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TOTALLY REAL SURFACE OF A COMPLEX SPACE FORM

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Abstract: In the present paper we show that a totally real surface of a 2-dimensional complex space form is Einstein.

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1. Introduction

Let $\overline{M}_2(c)$ be a 2-dimensional complex space form of constant holomorphic sectional curvature c. For each real number c there is exactly one complex space form in every dimension with holomorphic sectional curvature c. The complex space forms of holomorphic sectional curvature c are denoted by $P_2(C)$, C_2 and D_2 depending on whether c is positive, zero or negative, respectively. $P_2(C)$ is the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature c. C_2 is the complex Euclidean space. D_2 is the open unit ball in C_3 endowed with Bergman metric of constant holomorphic sectional curvature c. Let M be a 2-dimensional totally real (see Preliminaries) surface isometrially immersed in $\overline{M}_2(c)$.

In the present paper we would like to study whether a totally real surface $\overline{M}_2(c)$ is Einstein (see Preliminaries) and to show the following theorem.

Theorem. *Let M be a 2-dimensional connected totally real surface with nonzero mean curvature immersed in $\overline{M}_2(c)$. Then M is Einstein, i.e., the sectional curvature K of M is constant and M is either $K = \frac{1}{4}c$ or flat.*

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2. Preliminaries

Let M be a 2-dimensional surface isometrically immersed in a 4-dimensional Riemannian manifold $\overline{M}$. We denote by g the metric of $\overline{M}$ as well as the one induced on M. For an arbitrary point $x \in M$, we may choose a field ξ of unit normal vectors defined in a neighborhood U. The second fundamental form h and the corresponding symmetric operator A_ξ are defined and related to covariant differentiation $\overline{\nabla}, \nabla$ and D in $\overline{M}$, M and the normal bundle, respectively, by the following formulas (see [3], [4], [5]):

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

where X and Y are vector fields tangent to M.

The Gauss equation is:

$$\begin{aligned} g(R(X, Y)Z, W) &= g(\overline{R}(X, Y)Z, W) \\ &\quad + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{3}$$

where $X, Y, Z, W \in T_xM$ and $T_xM, \overline{R}$ and R denote the tangent space of M, the curvature tensor of $\overline{M}$ and M, respectively.

The normal connection form $s_{\alpha\beta}$ in U is defined by

$$D_X \xi_\alpha = \sum_\beta s_{\alpha\beta}(X)\xi_\beta, \quad s_{\alpha\beta} + s_{\beta\alpha} = 0 \tag{4}$$

for $X \in T_xM$ (in this paper Greek indices run from 1 to 2).

The Codazzi equation is:

$$(\nabla_X A_\alpha)Y - \sum_\beta s_{\alpha\beta}(X) A_\beta Y = (\nabla_Y A_\alpha)X - \sum_\beta s_{\alpha\beta}(Y) A_\beta X. \tag{5}$$

Let ∇^* denote the Whitney sum of the tangential and the normal connection. Then we have

$$\nabla^*_X A_\alpha = \nabla_X A_\alpha - \sum_\beta s_{\alpha\beta}(X) A_\beta, \tag{6}$$

$$(\nabla^*_X A_\alpha)Y = (\nabla^*_Y A_\alpha)X. \tag{7}$$

If we denote by S the Ricci tensor of M, then from the Gauss equation we have

$$S(X,Y) = \sum_i g(\overline{R}(X,e_i)e_i,Y) + \sum_\alpha \text{trace}A_\alpha g(A_\alpha X,Y) - \sum_\alpha g(A_\alpha^2 X,Y),$$

where $\{e_i\}, i = 1,2$ is an orthonormal basis for T_xM.

Also, if S satisfies the equation

$$S(X,Y) = kg(X,Y)$$

for some constant k, then M is called Einstein.

The normal vector $H = \sum_\alpha (\text{trace}A_\alpha)\xi_\alpha$ is said the mean curvature vector. $|H|$ is also said the mean curvature.

We call M an totally real submanifold of a Kaehler manifold $\overline{M}$ if M admits an isometric immersion into $\overline{M}$ such that for all x, $J(T_xM) \subset T_x^\perp M$, where $T_x^\perp M$ denotes the normal space at x and J the complex structure of $\overline{M}$ ([1]).

3. Proof of Theorem

Let $\overline{M}_2(c)$ be a 2-dimensional complex space form of constant holomorphic sectional curvature c and M a 2-dimensional totally real surface with non-zero mean curvature isometrically immersed in $\overline{M}_2(c)$.

Let H be the mean curvature vector. Then we can choose an orthogonal normal basis e_1, e_2 such that $Je_1 = \frac{H}{|H|}$ and

$$A_{Je_1} = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} & \mu \\ \mu & \end{pmatrix},$$

since $A_{Je_1}e_2 = A_{Je_2}e_1$.

