THE GREEN FUNCTION AND COVARIANCE FUNCTION OF STOCHASTIC FRACTIONAL PSEUDODIFERENTIAL MODELS

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Abstract: The problem of stochastic pseudodifferential representation of fractional spatial and spatiotemporal random fields is studied. This leads to a formulation of elliptic, parabolic and hyperbolic pseudodifferential equations driven by fractionally integrated white noise. The geometry of fractional Sobolev spaces is then considered to define the solutions to such equations as well as to derive their spectral representation. The spectral relationship between the covariance function of the solution and the Green function associated with the deterministic problem is derived for both bounded and unbounded domain cases.

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1. Introduction

The formulation of an appropriate stochastic differential model is a key aspect in the solution of the filtering, estimation, and prediction problems (Dolph and Woodbury, 1952; Krein, 1953, 1954; Dym and McKean, 1976; Anderson, 1983; Anh and Spencer, 1995, etc.). The spectral decomposition is a useful tool in the derivation of stochastic differential representations of homogeneous random fields (see, for example, Yaglom, 1962, Yadrenko, 1983; Walsh, 1983; Ivanov and Leonenko, 1989, etc.). In the non-homogeneous case, the covariance factorization plays a similar role in the definition of stochastic differential/pseudodifferential equations in the second-order moment sense, and in the sample-path sense for the Gaussian case. Such factorization provides a relationship between the covariance function and the Green function associated with the deterministic differential/pseudodifferential model.

The concept of generalized random fields, based on the theory of distributions, is another important technique in the theory of stochastic differential models. Indeed, the theories of Sobolev spaces and generalized random functions play a key role in the solution of stochastic partial differential equations. Conversely, the problem of deriving an abstract representation for a given class of generalized random fields leads to the formulation of stochastic partial differential models for the corresponding class of ordinary random fields. In Anh et al (2000; 2001), the abstract representation of generalized random fields defined on a Sobolev space of integer order is obtained under the existence of the dual generalized random fields. This representation is given in terms of a generalized white noise relative to a Hilbert space. Assuming the local property of the operator defining the stochastic representation, specialization to the ordinary case leads to the introduction of Gaussian random fields as solutions of stochastic partial differential equations of elliptic type driven by white noise. The fractal version of these results is obtained in Ruiz-Medina et al (2001; 2002; 2003; 2004). Specifically, in Ruiz-Medina et al (2001; 2002; 2003; 2004), the covariance factorization and the fractional pseudodifferential representation of a generalized random field defined on a fractional Sobolev space is derived for the cases of regular $C^\infty$-bounded domains, domains with fractal boundary and fractal domains. The fractional pseudodifferential representation of the sample paths of a class of elliptic fractal Gaussian random fields is then established as a special case of these weak-sense results.

In this paper, we extend to the spatiotemporal case the generalized formulation of spatial differential and pseudodifferential models derived in Anh et al (2000; 2001) and Ruiz-Medina et al (2001; 2002; 2003; 2004). We first
present the results related to the spatial elliptic case, then extend the formulation to the case where the driving process is fractionally integrated white noise. We then provide the ordinary and generalized solutions to spatiotemporal differential/pseudodifferential models of parabolic and hyperbolic types. Their spectral properties are analyzed, and the spectral relationship between the covariance function of the solution and the Green function associated with the deterministic problem is given.

In Ramm (1990), the Wiener-Kolmogorov prediction theory is generalized to the nonstationary case using the properties of Sobolev spaces of integer order. In particular, a key result is based on the isomorphism defined by the covariance operator between a Sobolev space of negative order and its dual. An extension of these results to the fractal case is obtained in Angulo et al (2000) based on the theory of linear self-adjoint operators on fractional Sobolev spaces. The results derived in the present paper provide the needed tools for an extension of the results of Ramm (1990) and Angulo et al (2000) to the spatiotemporal case.

2. Preliminaries

In this section, we describe the main results of the development in Ruiz-Medina et al (2003), where an abstract representation for a class of fractional generalized random fields is derived. Under suitable conditions, such a representation provides a pointwise fractional pseudodifferential representation.

