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A CLASS OF *I*-TOPOLOGICAL SPACES INDUCED FROM METRIC SPACES USING FUZZY POINTS

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Abstract: In this article we discuss the corresponding relation between I-topology and crisp topology. We also establish some characteristic of the I-open and I-closed sets. The relations between continuous mappings and I-continuous mappings are also discussed.

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1. Introduction

The fundamental concepts of a fuzzy set, introduced by Zadeh in 1965 [15], provides a natural foundation for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In the area of I-topology (here I denotes the interval $[0, 1]$ in R), much research has been carried Chang [2] since 1968, which is called fuzzy topology due to Chang [2].

Earlier work on I-topology was concentrated on generalizing topology to set-

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theoretic behavior almost identical to that of crisp sets. Therefore the concept of I-topology was collection of I-topology results valid in a fuzzy setting within this span of time, or has been able to not solve within the frame work of general topology. For the detail, see Chang [3], [4], [11], [12], etc.

According to Chang's definition, the I-topology is a direct generalization of the fuzzification crisp topology. But it is difficult to discuss the topological properties of the space. In order to avoid these difficulties, we derive the I-topology as the product topology by means of fuzzy points, it depends on the topological properties of the underlying interval I. We construct a class of I-topology which is based on corresponding relation between I-topology and crisp topology (for the I-topology and fuzzy points, we refer the reader to the excellent book of Höhle and Rodabaugh(Eds) [9]). Using the crisp topology as a stepping-stone, it is more convenient to discuss the topological properties of this I-topology.

In Section 2, we construct the I-topology using fuzzy points, and establish some characteristics of I-open and I-closed sets. In Section 3, we define the class of I-continuous functions and discuss the relation of continuous functions and I-continuous functions.

2. The Construction of *I*-Topology

Suppose that (X, d_X) is a metric space. A fuzzy set is a function from X into I. The class of all fuzzy sets is denoted by I^X. Next, we give the definition of fuzzy points which play an important role in our paper.

Definition 1. We say that x^*_α, $0 < \alpha \leq 1$, is a fuzzy point of X if

$$x^*_\alpha(y) = \begin{cases} \alpha, & y = x, \\ 0, & y \neq x, \end{cases}$$

for each $x \in X$. We use X^* to denote the set of all fuzzy points.

It is obvious that X^* is a subset of I^X.

Let us define $B_{\epsilon,\delta}(x^*_\alpha) = \{y^*_\beta | d_X(x, y) < \epsilon, |\beta - \alpha| < \delta\}$, where $x^*_\alpha \in X^*$ and $\epsilon, \delta > 0$. We have the following crisp topological property.

Theorem 2. *Let* $\Gamma = \{B_{\epsilon,\delta}(x^*_\alpha) | x^*_\alpha \in X^* \text{and } \epsilon, \delta > 0\}$. *Then* Γ *forms a crisp topology base on* X^*.

Proof. Let $z^*_\gamma \in B_{\epsilon_1,\delta_1}(x^*_\alpha) \cap B_{\epsilon_2,\delta_2}(y^*_\beta)$. We have $d_X(z, x) < \epsilon_1, d_X(z, y) < \epsilon_2, |\gamma - \alpha| < \delta_1$ and $|\gamma - \beta| < \delta_2$. Let $\epsilon_3 = \min(\epsilon_1 - d_X(z, x), \epsilon_2 - d_X(z, y))$

and let $\delta_3 = \min(\delta_1 - |\gamma - \alpha|, \delta_2 - |\gamma - \beta|)$. We shall show that $B_{\epsilon_3,\delta_3}(z^*_\gamma) \subset B_{\epsilon_1,\delta_1}(x^*_\alpha) \cap B_{\epsilon_2,\delta_2}(y^*_\beta)$. Then Γ forms a crisp topology base. Suppose that $t^*_\lambda \in B_{\epsilon_3,\delta_3}(z^*_\gamma)$, then $d_X(t,z) < \epsilon_3$ and $|\gamma - \lambda| < \delta_3$. It follows that $d_X(t,x) \leq d_X(t,z) + d_X(z,x) \leq \epsilon_3 + d_X(z,x) \leq \epsilon_1 - d_X(z,x) + d_X(z,x) = \epsilon_1$. Similarly, we can show that $d_X(t,y) < \epsilon_2, |\lambda - \alpha| < \delta_1$ and $|\beta - \lambda| < \delta_2$. It follows that Γ forms a crisp topology base. □

