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DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract: Dynamic inequalities are not only important, but also the fundamental tool of the study of dynamic system. In the last few years, dynamic equations on time scale become the object of investigation in mathematical research papers. In this paper we obtained some extensions of Gronwall-Bellman inequality, and present weighted Jensen and Opial inequalities on time scales.

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1. Preliminaries

In this paper we present a number of dynamic inequalities on time scales. For convenience, we list here the following definitions which will be necessary later.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T},$$
$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}$$

are called the *forward jump operator* and the *backward jump operator*, respectively. In this definition

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$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

The point $t \in \mathbb{T}$ is *left-dense, left-scattered, right-dense, right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), then the *delta derivative of f at the point t* is defined to be the number $f^{\Delta}(t)$ (provided it exists!) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We say that $f : \mathbb{T} \to \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-sided limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted in this paper by C_{rd}. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. In this case we define the integral of f by

$$\int_a^b f(t) \triangle t = F(b) - F(a).$$

We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \ne 0$ for all $t \in \mathbb{T}$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. If $p, q \in \mathcal{R}$, then we define

$$p \oplus q = p + q + \mu pq, \ \ominus q = -\frac{q}{1 + \mu q}, \text{ and } p \ominus q = p \oplus (\ominus q).$$

If $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and regressive, then the exponential function $e_p(, t_0)$ is for each fixed $t_0 \in \mathbb{T}$ the unique solution of the initial value problem

$$x^{\triangle} = p(t)x, \ x(t_0) = 1 \text{ on } \mathbb{T}.$$

2. Gronwall-Bellman Inequality

On the basis of various motivations, the Gronwall-Bellman Inequality has been extended and used considerably in various contexts. Beesack's inequality and Pachpatte's inequality are also the generalizations and variants of Gronwall-Bellman Inequality.

Lemma 2.1. (see [3]) *Let* $y, f \in C_{rd}$, *and* $p \in \mathcal{R}^+$. *Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \quad \text{for all} \quad t \in \mathbb{T}$$

implies that

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s \quad \text{for all } t \in \mathbb{T}.$$

Theorem 2.2. (Beesack's Inequality) *Let* y, p, f, q *be rd-continuous with* $p(t), q(t) \geq 0$ *and*

$$y(t) \leq f(t) + q(t)\int_{t_0}^{t} y(s)p(s)\Delta s \text{ for all } \ t \in \mathbb{T}\,. \tag{2.1}$$

Then

$$y(t) \leq f(t) + q(t)\int_{t_0}^{t} f(s)p(s)e_{pq}(t, \sigma(s))\Delta s \text{ for all } t \in \mathbb{T}. \tag{2.2}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} y(s)p(s)\Delta s\,,$$

then $z(t_0) = 0$ and

$$z^{\Delta}(t) = y(t)p(t) \leq (f(t) + q(t)z(t))p(t) = f(t)p(t) + q(t)p(t)z(t)\,.$$

By Lemma 2.1, we have

$$z(t) \leq \int_{t_0}^{t} e_{pq}(t, \sigma(s))f(s)p(s)\Delta s\,.$$

Since $y(t) \leq f(t) + q(t)z(t)$, we get the required inequality (2.2). □

The inequality that had been proved by Pachpatte [6], have played a significant role in the study of the end value problem.

Theorem 2.3. (Pachpatte's Inequality) *Let* $0 \in \mathbb{T}$, $y, f, p : [0, \infty) \cap \mathbb{T} \to \mathbb{R}^+$ *be continuous,* $f(t)$ *be decreasing function. If*

$$y(t) \leq f(t) + \int_{t}^{\infty} p(s)y(s)\Delta s\,, \tag{2.3}$$

then

$$y(t) \leq f(t)e_p(\infty, t), \quad t \in [0, \infty) \cap \mathbb{T}\,. \tag{2.4}$$

