International Journal of Pure and Applied Mathematics

Volume 22 No. 1 2005, 45-47

AN INTERPOLATION PROBLEM FOR PSEUDOCONVEX DOMAINS OF CERTAIN INFINITE-DIMENSIONAL COMPLEX VECTOR SPACES

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let U be a pseudoconvex domain of an infinite-dimensional complex vector space equipped with the finite topology. Here we show the surjectivity of the restriction map $H^0(U, \mathcal{O}_U) \to H^0(Z, \mathcal{O}_Z)$ for certain "zero-dimensional" closed subschemes of U (e.g. we may take as Z any countable discrete subset of U). Due to the chosen topology of V this is essentially an interpolation problem for Gâteaux analytic functions.

AMS Subject Classification: 32K05
Key Words: pseudoconvex domains, interpolation, pseudoconvexity in infinite-dimensional vector spaces, Gâteaux analitic function

*

Let V be an infinite-dimensional complex vector space equipped with the strongest vector topology τ_f which induces the euclidean topology on each of its finite-dimensional linear subspaces ([3], Chapter 2, or [1] or [2]). This is a Hausdorff topology but with this topology V is paracompact (or metrizable) if and only if it has countable algebraic dimension, i.e. if and only if $V \cong \mathbf{C}^{(\mathbf{N})}$ and in this case it is neither sequentially complete nor Baire ([3], Proposition 2.3.2). Essen-

Received: April 12, 2005

tially, a holomorphic function on an open subset D of (V, τ_f) is just a Gâteaux analytic function for any weaker vector topology on V for which D is open. An open subset U of (V, τ_f) is said to be pseudoconvex if its intersection with any finite-dimensional linear subspace of V is a pseudoconvex domain in the classical sense or it is empty ([1], [3] or [2]). In this paper we study interpolation problems for holomorphic functions on pseudoconvex Let $Z \subseteq U$ be a closed analytic subspace such that its support Z_{red} is a countable discrete subset of U. We will say that Z is a locally finitely embedded zero-dimensional subscheme of U if every connected component W of Z is contained in a finite-dimensional linear subspace $\langle W \rangle$ of V; this implies that W is a zero-dimensional analytic subset of the finite-dimensional domain $U \cap \langle W \rangle$ of $\langle W \rangle$. Roughly speaking, for any holomorphic function in a neighborhood Ω of W fixing its restriction to W means that we fix the value of f of f at the point W_{red} and some of its partial derivatives. Hence the restriction map $\rho_Z : H^0(U, \mathcal{O}_U) \to H^0(Z, \mathcal{O}_Z)$ is surjective if and only if we may always find a global holomorphic solution on U for all such prescribed data simultaneously for all connected components of Z. Fix $P \in U$, a linear subspace M of V and an integer $m > 0$. Let $\mathcal{I}_{mP,P+M;M}$ be the germ at P of the sheaves of all Gâteaux analytic functions on the affine space $P + M$ vanishing to order at least m at P, i.e. for which the Taylor series ([3], Theorem 2.3.5) starts with the derivatives of order at least m. Let $(mP, P+M)$ denote the ringed space $(P, \mathcal{O}_{P+M}/\mathcal{I}_{mP,P+M;M})$. This ringed space is a closed analytic subspace of $U \cap (P+M)$ equipped with the induded topology (i.e. with its τ_f-topology) and hence a closed analytic subspace of U. Here we prove the following interpolation result.

Theorem 1. *Let U be a pseudoconvex domain of the infinite-dimensional topological vector space (V, τ_f). Fix a countable discrete subset $S = \{P_n\}_{n \geq 1}$. For each $P_n \in S$ fix a linear subspace V_n of V with countable algebraic dimension. If $\dim(M_n)$ is finite, let Z_n be any zero-dimensional closed subscheme of V_n such that $(Z_n)_{red} = \{P_n\}$. If $\dim(M_n)$ is infinite, fix an integer $m_n > 0$ and set $Z_n := (m_n P_n, P_n + V_n)$. Set $Z := \cup_{n \geq 1} Z_n$. Then the restriction map $\rho_Z : H^0(U, \mathcal{O}_U) \to H^0(Z, \mathcal{O}_Z)$ is surjective.*

Proof. Let M be the minimal linear subspace of V containing S and all vector spaces V_n, $n \geq 1$. Thus V is countable. Just to fix the notation, we assume that M is not finite-dimensional; if $\dim(M)$ is finite, then just drop the central part of the proof. Fix a Hamel basis e_i, $i \geq 1$, of M and call M_i, $i \geq 1$, the linear span of $e_1, \dots, e_i$. Notice that $Z = Z \cap M$ even as ringed spaces by the very definition of M and Z. Set $U_i := U \cap M_i$ and $Z_i := Z \cap M_i$. Thus U_i is pseudoconvex and Z_i is a closed analytic subspace of U_i. By Thorem B of Cartan-Serre the restriction map $H^0(U_i, \mathcal{O}_{U_i}) \to H^0(Z_i, \mathcal{O}_{Z_i})$ is surjective. Furthermore, U_i (resp: Z_i) is the intersection of U_{i+1} (resp. Z_{i+1}) as ringed spaces with the hypeplane M_i of M_{i+1}. Thus by Theorem B of Cartan-Serre the restriction maps $H^0(U_{i+1}, \mathcal{O}_{U_{i+1}}) \to H^0(U_i, \mathcal{O}_{U_i})$ and $H^0(Z_{i+1}, \mathcal{O}_{Z_{i+1}}) \to H^0(Z_i, \mathcal{O}_{Z_i})$. Since $M \cong \mathbf{C}^{(\mathbf{N})}$, step by step we obtain the surjectivity of the restriction map $H^0(U \cap M, \mathcal{O}_{U\cap M}) \to H^0(Z, \mathcal{O}_Z)$. Since by transfinite induction it is very easy to check the surjectivity of the restriction map $H^0(U, \mathcal{O}_U) \to H^0(U \cap M, \mathcal{O}_{U\cap M})$ (see [1], pp. 339–340, or [2], Section 2, for a more difficult case), we are done. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] S. Dineen, Sheaves of holomorphic functions on infinite dimensional vector spaces, *Math. Ann.*, **202** (1973), 337-345.

[2] L. Gruman, The Levi problem in certain infinite dimensional vector spaces, *Illinois J. Math.*, **18**, No. 1 (1974), 20-26.

[3] M. Hervé, *Analyticity in Infinite Dimensional Spaces*, Walter de Gruyter, Berlin-New York (1989).