Let $(\Omega, A, P)$ be a complete probability space, and let $L^2(\Omega, A, P)$ be the Hilbert space of real-valued zero-mean random variables defined on $(\Omega, A, P)$ with finite second-order moments and with the inner product

$\langle Y, Z \rangle_{L^2(\Omega)} = E[YZ], \quad \forall Y, Z \in L^2(\Omega, A, P).$

Let $X$ be a generalized random field (GRF) defined on $C_0^\infty (T)$, the space of infinitely differentiable functions with compact support contained in $T$, with $T$ a $C^\infty$-bounded domain. That is, $X$ is a continuous linear mapping from $C_0^\infty (T)$ into $L^2(\Omega, A, P)$, considering in $C_0^\infty (T)$ the topology defined as the intersection of the topologies associated with the Sobolev spaces of fractional order $\{H^\alpha (T) : \alpha \in \mathbb{Q}\}$. Hereafter $U$ denotes the space $C_0^\infty (T)$ with this topology, and $U_{\alpha}$ the fractional Sobolev space $H^{-\alpha} (T)$ for $\alpha \in \mathbb{Q}$, the set of rational numbers. $V_\alpha$ represents the dual space $H^{-\alpha} (T)$ of $U_{\alpha}$, for each $\alpha \in \mathbb{Q}$ (see, for example, Triebel, 1978; Dautray and Lions, 1985).

The covariance functional $B(\varphi, \phi) = E[X(\varphi)X(\phi)]$ of the GRF $X$ is a continuous bilinear functional on $U \times U$. From the Kernel Theorem (see Gel’fand
and Vilenkin, 1964, p. 74), the covariance functional $B$ of $X$ admits the following representation, for a certain $\alpha \in \mathbb{Q}$:

$$B(\varphi, \phi) = \langle (R_{\alpha}\varphi)^*, \phi \rangle_{U_{\alpha}}, \quad \varphi, \phi \in U,$$

where $R_{\alpha}$ is a symmetric positive continuous linear operator from $U_{\alpha}$ into $V_{\alpha}$. $B$ and $X$ can then be extended to the space $U_{\alpha}$. From this extension we can define an $\alpha$-GRF $X_{\alpha}$ as a mean-square continuous linear random function from $U_{\alpha}$ into $L^2(\Omega, A, P)$. The closed spans of $\{X_{\alpha}(\varphi) : \varphi \in U_{\alpha}\}$ and $\{B_{\alpha}(\phi) : \phi \in U_{\alpha}\}$ in the $L^2(\Omega, A, P)$-topology are respectively denoted by $H(X_{\alpha})$ and $\mathcal{H}(X_{\alpha})$.

An $\alpha$-GRF $\widetilde{X}_{\alpha}$ defined from $[U_{\alpha}]^*$ into $L^2(\Omega, A, P)$ is said to be the $\alpha$-dual of the $\alpha$-GRF $X_{\alpha}$ if:

(i) $H(X_{\alpha}) = H\left(\widetilde{X}_{\alpha}\right)$, with $H\left(\widetilde{X}_{\alpha}\right)$ being the closed span of $\{\widetilde{X}_{\alpha}(\phi) : \phi \in [U_{\alpha}]^*\}$ in the $L^2(\Omega)$-topology, and

(ii) $\left<X(\phi), \widetilde{X}(g)\right>_{H(X)} = \langle \phi, g^* \rangle_{U_{\alpha}}$, for $\phi \in U_{\alpha}$, and $g \in [U_{\alpha}]^*$, with $g^*$ being the dual element of $g$ with respect to the $U_{\alpha}$-topology.

The existence of the $\alpha$-dual GRF $\widetilde{X}_{\alpha}$ of $X_{\alpha}$ is fundamental for deriving the covariance factorisation and an abstract representation of $X_{\alpha}$ on $U_{\alpha}$.

