (\sigma_1, ..., \sigma_{\iota-1}, \sigma_{\iota+1}, ..., \sigma_n)$ and, with an abuse of notation, $\sigma = (\sigma_\iota, \sigma_{-\iota})$. For a strategy profile $\sigma_{-\iota}$, we write $\sigma_{-\iota} = \prod_{j \in \mathcal{N}|\{\iota\}} \sigma_j$, the probability identified with the routing policy π that the opponents of player ι play strategy profile $s_{-i} \in S_{-i}$. We restrict our attention to independent strategy profiles. For our construction of the GEDPPN a strategy profile determines an outcome representing the corresponding utility of each player.

Then, formally we define the utility function H as follows.

Definition 4.6. The utility function H with respect a game entropy decision process Petri net GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ is represented by the equation

$$H_{k,\iota}^{((\sigma_{hj})_\iota,(\sigma_{hj})_{-\iota})}(p_i) = \begin{cases} H_{k,\iota}(p_0) & \text{if} \quad i=0,\ k=0\,, \\ L(\alpha) & \text{if} \quad i>0,\ k=0\ i\geq 0,\ k>0\,, \end{cases} \tag{8}$$

$$\alpha = \begin{bmatrix} \sum\limits_{h\in\eta_{ij_0}} \left(\left\langle ((\sigma_{hj_0})_\iota,(\sigma_{hj_0})_{-\iota})(p_i) * H_{k,\iota}^{((\sigma_{hj_0})_\iota,(\sigma_{hj_0})_{-\iota})}(p_h)\right\rangle\right)_\iota, \\ \sum\limits_{h\in\eta_{ij_1}} \left(\left\langle ((\sigma_{hj_1})_\iota,(\sigma_{hj_1})_{-\iota})(p_i) * H_{k,\iota}^{((\sigma_{hj_1})_\iota,(\sigma_{hj_1})_{-\iota})}(p_h)\right\rangle\right)_\iota, ..., \\ \sum\limits_{h\in\eta_{ij_f}} \left(\left\langle ((\sigma_{hj_f})_\iota,(\sigma_{hj_f})_{-\iota})(p_i) * H_{k,\iota}^{((\sigma_{hj_f})_\iota,(\sigma_{hj_f})_{-\iota})}(p_h)\right\rangle\right)_\iota \end{bmatrix}, \tag{9}$$

where $\iota \in \mathcal{N}$ represents a given the player, the vector function $L : D \subseteq \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a vector Lyapunov-like function which optimizes the utility through all possible strategies (i.e. trough all the possible trajectories defined by the different q_js), D is the decision set formed by the j́s ; $0 \leq j \leq f$ of all those possible transitions $(q_j, p_i) \in O$, $((\sigma_{hj})_\iota, (\sigma_{hj})_{-\iota})(p_i) = (\pi(p_{h1}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)}, \pi(p_{h2}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)}, ..., \pi(p_{hn}, q_j) * \frac{FN(q_j,p_i)}{FN(p_h,q_j)})$, $(\langle * \rangle)_\iota$ represent the product of the vector element by element, i.e. $(\langle (a_1, a_2, ..., a_n) * (b_1, b_2, ..., b_n)\rangle)_\iota = (a_1 b_1, a_2 b_2, ..., a_n b_n)$, η_{ij} is the index sequence of the list of previous places to p_i through transition q_j, p_h $(h \in \eta_{ij})$ is a specific previous place of p_i through transition q_j.

Property 4.2. The utility function $H : P \to \mathbb{R}_+^n$ is a vector Lyapunov-like function.

5. The Equilibrium Point and the Entropy Properties

The interaction among players (see [1]) obligates each player to develop a belief about the possible strategies of the other players. Nash equilibria (see [8], [9], [10]) are supported by two premises: i) each player behaves rationally given the beliefs about the other players' strategies; and 2) these beliefs are correct. Both premises allow us to regard the Nash equilibrium point as a steady-state of the strategic interaction. In particular, the second premise makes this an equilibrium concept, because when every individual is acting in agreement with the Nash equilibrium no one has the need to take another strategy.

The best-reply strategy for a player is relative to the strategy profile chosen by the opponents. The strategy profile is said to contain a best reply for a given player if cannot increase the utility by playing another strategy with respect to the opponents strategies. A strategy profile is a Nash equilibrium point if none of the players can increase the utility by playing another strategy, in other words each player's choice of strategy is a best reply to the strategies taken by his opponents. When a player is acting in accordance with the Nash equilibrium has no motivation to unilaterally deviate and take another strategy. Formally, we have the following definitions.

