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THE SUBGROUP OF THE JACOBIAN OF A PROJECTIVE CURVE GENERATED BY A GENERIC HYPERPLANE SECTION

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Abstract: Let $C \subset \mathbf{P}^r$ be an integral non-degenerate curve and $f : X \to C$ its normalization. If $\mathrm{char}(\mathbb{K}) > 0$ assume that C is reflexive. Set $d := \deg(C)$. Let $E_C \subset \mathrm{Pic}^0(X)$ be the subgroup induced by the differences of the points of a generic hyperplane section of C. Here we prove that $E_C \cong \mathbb{Z}^{d-1}$. A similar result is proven when we only consider the hyperplane sections tangent to a general (but fixed) $P \in C$.

AMS Subject Classification: 14H50, 14H40
Key Words: projective curves, hyperplane section, Jacobian

1. Torsion Free Subgroups of the Jacobian of a Curve

We work over an algebraically closed field $\mathbb{K}$ and prove the following results.

Theorem 1. *Let $C \subset \mathbf{P}^r$ be an integral non-degenerate curve and $f : X \to C$ its normalization. If $\mathrm{char}(\mathbb{K}) > 0$ assume that C is reflexive. Set $d := \deg(C)$. Let $E_C \subset \mathrm{Pic}^0(X)$ be the subgroup induced by the differences of the points of a generic hyperplane section of C. Then $E_C \cong \mathbb{Z}^{d-1}$.*

Theorem 2. *Let $C \subset \mathbf{P}^r$ an integral non-degenerate curve and $f : X \to C$ its normalization. If $\mathrm{char}(K) > 0$ assume that C is reflexive and that, with the*

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notation of [4], its generic order of contact, b_2*, with its generic osculating plane is equal to* 3*. Set* $d := \deg(C)$*. Fix a general* $P \in X$*. Let* $F_C \subset \mathrm{Pic}^0(X)$ *be the subgroup induced by the differences with* P *of all points (different from* P*) of a generic hyperplane section of* C *tangent to* C *at* P*. Then* $F_C \cong \mathbb{Z}^{d-2}$*.*

Remark 1. Let $C \subset \mathbf{P}^r$, $r \geq 3$, an integral non-degenerate curve and $f : X \to C$ its normalization. If $\mathrm{char}(\mathbb{K}) > d$, then C is reflexive ([6], first line of page 571). Hence the assumption that C is reflexive in the statements of Theorem 1 and Theorem 2 is rather mild. By [4], Theorem 15, the assumption "$b_2 = 3$" in the statement of Theorem 2 is satisfied if $\mathrm{char}(\mathbb{K}) > d$.

Remark 2. In the proof of Theorem 2 (resp. Theorem 1) we used that a certain permutation group of $\{1, \dots, d\}$ (resp. $\{1, \dots, d-2\}$) is the full symmetric group. As in [6] it is sufficient that this group contains the alternating group. See [6], Theorem 2.5, for many cases in which C is not reflexive, but this weaker assumption is satisfied.

Our work was ignited from the reading of [2]. See [2], Theorem 7 (i.e. the quotation of [5] and [7], p. 152), for the tools needed to work over a subfield of $\mathbb{K}$.

Remark 3. It is easy to construct a smooth non-degenerate curve $C \subset \mathbf{P}^r$ such that there is a hyperplane H intersecting transversally C and such that $C \cap H$ induces a torsion subgroup of the Jacobian. It is not difficult to cnstruct a smooth non-degenerate curve $C \subset \mathbf{P}^r$ such that $C \cap H$ induces a torsion-free subgroup of the Jacobian for all hyperplane H intersecting transversally C.

Proof of Theorem 1. By our assumptions on C the monodromy of a generic hyperplane sections of C is the full symmetric group (see [6], Proposition 2.1, or (in characteristic zero) [1], p. 109, or [3], Chapter III). Hence either $E_C \cong \mathbb{Z}^{d-1}$ or E_C is torsion. Fix general $P, Q \in C(\mathbb{K})$. Hence $P' := f^{-1}(P)$ and $Q' := f^{-1}(Q)$ are general in X. Hence $\mathcal{O}_X(P' - Q')$ is not a torsion point of $\mathrm{Pic}^0(X)$. Since there is a hyperplane $H \subset \mathbf{P}^r$ transversal to C and containing $\{P, Q\}$, E_C is not torsion. □

Proof of Theorem 2. Let H be a general hyperplane tangent to C at P. Since P is general in C and C is reflexive and H is general, H has order of contact 2 with C at P, $H \cap \mathrm{Sing}(C) = \emptyset$ and $(C \cap H) \backslash \{P\}$ is formed by $d-2$ discinct points. Furthermore the linear projection of C from P into $\mathbf{P}^{r-1}$ is birational onto its image C' which is an integral degree $d-2$ curve. The assumption "$b_2 = 3$" for C gives the reflexivity of C'. Applying Theorem 1 to C' would give a weaker result. Instead, we apply directly that the monodomy group of a general hyperplane section tangent to C at P is the full symmetric group of permutation of the remaining $d-2$ points of intersection. $\square$

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