The derivation of the abstract representation of $X_{\alpha}$, under the existence of the $\alpha$-dual $\widetilde{X}_{\alpha}$, consists of the definition of an isomorphism $L$ on $U_{\alpha}$ such that the geometry induced by the covariance function of the generalized random field $X_{\alpha}L$ coincides with the geometry of the parameter space $U_{\alpha}$ of $X_{\alpha}$. Thus,

$$\langle X_{\alpha}L(\phi), X_{\alpha}L(\varphi) \rangle_{H(X_{\alpha})} = \langle \phi, \varphi \rangle_{U_{\alpha}} = \langle \mathcal{I}_{\alpha}(\phi), \mathcal{I}_{\alpha}(\varphi) \rangle_{L^2(T)}, \forall \phi, \varphi \in U_{\alpha},$$

(1)

where $\mathcal{I}_{\alpha}$ denotes the inverse of the trace on domain $T$ of the Bessel potential of order $\alpha$ (see Stein, 1970). The right-hand side of equation (1) is the covariance functional of an $\alpha$-generalized white noise ($\alpha$-GWN) $\varepsilon_{\alpha}$ (see Ruiz-Medina et al 2003). That is,

$$E[\varepsilon_{\alpha}(\phi) \varepsilon_{\alpha}(\varphi)] = \langle \varepsilon_{\alpha}(\phi) \varepsilon_{\alpha}(\varphi) \rangle_{H(\mathcal{I}_{\alpha})} = \langle \phi, \varphi \rangle_{U_{\alpha}}, \forall \phi, \varphi \in U_{\alpha}.$$ 

Equation (1) can then be expressed as

$$X_{\alpha}L(\phi) = \varepsilon_{\alpha}(\phi) = \varepsilon_{L^2(T)}[\mathcal{I}_{\alpha}(\phi)], \quad \forall \phi \in U_{\alpha},$$

(2)

where $\varepsilon_{L^2(T)}$ is a GWN relative to $L^2(T)$ (see Anh et al 2000). The $\alpha$-dual GRF of $\varepsilon_{\alpha}$ is the $\alpha$-GWN $\widetilde{\varepsilon}_{\alpha}$ defined as $\widetilde{\varepsilon}_{\alpha} = \varepsilon_{\alpha}I_{[U_{\alpha}]^*}$ with $[U_{\alpha}]^*$ being the dual space of $U_{\alpha}$ and $I_{[U_{\alpha}]^*} : [U_{\alpha}]^* \rightarrow U_{\alpha}$ being the isometric isomorphism defined by the Riesz Representation Theorem. Depending on whether the $\alpha$-GWN $\varepsilon_{\alpha}$
is or is not predetermined, equation (2) can be interpreted as a *weak-sense* or a *strong-sense* abstract representation for $X_\alpha$ (see Anh et al 2000). The conditions under which the $\alpha$-GRF $X_\alpha$ has a weak-sense (unique except for isometric isomorphisms) or a strong-sense (unique) abstract representation are given in Ruiz-Medina et al (2003).

The $\alpha$-dual GRF $\tilde{X}_\alpha$ also satisfies the following equation: for all $f,g \in [U_\alpha]^*$,

\[
\langle \tilde{X}_\alpha L' (g), \tilde{X}_\alpha L' (f) \rangle_{H(X_\alpha)} = \langle \mathcal{I}_{-2\alpha} (g), \mathcal{I}_{-2\alpha} (f) \rangle_{U_\alpha} = \langle g, f \rangle_{[U_\alpha]^*} = \langle \mathcal{I}_{-\alpha} (g), \mathcal{I}_{-\alpha} (f) \rangle_{L^2(T)},
\]

(3)

with $L' = R_\alpha LI_{[U_\alpha]^*}$, and with $R_\alpha$ and $L$ being, respectively, the covariance operator of $X_\alpha$ and the isomorphism defining the abstract representation of $X_\alpha$. Equation (3) can be equivalently expressed as

\[
\tilde{X}_\alpha (L' g) = \tilde{\varepsilon}_\alpha (g) = \varepsilon_{L^2(T)} [\mathcal{I}_{-\alpha} (g)], \quad \forall g \in [U_\alpha]^*,
\]

(4)

where $\tilde{\varepsilon}_\alpha = \varepsilon_{I_{[U_\alpha]^*}}$, and $\varepsilon_\alpha$ is the $\alpha$-dual GWN of $\tilde{\varepsilon}_\alpha$ appearing in the abstract representation of $X_\alpha$.