Consider the game GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$. Denote for each player $\iota \in \mathcal{N}$ and each profile $\sigma_{-\iota} \in \Gamma_{-\iota}$ of strategies of his opponent the set of best replies, i.e. the strategies that player ι cannot improve upon, and it is defined as follows:

$$B_\iota(\sigma_{-\iota}) := \left\{ \sigma_\iota^\triangle \in \Gamma_\iota | \forall \sigma'_\iota \in \Gamma_\iota : H_\iota^{(\sigma_\iota^\triangle, \sigma_{-\iota})}(p^\triangle) \geq H_\iota^{(\sigma'_\iota, \sigma_{-\iota})}(p) \right\}.$$

Opposite to what we define in the GEDPPN, in game theory we look for maximizing the utility and we change $\leq$ by $\geq$. Since Γ_ι is finite and u_ι establish an acyclic order, $B_\iota(\sigma_{-\iota})$ is not empty.

We call a strategy $\sigma_\iota \in \Gamma_\iota$ a *never best reply* if

$$B_\iota^{-1}(\sigma_{-\iota}) := \{\sigma_{-\iota} \in \Gamma_{-\iota} | \sigma_\iota \in B_\iota(\sigma_{-\iota})\} = \emptyset.$$

Alternatively, we have

$$B_\iota^{-1}(\sigma_{-\iota}) := \left\{ \sigma_{-\iota} \in \Delta_{-\iota} | \max_{\sigma'_\iota \in \Gamma_\iota} H_\iota^{(\sigma'_\iota, \sigma_{-\iota})}(p) > H_\iota^{(\sigma_\iota, \sigma_{-\iota})}(p) \right\}.$$

A Nash equilibrium is a profile of strategies such that each player's strategy is an optimal response to the other players' strategies.

Definition 5.1. A strategy profile $\sigma_\iota^\triangle$ is a Nash equilibrium point if, for all players ι

$$H_\iota^{(\sigma_\iota^\triangle, \sigma_{-\iota}^\triangle)}(p^\triangle) \geq H_\iota^{(\sigma'_\iota, \sigma_{-\iota}^\triangle)}(p) \quad \forall \sigma'_\iota \in \Gamma_\iota.$$

Remark 5.1. It is important to note that in case the strategy is implemented as a chain of transitions $\geq$ does not represent a vectorial inequality, the interpretation is obtained from calculating the best reply B_ι.

Definition 5.2. A strategy σ has the fixed point property if it leads to the optimum point $(H_\iota^{(\sigma_\iota^\triangle, \sigma_{-\iota}^\triangle)}(p^\triangle))$.

Remark 5.2. From the two previous definitions the following characterization is obtained: A strategy which has the fixed point property is equivalent to being a Nash equilibrium point.

Theorem 5.1. *A non blocking (unless $p \in P$ is an equilibrium point) game entropy decision process Petri net GEDPPN=* $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ *has a strategy σ which has the fixed point property.*

Proof. The conclusion is a direct consequence of Theorem 3.5 outlined in [2] and its proof (where the existence of p^{Δ} is guaranteed by the first property given in the definition of the vector Lyapunov-like function, given in [6]). □

Corolary 5.1. *If in addition to the hypothesis of the theorem the game GEDPPN is finite, the strategy σ leads to an equilibrium point .*

Proof. The proof is outlined in [2] Corollary 3.1. □

Theorem 5.2. *The optimum point*[1] *coincides with the Nash equilibria.*

Proof. This is immediate from the definition of optimum point and Remark 5.2 □

Remark 5.3. The potential of the previous theorem remains in its formal proof simplicity for the existence of an equilibrium point.

Example 5.1. Let us consider the famous "Prisoner's dilemma" game, where two men are arrested for a crime. The police tell each suspect separately that if he testifies against the other, he will be rewarded for testifying. Each prisoner has two possible strategies (Table 1): to testify (not cooperate with other player, defect from cooperation) or not testify (cooperate). If both players defect, there is a mutual punishment resulting in a pay-off P (which the players obtain by collaborating with each other, e.g. in that the law enforcement does not obtain enough information to give regular punishments). If both cooperate, there is a mutual reduction of punishment, resulting in a pay-off value of R. However, if one testifies and the other collaborate, the defector which testifies receives a considerable punishment reduction (pay-off of T, the temptation for testifying), and the other player receives the regular punishment (pay-off of S, the "sucker" pay-off for attempting to cooperate against non-cooperation). We use the common pay-offs 5, 3, 1, and 0 for T, R, P, and S, respectively. We have

[1] The definition of optimum point is equivalent to the definition of "steady state" equilibrium point in the Lyapunov sense given by [4].