Notations. We use $\mathcal{T}^*_X$ to denote the crisp topology generated by Γ on X^* and $(X^*, \mathcal{T}^*_X)$ denotes the crisp topological space X^* with crisp topology $\mathcal{T}^*_X$.

Now we recall the definition of I-topology on X.

Definition 3. A family $\mathcal{T}_X \subseteq I^X$ is called an I-topology on X if $\mathcal{T}_X$ satisfies the following conditions:

(1) $0_x, 1_x \in \mathcal{T}_X$, where $0_x : x \mapsto 0,\ \forall x \in X$ and $1_x : x \mapsto 1,\ \forall x \in X$,

(2) $A \bigwedge B \in \mathcal{T}_X$, whenever $A,\ B \in \mathcal{T}_X$,

(3) $\bigvee_{i \in J} \{A_i\} \in \mathcal{T}_X$, whenever each $A_i \in \mathcal{T}_X, i \in J$ for any index family J.

Definition 4. Let $\mathcal{T}_X \subseteq I^X$ be an I-topology on X. We call every $A \in \mathcal{T}_X$ an I-open set. The complement $A' \in I^X$ of A defined by $A'(x) := 1 - A(x)$ is called an I-closed set.

The following theorem is the main result of this section. It establishes a connection of I-topology on X and the crisp topological space $(X^*, \mathcal{T}^*_X)$.

Theorem 5. *Suppose that $A \in I^X$ and $E(A)$ denotes the set $E(A) := \{x^*_\alpha | x^*_\alpha \in X^*, \alpha < A(x)\}$. Then the collection $\mathcal{T}_X := \{A | A \in I^X,\ E(A)$ is open in $(X^*, \mathcal{T}^*_X)\}$ forms an I-topology on X.*

Proof. The fuzzy set 0_x and $1_x \in \mathcal{T}_X$ since $E(0_x) = \emptyset$ and $E(1_x) = X^* \backslash \{x^*_1\}$ are open in $(X^*, \mathcal{T}^*_X)$. Next, suppose that A and B belong to $\mathcal{T}_X$, then $E(A)$ and $E(B)$ are open in $(X^*, \mathcal{T}^*_X)$. Moreover,

$$\begin{aligned} E(A \wedge B) &= \{x^*_\alpha | \alpha < (A \wedge B)(x), x \in X\} \\ &= \{x^*_\alpha | \alpha < A(x) \quad \text{and} \quad \alpha < B(x), x \in X\} \\ &= \{x^*_\alpha | \alpha < A(x), x \in X\} \cap \{x^*_\alpha | \alpha < B(x), x \in X\} = E(A) \cap E(B) \end{aligned}$$

is open in $(X^*, \mathcal{T}^*_X)$.

Finally, we claim that $E(\bigvee_{i \in J} A_i) = \bigcup_{i \in J} E(A_i)$ for each $A_i \in \mathcal{T}_X, i \in J$ and any index set J. If it is true, then $E(\bigvee_{i \in J} A_i)$ will be open in $(X^*, \mathcal{T}^*_X)$ since $\bigcup_{i \in J} E(A_i)$ is open in $(X^*, \mathcal{T}^*_X)$. It follows that $\mathcal{T}_X$ forms an I-topology on X.