Proof. Now suppose, without loss of generality, that $f(t) > 0$, let $z(t) = 1 + \int_t^\infty p(s)\frac{y(s)}{f(s)}\Delta s$. Since $f(t)$ is decreasing, we have

$$\frac{y(t)}{f(t)} \leq z(t), \quad \lim_{t\to\infty} z(t) = 1\,,$$

then $z^\Delta(t) = -p(t)\frac{y(t)}{f(t)} \geq -p(t)z(t)$, thus

$$z^\Delta(t)e^\sigma_{\ominus p}(\infty, t) + z(t)pe^\sigma_{\ominus p}(\infty, t) \geq 0\,,$$

i.e.,

$$z^\Delta(t)e^\sigma_{\ominus p}(\infty, t) + z(t)(-\ominus p)e^\sigma_{\ominus p}(\infty, t)(1 + \mu p) \geq 0\,.$$

Since $p(t) \geq 0$, we have $1 + \mu(t)p(t) \geq 1$, So

$$z^\Delta(t)e^\sigma_{\ominus p}(\infty, t) + z(t)(e_{\ominus p}(\infty, t))^\Delta \geq 0\,, \quad (z(t)e_{\ominus p}(\infty, t))^\Delta \geq 0\,,$$

and integrate from t to ∞, we obtain

$$\lim_{t\to\infty} z(t)e_{\ominus p}(\infty, t) - z(t)e_{\ominus p}(\infty, t) \geq 0\,,$$

i.e., $z(t)e_{\ominus p}(\infty, t) \leq 1$ and hence

$$z(t) \leq e_p(\infty, t)\,.$$

By $y(t) \leq f(t)z(t)$, we have

$$y(t) \leq f(t)e_p(\infty, t), \quad t \in [0, \infty) \cap \mathbb{T}.$$

□

3. Weighted Jensen's Inequality

In [3] the authors proved $\varphi(\frac{\int_a^b g(t)\Delta t}{b-a}) \leq \frac{\int_a^b \varphi(g(t))\Delta t}{b-a}$, where φ is convex on $\mathbb{R}$. We would introduce the classical weighted Jensen's inequality on time scale.

Theorem 3.1. *Let $a, b \in \mathbb{T}$, $c, d \in \mathbb{R}$, If $g : [a, b] \to (c, d)$ is rd-continuous and $\varphi : (c, d) \to \mathbb{R}$ is a convex function, $p : \mathbb{T} \to \mathbb{R}$ is nonnegative and rd-continuous, $\int_a^b p(t)\Delta t > 0$, we have*

$$\varphi(\frac{\int_a^b p(t)g(t)\Delta t}{\int_a^b p(t)\Delta t}) \leq \frac{\int_a^b p(t)\varphi(g(t))\Delta t}{\int_a^b p(t)\Delta t}. \tag{3.1}$$

Proof. Let $x_0 \in (c,d)$, then there exists $\beta \in \mathbb{R}$, with

$$\varphi(x) - \varphi(x_0) \geq \beta(x - x_0) \quad \text{for all } x \in (c,d)\,. \tag{3.2}$$

Let $f(t) = \frac{p(t)}{\int_a^b p(t)\Delta t}$, then $\int_a^b f(t)\Delta t = 1$. Since $g(t)$ is rd-continuous,

$$x_0 = \frac{\int_a^b p(t)g(t)\Delta t}{\int_a^b p(t)\Delta t} = \int_a^b f(t)g(t)\Delta t$$

is well defined. $\varphi \circ g$ is also rd-continuous, and we apply (3.2) with $x = g(t)$ and integrate from a to b to obtain

$$\begin{aligned}
&\int_a^b p(t)\varphi(g(t))\Delta t - \varphi(\int_a^b f(t)g(t)\Delta t)\int_a^b p(t)\Delta t \\
&\quad = \int_a^b p(t)\varphi(g(t))\Delta t - \int_a^b p(t)\varphi(x_0)\Delta t = \int_a^b p(t)(\varphi(g(t)) - \varphi(x_0))\Delta t \\
&\qquad \geq \beta\int_a^b p(t)(g(t) - x_0)\Delta t = \beta(\int_a^b p(t)g(t)\Delta t - x_0\int_a^b p(t)\Delta t) = 0\,,
\end{aligned}$$

i.e., $\int_a^b p(t)\varphi(g(t))\Delta t \geq \varphi(\int_a^b f(t)g(t)\Delta t)\int_a^b p(t)\Delta t$, hence (3.1) holds. □