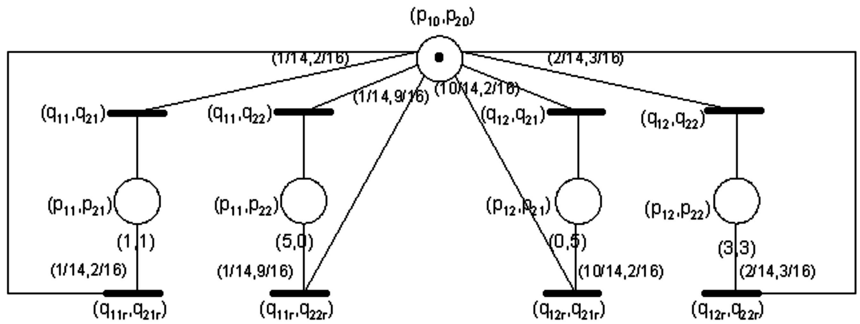

Figure 1: GEDPPN - Iterated Prisoner's Dilemma

$B_1(\sigma_{21}) = \sigma_{11}$	$B_2(\sigma_{11}) = \sigma_{21}$
$B_1(\sigma_{22}) = \sigma_{11}$	$B_2(\sigma_{12}) = \sigma_{21}$

The best strategy is given by the strategy profile $\sigma = (\sigma_{11}, \sigma_{21})$. Alternatively, for player 1 we have that

		$H_1^{(\sigma_{11},\sigma_{22})}((p_{11}, p_{22}))$
$H_1^{(\sigma_{11},\sigma_{21})}((p_{11}, p_{21}))$	$\geq$	$H_1^{(\sigma_{12},\sigma_{21})}((p_{12}, p_{21}))$
		$H_1^{(\sigma_{12},\sigma_{22})}((p_{12}, p_{22}))$

calculating H_2 we obtain the same strategy profile $\sigma = (\sigma_{11}, \sigma_{21})$.

5.1. Entropy Properties and the Lyapunov ("Steady State") Equilibrium Point

Equilibrium refers to the order state or minimum entropy. The order state is the opposite of entropy (disorder measure). Let us consider $H_k^{\sigma}(p) = -\sum_{i=1}^{n} \mathcal{P}_i(\sigma) \ln \mathcal{P}_i(\sigma)$.

Theorem 5.3. *Let GEDPPN=* $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ *a game entropy decision process Petri net. Then, the entropy utility function* H *is minimum in a Lyapunov equilibrium point.*

Proof. The minimum of the entropy is given when

$$\frac{\partial H_k^{\sigma}(p)}{\partial \mathcal{P}_i(\sigma)} = -\sum_{i=1}^{n} (\ln \mathcal{P}_i(\sigma) + 1) \partial \mathcal{P}_i(\sigma) = 0 \,,$$

with the constrain $\sum_{i=1}^{n} \mathcal{P}(\sigma) = 1$. Using over the constrain the Lagrangian's multiplier $\lambda \sum_{i=1}^{n} \partial\mathcal{P}_i(\sigma) = 0$ we have that

$$\frac{\partial H_k^\sigma(p)}{\partial \mathcal{P}_i(\sigma)} = -\sum_{i=1}^{n} (\ln \mathcal{P}_i(\sigma) + 1 + \lambda)\partial\mathcal{P}_i(\sigma) = 0\,,$$

whose solution is $\mathcal{P}_i(\sigma) = e^{-(1+\lambda)}$. We have that $\sum_{i=1}^{n} \mathcal{P}_i(\sigma) = \sum_{i=1}^{n} e^{-(1+\lambda)} = ne^{-(1+\lambda)} = 1$ then with $\lambda = -\ln\frac{1}{n} - 1$ we have $\mathcal{P}_i(\sigma) = \frac{1}{n}$ such that $H_k^\sigma(p) = 0$. But, by convention it is true if and only if σ is a Lyapunov equilibrium point, i.e. $\sigma = \sigma^\triangle$. □

Corolary 5.2. *Let GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ a game entropy decision process Petri net. Then, the Gaussian Density Function minimizes the entropy H in the Lyapunov equilibrium point.*