Let $x_\alpha^* \in E(\bigvee_{i\in J} A_i)$. Then $\alpha < \bigvee_{i\in J}(A_i)(x)$. It follows that $\alpha < (A_i)(x)$ for some $i \in J$ by the definition of union of fuzzy sets. Thus $x_\alpha^* \in E(A_i)$ for some $i \in J$. Then $x_\alpha^* \in \bigcup_{i\in J} E(A_i)$. Hence, $E(\bigvee_{i\in J} A_i) \subset \bigcup_{i\in J} E(A_i)$. Conversely, let $x_\alpha^* \in \bigcup_{i\in J} E(A_i)$. Then $x_\alpha^* \in E(A_i)$ and $\alpha < (A_i)(x)$ for some $i \in J$. Thus, $\alpha < (\bigvee_{i\in J}(A_i))(x)$, i.e. $x_\alpha^* \in E(\bigvee_{i\in J} A_i)$ and $\bigcup_{i\in J} E(A_i) \subseteq E(\bigvee_{i\in J} A_i)$. Therefore, we complete the proof. $\square$

Remark 6. Theorem 5 does not hold for general lattice L. Since $E(\bigvee_{i\in J} A_i) = \bigcup_{i\in J} E(A_i)$ may not be true.

Notation. We use $(I^X, \mathcal{T}_X)$ to denote the I-topological space X with the I-topology $\mathcal{T}_X$ in Theorem 5.

In the Theorem 5, we introduce a new technique which can replace the I-open set by the open set of crisp topology. By similar tools we also can establish the relation between I-closed sets by means of the closed sets in $(X^*, \mathcal{T}_X^*)$.

Definition 7. Let $A \in I^X$. Define $C(A) = \{x_\alpha^* | x_\alpha^* \in X^*, \alpha \leq A(x)\}$.

Remark 8. (1) Define $\bigvee \emptyset = 0$.Then $A(x) = \bigvee\{\alpha | x_\alpha^* \in C(A)\} = \bigvee\{\alpha | x_\alpha^* \in E(A)\}$ for fixed $x \in X$ and each $A \in I^X$.

(2) $E(A) = E(B) \Leftrightarrow C(A) = C(B) \Leftrightarrow A = B$ for $A, B \in I^X$.

(3) It is obvious that $(x_n)_{\alpha_n}^* \to x_\alpha^*$ in (X^*, τ_X^*) iff $x_n \to x$ in (X, d) and $\alpha_n \to \alpha$ in $(I, |\cdot|)$ for each sequence $\{(x_n)_\alpha^*\} \subseteq X^*$ (Here $|\cdot|$ denotes the absolute value function).

Theorem 9. *Let $A \in I^X$. A is I-closed in $(I^X, \mathcal{T}_X)$ iff $C(A)$ is closed in $(X^*, \mathcal{T}_X^*)$.*

Proof. "$\Rightarrow$" Suppose that $A \in I^X$ is an I-closed set and $\{(x_n)_{\alpha_n}^*\}$ is a convergent sequence in $C(A)$ such that $\{(x_n)_{\alpha_n}^*\}$ converges to x_α^* in the crisp topology $(X^*, \mathcal{T}_X^*)$. We shall show that $x_\alpha^* \in C(A)$, and it will imply $C(A)$ is closed in $(X^*, \mathcal{T}_X^*)$. Suppose not, $x_\alpha^* \notin C(A)$, then it will imply $\alpha > A(x)$. So, $1-\alpha < A'(x)$ and $x_{1-\alpha} \in E(A')$. By the definition of A and Theorem 5, $E(A')$ is open. Hence, there exists an open ball $B_{\epsilon,\delta}(x_{1-\alpha}^*)$ such that $B_{\epsilon,\delta}(x_{1-\alpha}^*) \subset E(A')$. Since $\{(x_n)_{\alpha_n}^*\}$ converges to x_α^*, we see that $\{(x_n)_{1-\alpha_n}^*\}$ converges to $x_{1-\alpha}^*$. Hence, there exists a nature number N such that $(x_n)_{1-\alpha_n}^* \in B_{\epsilon,\delta}(x_{1-\alpha}^*) \subset E(A')$ for each $n > N$. That is, $1-\alpha_n < A'(x_n)$ for $n > N$. Hence, $\alpha_n > A(x_n)$, i.e., $(x_n)_{\alpha_n}^* \notin C(A)$ for $n > N$, which contradicts that $\{(x_n)_{\alpha_n}^*\}$ is a convergent sequence in $C(A)$. Therefore $x^*_\alpha \in C(A)$. We derive that $C(A)$ is closed in $(X^*, \mathcal{T}^*_X)$.