4. Weighted Opial Inequality

Opial inequality have various significant applications in the study of differential equality. In this section we present several weighted opial inequalities that a valid on time scale. We assume $0 \in \mathbb{T}, h \in \mathbb{T}$ with $h > 0$.

Theorem 4.1. *Let $\omega(t)$ be positive and continuous on $(0,h)$ with $\int_0^h \omega^{1-q}(t)\Delta t < \infty$, $q > 1$. For differentiable $f : [0,h] \to \mathbb{R}$ with $f(0) = 0$, we have*

$$\int_0^h |(f + f^\sigma)f^\Delta|\Delta t \leq (\int_0^h \omega^{1-q}\Delta t)^{\frac{2}{q}}(\int_0^h \omega \mid f^\Delta \mid^p \Delta t)^{\frac{2}{p}}\,, \tag{4.1}$$

where

$$p > 1 \ \text{ and } \ \frac{1}{p} + \frac{1}{q} = 1\,,$$

and with equality when $f(t) = c\int_0^t \omega^{1-q}\Delta\tau$ for some constant c.

Proof. We consider

$$y(t) = \int_0^t |f^\Delta| \Delta\tau, \quad t \in \mathbb{T} \text{ and } 0 \le t \le h\,.$$

Then we have $|f(t)| \le y(t)$, $y^\Delta(t) = |f^\Delta(t)|$, so that

$$\begin{aligned}
&\int_0^h |(f+f^\sigma)f^\Delta|\Delta t \le \int_0^h (|f|+|f^\sigma|)|f^\Delta|\Delta t \le \int_0^h (y+y^\sigma)y^\Delta \Delta t \\
&= \int_0^h (y^2)^\Delta \Delta t = y^2(h) - y^2(0) = (\int_0^h |f^\Delta|\Delta t)^2 = (\int_0^h \omega^{-\frac{1}{p}}\omega^{\frac{1}{p}}|f^\Delta|\Delta t)^2 \\
&\le (\int_0^h (\omega^{-\frac{1}{p}})^q \Delta t)^{\frac{2}{q}} (\int_0^h \omega |f^\Delta|^p \Delta t)^{\frac{2}{p}} = (\int_0^h \omega^{1-q}\Delta t)^{\frac{2}{q}} (\int_0^h \omega |f^\Delta|^p \Delta t)^{\frac{2}{p}}\,,
\end{aligned}$$

where we have used the Hölder's inequality. This directly yields weighted Opial inequality. □

Remark 4.2. If $\omega(t) \equiv 1$, $p = 2, q = 2$ then (4.1) is the same as Theorem 6.1 of [1].

Theorem 4.3. *Let* $f : [o,h] \bigcap \mathbb{T} \to \mathbb{R}$ *is differentiable,* $\omega(t)$ *be positive and continuous on* $(0,h)$ *with* $\int_0^h \omega^{1-q}(t)\Delta t < \infty$, $q > 1$. *Then*

$$\int_0^h |(f+f^\sigma)f^\Delta|\Delta t \le \nu^{\frac{2}{q}} (\int_0^h \omega |f^\Delta|^p \Delta t)^{\frac{2}{p}} + 2\beta \int_0^h |f^\Delta|\Delta t\,, \tag{4.2}$$

where

$$p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1$$

and

$$\nu = \max(\int_o^\alpha \omega^{1-q}\Delta t, \int_\alpha^h \omega^{1-q}\Delta t),$$

$$\alpha \in \mathbb{T} \text{ with } |\frac{h}{2} - \alpha| = \operatorname{dist}(\frac{h}{2}, \mathbb{T}), \quad \beta = \max(f(0), f(h)).$$