Proof. The minimum of the entropy is given when

$$\frac{\partial H_k^\sigma(p)}{\partial \mathcal{P}_i(\sigma)} = -\sum_{i=1}^{n} (\ln \mathcal{P}_i(\sigma) + 1)\partial\mathcal{P}_i(\sigma) = 0\,,$$

with the constrain $\sum_{i=1}^{n} \mathcal{P}(\sigma) = 1$ and $\sigma^2 = \sum_{i=1i}^{n} \sigma_i^2\mathcal{P}_i(p)$. Using over the constrains the Lagrangian's multipliers $\alpha \sum_{i=1}^{n} \partial\mathcal{P}_i(\sigma) = 0$ and $\beta \sum_{i=1}^{n} \sigma_i^2\partial\mathcal{P}_i(\sigma) = 0$ we have that

$$\frac{\partial H_k^\sigma(p)}{\partial \mathcal{P}_i(\sigma)} = -\sum_{i=1}^{n} (\ln \mathcal{P}_i(\sigma) + 1 + \alpha + \beta)\partial\mathcal{P}_i(\sigma) = 0\,,$$

whose solution is $\mathcal{P}_i(\sigma) = Ae^{-\beta p_i^2}$. Considering

$$\frac{1}{\sqrt{2\pi(D_i)^2}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_i^2}{2(D_i)^2}} d\sigma = 1 \text{ and let } \frac{1}{\sqrt{2\pi(D_i)^2}} \int_{-\infty}^{\infty} \sigma_i^2 e^{-\frac{\sigma_i^2}{2(D_i)^2}} d\sigma = D_i^2\,,$$

integrating we have

$$\mathcal{P}_i(\sigma) = \frac{1}{\sqrt{2\pi(D_i)^2}} e^{-\frac{\sigma_i^2}{2(D_i)^2}}\,,$$

which is the Gaussian density function, such that $H_k^\sigma(p) = 0$. But, by convention it is true if and only if σ is a Lyapunov equilibrium point, i.e. $\sigma = \sigma^\triangle$. □

Corolary 5.3. *Let GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ a game entropy decision process Petri net. If the probability density function of a set of variables $\sigma \in \Gamma$ is bounded by a normal distribution in game sense then the standard deviation is $D^* = \frac{1}{\sqrt{2\pi e}}$ in the entropy Lyapunov equilibrium point.*

Proof. Let $\sigma^{\triangle} \in \Gamma$ be an optimum strategy such that $H(\sigma^{\triangle}) = 0$ and let entropy under the normal distribution defined by

$$H_k^{\sigma}(p) = \int_{-\infty}^{\infty} \mathcal{P}(\sigma) \ln \mathcal{P}(\sigma)$$

$$= \frac{1}{\sqrt{2\pi (D)^2}} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2(D)^2}} \left(-\frac{1}{2} \ln 2\pi (D)^2 - \frac{\sigma^2}{2(D)^2} \ln e \right) d\sigma \,,$$

where $\frac{1}{\sqrt{2\pi (D)^2}} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2(D)^2}} d\sigma = 1$ and $\frac{1}{\sqrt{2\pi (D)^2}} \int_{-\infty}^{\infty} (\sigma)^2 e^{-\frac{\sigma^2}{2(D)^2}} d\sigma = (D)^2$, then in a Lyapunov equilibrium point (0 by convention)we have

$$H_k^{\sigma^{\triangle}}(p) = \frac{1}{2} \ln 2\pi (D^*)^2 + \frac{1}{2} \ln e = \ln \sqrt{2\pi e (D^*)^2} = 0 \,,$$

where $(D^*)^2$ is the variance. Because $\ln \sqrt{2\pi e (D^*)^2} = 0$ we have that standard deviation $D^* = \frac{1}{\sqrt{2\pi e}}$. □

Corolary 5.4. *Let GEDPPN= $(\mathcal{N}, P, Q, F, W, M_0, \pi, H)$ a game entropy decision process Petri net. If the probability density function of a set of variables $\sigma \in \Gamma$ is bounded by a normal distribution in game sense then then the entropy is minimum if and only if the standard deviation is minimum in the entropy Lyapunov equilibrium point.*