"$\Leftarrow$" Suppose that $C(A)$ is closed. If A' is not an I-open set, then $E(A')$ will be not open in $(X^*, \mathcal{T}^*_X)$. There exist $x^*_\alpha \in E(A')$ and $N \in \mathcal{N}$ such that $B_{\frac{1}{n},\frac{1}{n}}(x^*_\alpha)$ is not contained in $E(A')$ for each $n > N$. Choose $(x_n)^*_{\alpha_n} \in B_{\frac{1}{n},\frac{1}{n}}(x^*_\alpha)$ but $(x_n)^*_{\alpha_n} \notin E(A')$. Then it is clear that $\{(x_n)^*_{\alpha_n}\}$ converges to x^*_α. Thus $\{(x_n)^*_{1-\alpha_n}\}$ converges to $x^*_{1-\alpha}$ in $(X^*, \mathcal{T}^*_X)$. Since $(x_n)^*_{\alpha_n} \notin E(A')$ for $n > N$, it follows that $1 - \alpha_n < A(x_n)$. So, $(x_n)^*_{1-\alpha_n} \in E(A)$ for each $n \in \mathcal{N}$. Since $C(A)$ is closed and $\{(x_n)^*_{1-\alpha_n}\}$ converges to $x^*_{1-\alpha}$, we have that $x^*_{1-\alpha} \in C(A)$. It implies that $1 - \alpha \leq A(x)$, i.e. $\alpha \geq A'(x)$. Thus $x^*_\alpha \notin E(A')$. It contradicts to $x^*_\alpha \in E(A')$. So, $E(A')$ is open. □

Finally, it follows from Theorem 5 and Theorem 9 that

Theorem 10. *Let $(X^*, \mathcal{T}^*_X)$ be the crisp topology generated by the base Γ and let $\mathcal{T}_X$ be the I-topology generated by $(X^*, \mathcal{T}^*_X)$. Then the followings are equivalent:*

(a) A is I-open in $(I^X, \mathcal{T}_X)$.

(b) $E(A)$ is open in $(X^*, \mathcal{T}^*_X)$.

(c) A' is I-closed in $(I^X, \mathcal{T}_X)$.

(d) $C(A')$ is closed in $(X^*, \mathcal{T}^*_X)$.

In the following theorem, we shall show that an I-open set A in $(I^X, \mathcal{T}_X)$ can be viewed as a crisp open set in $X \times (0, 1]$ by using $E(A)$.

Theorem 11. *Let A be a fuzzy set of X. Then $S = \{(x, \alpha) | x^*_\alpha \in E(A)\}$ is open in the crisp topological space $X \times (0, 1]$ with the product topology iff A is an I-open set in $(I^X, \mathcal{T}_X)$.*

Proof. By the definition of $(I^X, \mathcal{T}_X)$ and crisp product topology, we see that

$$
\begin{aligned}
&S \text{ is open in } X \times (0, 1] \\
&\qquad \Longleftrightarrow \text{for each } (x, \alpha) \in S \text{ there exist } \delta, \epsilon > 0 \text{s.t.} \\
&\qquad\quad \{(y, \beta) | d_X(x, y) < \epsilon \text{ and } |\alpha - \beta| < \delta\} \subset S \\
&\qquad \Longleftrightarrow \text{ there exist } \delta, \epsilon > 0 \text{ s.t } B_{\epsilon,\delta}(x^*_\alpha) \subset E(A) \\
&\qquad\qquad \Longleftrightarrow A \text{ is an } I\text{-open set in } (I^X, \mathcal{T}_X).
\end{aligned}
$$
□

Finally, we give some examples.