If $\lambda \neq \mu$ at some point x_0, then there exits a neighborhood U of x_0 which satisfies that $\lambda \neq \mu$ on U. Since it holds

$$A_{Je_1} = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} & \mu \\ \mu & \end{pmatrix},$$

we have

$$S = \frac{1}{4}cI + (\lambda\mu - \lambda^2)I. \tag{8}$$

Noting that $J\nabla_X Y = D_X JY$, we get

$$(\nabla^*_{e_1} A_{Je_1})e_2 \quad = \quad (e_1\mu)e_2 + (2\mu - \lambda) < \nabla_{e_1}e_2, e_1 > e_1,$$

$$\begin{aligned}
(\nabla^*_{e_2} A_{Je_1})e_1 &= (e_2\lambda)e_1 + (\lambda - 2\mu) < \nabla_{e_2} e_1, e_2 > e_2, \\
(\nabla^*_{e_1} A_{Je_2})e_2 &= (e_1\mu)e_1 - 3\mu < \nabla_{e_1} e_2, e_1 > e_2, \\
(\nabla^*_{e_2} A_{Je_2})e_1 &= (e_2\mu)e_2 + (\lambda - 2\mu) < \nabla_{e_2} e_1, e_2 > e_1, \\
(\nabla^*_{e_1} A_{Je_1})e_1 &= (e_1\lambda)e_1 + (2\mu - \lambda) < \nabla_{e_1} e_2, e_1 > e_2, \\
(\nabla^*_{e_2} A_{Je_1})e_2 &= (e_2\mu)e_2 + (\lambda - 2\mu) < \nabla_{e_2} e_1, e_2 > e_1, \\
(\nabla^*_{e_1} A_{Je_2})e_1 &= (e_1\mu)e_2 + (2\mu - \lambda) < \nabla_{e_1} e_2, e_1 > e_1, \\
(\nabla^*_{e_2} A_{Je_2})e_2 &= (e_2\mu)e_1 + 3\mu < \nabla_{e_2} e_1, e_2 > e_2.
\end{aligned}$$

From the Codazzi equation $(\nabla^*_{e_1} A_{Je_1})e_2 = (\nabla^*_{e_2} A_{Je_1})e_1$ we obtain

$$\begin{aligned}
e_1\mu &= (\lambda - 2\mu) < \nabla_{e_2} e_1, e_2 >, \\
e_2\lambda &= -(\lambda - 2\mu) < \nabla_{e_1} e_2, e_1 > .
\end{aligned}$$

Similarly, from $(\nabla^*_{e_1} A_{Je_2})e_2 = (\nabla^*_{e_2} A_{Je_2})e_1$

$$e_2\mu = -3\mu < \nabla_{e_1} e_2, e_1 > .$$

In terms of (8)

$$\sum_i < h(e_2, e_i), h(e_1, e_i) >= 0.$$

Hence we have

$$\sum_i < (\nabla h)(e_1, e_2, e_i), h(e_1, e_i) > + \sum_i < h(e_2, e_i), (\nabla h)(e_1, e_1, e_i) >= 0, \quad (9)$$

$$\sum_i < (\nabla h)(e_2, e_2, e_i), h(e_1, e_i) > + \sum_i < h(e_2, e_i), (\nabla h)(e_2, e_1, e_i) >= 0. \quad (10)$$

The equations (9) and (10) are equivalent to the following:

$$\begin{aligned}
&< (\nabla^*_{e_1} A_{Je_1})e_2, A_{Je_1} e_1 > + < A_{Je_1} e_2, (\nabla^*_{e_1} A_{Je_1})e_1 > \\
&\quad + < (\nabla^*_{e_1} A_{Je_2})e_2, A_{Je_2} e_1 > + < A_{Je_2} e_2, (\nabla^*_{e_1} A_{Je_2})e_1 >= 0, \quad (11)
\end{aligned}$$

$$< (\nabla^*_{e_2} A_{Je_1})e_2, A_{Je_1}e_1 > + < A_{Je_1}e_2, (\nabla^*_{e_2} A_{Je_1})e_1 > \\ + < (\nabla^*_{e_2} A_{Je_2})e_2, A_{Je_2}e_1 > + < A_{Je_2}e_2, (\nabla^*_{e_2} A_{Je_2})e_1 >= 0, \quad (12)$$

respectively. From (11) and (12) we have

$$(\lambda + \mu)(\lambda - \mu) < \nabla_{e_1} e_2, e_1 >= 0, \quad (\lambda + \mu)(\lambda - \mu) < \nabla_{e_2} e_1, e_2 >= 0.$$

From the assumption of Theorem we obtain

$$e_1\mu = 0,$$
$$e_2\lambda = e_2\mu = 0,$$
$$< \nabla_{e_1} e_2, e_1 >=< \nabla_{e_2} e_1, e_2 >= 0.$$

Therefore

$$< R(e_1, e_2)e_2, e_1 >= 0,$$

i.e., flat. By the assumption of connectedness we see that M is either flat or $\lambda = \mu$ on M. Assume that $\lambda = \mu$ on M. Then we have

$$A_{Je_1} = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} & \lambda \\ \lambda & \end{pmatrix}.$$

Hence we have

$$S = \frac{1}{4}cI.$$

Thus we see that M is Einstein. This proves the theorem.

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