Proof. Consider $g(t) = f(t) - f(0)$, $g^\Delta(t) = f^\Delta(t)$. Then $g(t)$ satisfies Theorem 4.1, so that

$$\int_0^h |(g+g^\sigma)g^\Delta|\Delta t \le (\int_0^h \omega^{1-q}\Delta t)^{\frac{2}{q}} (\int_0^h \omega |g^\Delta|^p \Delta t)^{\frac{2}{p}}\,,$$

$$\int_0^h |f + f^\sigma - 2f(0)||f^\Delta|\Delta t \le (\int_0^h \omega^{1-q}\Delta t)^{\frac{2}{q}}(\int_0^h \omega|f^\Delta|^p\Delta t)^{\frac{2}{p}},$$

$$\int_0^h |f + f^\sigma||f^\Delta|\Delta t \le (\int_0^h \omega^{1-q}\Delta t)^{\frac{2}{q}}(\int_0^h \omega|f^\Delta|^p)^{\frac{2}{p}} + 2|f(0)|\int_0^h |f^\Delta|\Delta t\,.$$

Replacing h with $\alpha \in \mathbb{T}$, we have

$$\int_0^\alpha |f + f^\sigma||f^\Delta|\Delta t \le (\int_0^\alpha \omega^{1-q}\Delta t)^{\frac{2}{q}}(\int_0^\alpha \omega|f^\Delta|^p\Delta t)^{\frac{2}{p}} + 2|f(0)|\int_0^\alpha |f^\Delta|\Delta t\,. \quad (4.3)$$

Let $f(t) = f(h-t)$, we find

$$\int_0^\alpha |f(h-t) + f^\sigma(h-t)||f^\Delta(h-t)|\Delta t \le (\int_0^\alpha \omega^{1-q}(h-t)\Delta t)^{\frac{2}{q}}(\int_0^\alpha \omega(h-t)|f^\Delta(h-t)|^p\Delta t)^{\frac{2}{p}} + 2|f(h)|\int_0^\alpha |f^\Delta(h-t)|\Delta t\,.$$

Using the integral transforms, we have

$$\int_{h-\alpha}^h |f + f^\sigma||f^\Delta|\Delta t \le (\int_{h-\alpha}^h \omega^{1-q}\Delta t)^{\frac{2}{q}}(\int_{h-\alpha}^h \omega|f^\Delta|^p\Delta t)^{\frac{2}{p}} + 2|f(h)|\int_{h-\alpha}^h |f^\Delta|\Delta t\,. \quad (4.4)$$

By putting

$$\nu = \max(\int_o^\alpha \omega^{1-q}\Delta t, \int_\alpha^h \omega^{1-q}\Delta t),$$

$$\alpha \in \mathbb{T} \text{ with } |\frac{h}{2} - \alpha| = \operatorname{dist}(\frac{h}{2}, \mathbb{T}),\ \beta = \max(f(0), f(h))\,,$$

and adding (4.3) and (4.4), we obtain (4.2). □

Corollary 4.4. *Let $\omega(t)$ be positive and continuous on $(0,h)$, $\int_0^h \omega^{1-q}(t)\Delta t < \infty$, $q > 1$. For differentiable $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = f(h) = 0$, we have*

$$\int_0^h |(f + f^\sigma)f^\Delta|\Delta t \le \nu^{\frac{2}{q}}(\int_0^h \omega|f^\Delta|^p\Delta t)^{\frac{2}{p}},$$

where

$$p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1\,,$$

and $\nu = \max(\int_o^\alpha \omega^{1-q}\Delta t, \int_\alpha^h \omega^{1-q}\Delta t)$, $\alpha \in \mathbb{T}$ *with* $|\frac{h}{2} - \alpha| = \mathrm{dist}(\frac{h}{2}, \mathbb{T})$.

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