Example 1. Let $X = \mathcal{R}$ with the metric $d(x, y) = |x - y|$. Let A be the fuzzy set defined by $A(x) = \frac{1}{2}$ for each $x \in \mathcal{R}$. We can find $S = \{(x, \alpha) | x^*_\alpha \in E(A)\} = \mathcal{R} \times (0, \frac{1}{2})$. By Theorem 11, A is an I-open set in $(I^X, \mathcal{T}_X)$.

Example 2. Let $X = \mathcal{R}$ with the metric $d_X(x, y) = |x - y|$. Let A be the fuzzy set defined by

$$A(x) = \begin{cases} \frac{1}{2}, & x \le 0, \\ 1, & x > 0. \end{cases}$$

We can find that $S = \{(x, \alpha) | x^*_\alpha \in E(A)\} = (-\infty, 0] \times (0, \frac{1}{2}) \bigcup (0, \infty) \times (0, 1]$ is an open set in $\mathcal{R} \times (0.1]$. By Theorem 11, A is an I-open set in $(I^X, \mathcal{T}_X)$.

Define

$$B(x) = \begin{cases} \frac{1}{2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

We can find that $S = \{(x, \alpha) | x^*_\alpha \in E(B)\} = (-\infty, 0) \times (0, \frac{1}{2}) \bigcup (0, \infty) \times (0, \frac{1}{2}) \bigcup \{0\} \times (0, 1]$ is not an open set in $\mathcal{R} \times (0.1]$. By Theorem 11, B is not an I-open set in $(I^X, \mathcal{T}_X)$.

3. *I*-Continuous Functions

In this section, all topologies are the same as those in Section 2. Suppose that (X, d_X) and (Y, d_Y) are two metric spaces. If f is a function from X into Y, then we can define the function $f^{\rightarrow} : I^X \rightarrow I^Y$ by

$$f^{\rightarrow}(A)(y) = \begin{cases} \bigvee_{f(x)=y} A(x), & \\ 0, & y \neq f(x), \end{cases}$$

for $A \in I^X$ by Zadeh extension. In particular, we use $(f^{\rightarrow})^*$ to denote the restriction of $f^{\rightarrow}$ on X^*, i,e. $(f^{\rightarrow})^* : X^* \rightarrow Y^*$ is defined by $(f^{\rightarrow})^*(x^*_\alpha) = y^*_\alpha$ if $f(x) = y$.

We want to show that some crisp topological properties of functions will be preserved under Zadeh extension, for example, continuity and open mapping.

Theorem 12. *If $f : (X, d_X) \rightarrow (Y, d_Y)$ is a continuous function, then $(f^{\rightarrow})^*$ is a continuous function from $(X^*, \mathcal{T}^*_X)$ into $(Y^*, \mathcal{T}^*_Y)$.*

Proof. Let $\{(x_n)^*_{\alpha_n}\}$ be a sequence which converges to x^*_α in $(X^*, \mathcal{T}^*_X)$. So, $x_n \rightarrow x$ and $\alpha_n \rightarrow \alpha$. By the continuity of f, we see that $f(x_n) \rightarrow f(x)$. Hence, $(f^{\rightarrow})^*((x_n)^*_{\alpha_n}) = (f(x_n))^*_{\alpha_n} \rightarrow (f(x))^*_\alpha = (f^{\rightarrow})^*(x^*_\alpha)$ and it follows that $(f^{\rightarrow})^*$ is a continuous function from $(X^*, \mathcal{T}^*_X)$ into $(Y^*, \mathcal{T}^*_Y)$. □

Now, we recall the definition of the I-continuous functions.

Definition 13. Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function. f is called I-continuous if $f^{\leftarrow}(B) \in \mathcal{T}_X$ for each $B \in \mathcal{T}_Y$, where $f^{\leftarrow}(B) \in I^X$ is the fuzzy set in X defined by $f^{\leftarrow}(B)(x) = B(f(x))$ for each $x \in X$.