Proof. $\Longrightarrow$ For a game of $\mathcal{N}$ players and supposing that the equilibrium between the players is non-correlated we have that

$$H = \sum_{i=1}^{n} H_i = \sum_{i=1}^{n} \ln \sqrt{2\pi e} D_i = \ln(2\pi e)^{\frac{n}{2}} + \sum_{i=1}^{n} \ln D_i \,,$$

$$\frac{\partial H}{\partial D_i} = \frac{\partial}{\partial D_i} \left(\ln(2\pi e)^{\frac{n}{2}} + \sum_{i=1}^{n} \ln D_i \right) = \frac{1}{\sum_{i=1}^{n} \ln D_i} \partial D_i \,,$$

with the constrain $\sum\limits_{i=1}^{n} D_i = \frac{n}{\sqrt{2\pi e}}$. Using the Lagrangian's multiplier $\lambda \sum\limits_{i=1}^{n} \partial D_i(\sigma) = 0$ we have that

$$\frac{\partial H}{\partial D_i} = \left(\frac{1}{\sum\limits_{i=1}^{n} \ln D_i} + \lambda \right) \partial D_i \, ,$$

whose solution is $D_i = e^{\lambda}$. We have that $\sum\limits_{i=1}^{n} D_i = \sum\limits_{i=1}^{n} e^{\lambda} = ne^{\lambda} = \frac{n}{\sqrt{2\pi e}}$ then with $\lambda = \ln \frac{1}{\sqrt{2\pi e}}$ we have that $D_i = D_i^*$.

$\Longleftarrow$ The standard deviation is minimum at $D^* = \frac{1}{\sqrt{2\pi e}}$ then replacing in the equation of the entropy

$$\begin{aligned} H_i = \frac{1}{\sqrt{2\pi (D_i)^2}} \int\limits_{-\infty}^{\infty} e^{-\frac{\sigma_i^2}{2(D_i)^2}} \left(-\frac{1}{2} \ln 2\pi (D_i)^2 - \frac{\sigma_i^2}{2(D_i)^2} \ln e \right) d\sigma \\ = \frac{1}{2} \ln e^{-1} + \frac{1}{2} \ln e = 0 \end{aligned}$$

□

Remark 5.4. The Gaussian density function produces the optimal minimum dispersion with respect a $\sigma^{\triangle}$.

Remark 5.5. The Gaussian density function produces the minimum error with respect a $\sigma^{\triangle}$.

Remark 5.6. The entropy or measure of the incomplete information (disorder) is directly proportional to the standard deviation or measure of the uncertainty (see [5]) in $\sigma^{\triangle}$.

Remark 5.7. The game entropy variance D^2 converges in $\sigma^{\triangle}$ to $(D^*)^2$ by the Kolmogorov's law of large numbers.

6. Conclusions and Future Work

A theoretical approach for game representation using the well known concepts entropy for defining a utility function has been shown. As a result is introduced a modeling paradigm for developing a game representation called game entropy decision process Petri nets (GEDPPN). The advantage of this approach is that fixed-point conditions for the game are given by the definition of the entropy function. We proved that the equilibrium concept in an entropy sense coincides with the equilibrium concept of Nash, representing an alternative way to calculate the equilibrium and stability of the game. New properties related with the characterization of the entropy were introduced.

References

[1] J. Clempner, J. Medel, A. Cârsteanu, Extending games with local and robust Lyapunov equilibrium and stability condition, *International Journal of Pure and Applied Mathematics*, **19**, No. 4 (2005).

[2] J. Clempner, An optimum trajectory planning algorithm for decision process Petri net, *International Journal of Pure and Applied Mathematics*, To Appear.

[3] T.M. Cover, J.A. Thomas, *Elements of Information Theory*, Wiley Series in Telecommunications, Wiley (1991).

[4] R.E. Kalman, J.E. Bertram, Control system analysis and design via the "Second Method" of Lyapunov, *Journal of Basic Engineering*, (June 1960), 371-393.

[5] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag (1991).

[6] V. Lakshmikantham, V.M. Matrosov, S. Sivasundaram, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic Publ., Dordrecht (1991).

[7] J.C. Maxwell, *Theory of Heat*, Dover, Reprint (2001).

[8] J. Nash, Non-cooperative games, *Ann. Math.*, **54** (1951), 287-295.

[9] J. Nash, *Essays on Game Theory*, Elgar, Cheltenham (1996).

[10] J. Nash, *The essential John Nash*, (Ed-s: H.W. Kuhn, S. Nasar), Princeton UP (2002).

[11] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Second Edition, Princeton, N.J. Princeton University (1947).

[12] M. Osborne, A. Rubinstein, *A Course in Game Theory*, The MIT Press (1994).

[13] C.E. Shannon, A mathematical theory of communication, *Bell System Technical Journal*, **27** (July and October 1948), 379-423, 623-656,