We also define $(f^{\leftarrow})^*(y^*_\alpha)$ to be the crisp set $\{x^*_\alpha | f(x) = y\}$. Moreover, we define

$$(f^{\rightarrow})^*(E(A)) = \{(f^{\rightarrow})^*(x^*_\alpha) | x^*_\alpha \in E(A)\}$$

for each $A \in I^X$ and

$$(f^{\leftarrow})^*(E(B)) = \{x^*_\alpha | (f^{\rightarrow})^*(x^*_\alpha) \in E(B)\}$$

for each $B \in I^Y$.

Lemma 14. *Suppose that f is a function from X into Y. Then $f^{\rightarrow}$ has the following properties:*

(a) $E(f^{\rightarrow}(A)) = (f^{\rightarrow})^*(E(A))$ for each $A \in I^X$.

(b) $E(f^{\leftarrow}(B)) = (f^{\leftarrow})^*(E(B))$ for each $B \in I^Y$.

Proof. (a) Suppose that $y^*_\alpha \in (f^{\rightarrow})^*(E(A))$, then there exists an $x^*_\alpha \in E(A)$ such that $(f^{\rightarrow})^*(x^*_\alpha) = y_\alpha$. Hence, $f(x) = y$ and $\alpha < A(x)$. It implies $\alpha < \bigvee_{f(x)=y} A(x)$. Therefore, $\alpha < (f^{\rightarrow}(A))(y)$. Thus $y_\alpha \in E(f^{\rightarrow}(A))$ and $E(f^{\rightarrow}(A)) \supset (f^{\rightarrow})^*(E(A))$.

Conversely, suppose that $y_\alpha \in E(f^{\rightarrow}(A))$, we have $\alpha < (f^{\rightarrow}(A))(y)$. So, $\alpha < A(x)$ for some x with $f(x) = y$. Therefore, there exists an x^*_α such that $x^*_\alpha \in E(A)$ and $y^*_\alpha = (f^{\rightarrow})^*(x^*_\alpha) \in (f^{\rightarrow})^*(E(A))$. We obtain that $E(f^{\rightarrow}(A)) \subset (f^{\rightarrow})^*(E(A))$.

(b) Let $B \in I^Y$. Then

$$\begin{aligned} x^*_\alpha \in (f^{\leftarrow})^*(E(B)) &\Leftrightarrow (f^{\rightarrow})^*(x^*_\alpha) \in E(B) \\ &\Leftrightarrow (f(x))^*_\alpha \in E(B). \Leftrightarrow \alpha < B(f(x)) \\ &\Leftrightarrow \alpha < (f^{\leftarrow}(B))(x) \Leftrightarrow x^*_\alpha \in E(f^{\leftarrow}(B)). \end{aligned}$$

This shows that $E(f^{\leftarrow}(B)) = (f^{\leftarrow})^*(E(B))$ for each $B \in I^Y$. □

Now, we discuss the main results of this section.

Theorem 15. *If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, then $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is an I-continuous function.*

Proof. Suppose that $B \in \mathcal{T}_Y$, then $E(B) \in \mathcal{T}^*_Y$. We see that $(f^{\leftarrow})^*(E(B)) \in \mathcal{T}^*_X$ by Theorem 12. Moreover, by Lemma 14(b), we see that $E(f^{\leftarrow}(B)) = (f^{\leftarrow})^*(E(B)) \in \mathcal{T}^*_X$ and it shows that f is an I-continuous function. □

We say that $f : (X, d_X) \to (Y, d_Y)$ is an open mapping if $f(V)$ is open in (Y, d_Y) whenever V is open in (X, d_X). Similarly, we say that f is an I-open mapping if $f^{\to}(A)$ is an I-open set whenever A is an I-open set. In the rest of this section, we will show that $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is an I-open mapping whenever f is an open mapping from (X, d_X) to (Y, d_Y).

Lemma 16. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is an open mapping, then $(f^{\to})^* : (X^*, \mathcal{T}_X^*) \to (Y^*, \mathcal{T}_Y^*)$ is an open mapping.*

Proof. Let U be an open set in $(X^*, \mathcal{T}_X^*)$ and let $y_\alpha^* \in (f^{\to})^*(U)$. There exists an $x_\alpha^* \in U$ such that $(f^{\to})^*(x_\alpha^*) = y_\alpha^*$. Moreover, let $B_{\epsilon,\delta}(x_\alpha^*)$ be a open ball of x_α^* such that $B_{\epsilon,\delta}(x_\alpha^*) \subset U$ and let $V = \{b \in X |$ there exists γ such that $b_\gamma^* \in B_{\epsilon,\delta}(x_\alpha^*)\}$. Then V is an open set in (X, d_X) by the definition of $B_{\epsilon,\delta}(x_\alpha^*)$. Since f is an open mapping, we have that $f(V)$ is an open set in Y. So, we can obtain that the set $S = \{b_\gamma^*; b \in f(V),\ |\gamma - \alpha| < \delta\}$ is open in $(Y^*, \mathcal{T}_Y^*)$. We claim that $S \subset (f^{\to})^*(U)$. If it is true, then S is an open neighborhood of y_α^*, this shows that $(f^{\to})^*(U)$ is open. And then we can conclude that $(f^{\to})^*$ is an open mapping.

Let $b_\gamma \in S$. Then there exists $a \in V$ such that $f(a) = b$ and $|\gamma - \alpha| < \delta$. So, $a_\gamma \in B_{\epsilon,\delta}(x_\alpha^*)$. It follows that

$$b_\gamma^* = (f(a))_\gamma^* = (f^{\to})^*(a_\gamma^*) \in (f^{\to})^*(B_{\epsilon,\delta}(x_\alpha^*)) \subset (f^{\to})^*(U).$$

So, $S \subset (f^{\to})^*(U)$. □

Theorem 17. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is an open mapping, then f is also an I-open mapping from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$.*

Proof. Let A be an I-open set in $(X, \mathcal{T}_X)$. Then $E(A) \in \mathcal{T}_X^*$. By Lemma 16 it follows that $(f^{\to})^*(E(A)) \in \mathcal{T}_Y^*$. Moreover, by Lemma 14(a), we see that $E(f^{\to}(A)) \in \mathcal{T}_Y^*$ and it follows that $f^{\to}(A)$ is an I-open set in $(I^Y, \mathcal{T}_Y)$. So, f is an I-open mapping from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$. □

Corollary 18. *Suppose that X and Y are two Banach spaces over $\mathcal{R}$ and $f : X \to Y$ is a continuous linear surjection, then f is an I-open mapping.*

Proof. It is obvious from the Open Mapping Theorem and Theorem 17. □

Definition 19. Let A be a fuzzy set of a vector space over $\mathcal{R}$ and let t be a scalar. Then:

(i) for $t \neq 0$, $(tA)(x) = A(t^{-1}x)$ for all $x \in X$,

(ii) for $t = 0$,

$$(tA)(x) = \begin{cases} 0, & \text{if} \quad x \neq 0, \\ \bigvee_y A(y) & \text{if} \quad x = 0. \end{cases}$$

Corollary 20. *Suppose that X is a Banach space over $\mathcal{R}$ and define the mapping $f : X \to X$ by $f(x) = tx$, where $t \in \mathcal{R}$ is a scalar, then f is an I-open mapping.*

Proof. f is a linear surjection. It follows that f is an I-open mapping by Corollary 18. □

In the following corollary, we show that the translation is an I-open mapping.

Corollary 21. *Suppose that X is a Banach space over $\mathcal{R}$ and define the mapping $f_u : X \to X$ by $f_u(x) = u + x$, for any fixed $u \in X$. Then f is an I-open mapping.*

Proof. It is easy to see that $f_u(U) = u + U$ is an open set for each open set U. So, f_u is an open mapping. Thus it follows that f_u is an I-open mapping by Theorem 17 □

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