In the generalized ordinary case, equation (2) can be interpreted as

\[
\langle X_\alpha (\cdot), L \varphi (\cdot) \rangle_{L^2(T)} = \langle \varepsilon (\cdot), \mathcal{I}_\alpha (\varphi) (\cdot) \rangle_{L^2(T)}, \quad \forall \varphi \in U_\alpha,
\]

(5)

where \{X_\alpha (z) : z \in T\} denotes the ordinary random field associated with the $\alpha$-GRF $X_\alpha$ whose trajectories are in $V_\alpha$, in the Gaussian case, and \{\varepsilon (z) : z \in T\} represents white-noise.

Similarly, equation (4) can be interpreted in the generalized ordinary case as

\[
\langle \tilde{X}_\alpha (\cdot), L' g (\cdot) \rangle_{L^2(T)} = \langle \tilde{\varepsilon} (\cdot), \mathcal{I}_{-\alpha} (g) \rangle_{L^2(T)},
\]

(6)

for all $g \in [U_\alpha]^*$, where $\tilde{\varepsilon}(\cdot)$ denotes white noise, and $\tilde{X}_\alpha$ is the ordinary random field corresponding to the $\alpha$-GRF $\tilde{X}_\alpha$ whose trajectories are in $U_\alpha$, in the Gaussian case.

The specializations to the generalized ordinary case given in equations (5) and (6) lead to the definition of stochastic equations driven by weak-sense fractionally differentiated or integrated white noise. Such equations will be considered in the construction of generalized versions of ordinary differential equations in Section 4. Note that under suitable conditions, provided by the
embedding theorems between fractional Besov spaces (see Triebel, 1978), the ordinary solutions $X_\alpha$ and $\tilde{X}_\alpha$ satisfying

$$L^*X_\alpha = I_\alpha \varepsilon, \quad [L]^*\tilde{X}_\alpha = I_{-\alpha} \overline{\varepsilon}$$

are uniquely defined (pointwise) from equations (5) and (6).

3. Spectral Representation of Fractional Operators

The spectral theory of self-adjoint differential/pseudodifferential operators allows an investigation of the solutions of corresponding stochastic models. Particularly, the spectral theory of self-adjoint pseudodifferential operators is fundamental in the characterization of anomalous diffusion processes. In this section, we briefly review some basic results of the spectral theory of self-adjoint operators (see, for example, Hutson and Pym, 1980). In Section 3.2 we apply such a theory in the definition of a solution to stochastic differential/pseudodifferential models driven by fractionally integrated white noise, in the elliptic (spatial processes), parabolic and hyperbolic (spatiotemporal processes) cases.

3.1. Spectral Theory of Bounded Self-Adjoint Operators

The diagonalization of a Hermitian matrix by means of a suitable basis is a well-known result (see, for example, Halmos, 1948, for the finite-dimensional case). Its generalisation to self-adjoint operators on an infinite-dimensional Hilbert space is given by the Spectral Theorem (e.g. Hutson and Pym, 1980). The first step of this generalisation is the Hilbert-Schmidt Theorem for compact self-adjoint operators.

**Theorem 3.1.** Let $L$ be a compact and self-adjoint operator defined on an infinite-dimensional separable Hilbert space $H$. Then, its eigenfunctions form an orthonormal basis of the space $H$.

**Theorem 3.2.** (Canonical Form of Compact Self-Adjoint Operators) For a compact self-adjoint operator $L$ on a Hilbert space $H$, the following identity holds:

$$L(f) = \sum_{n=0}^{\infty} \lambda_n (f, \phi_n) \phi_n, \quad \forall f \in H,$$

where $\{\phi_n\}_{n \in \mathbb{N}}$ is the system of eigenfunctions associated with the sequence of eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$, and, for each $n \in \mathbb{N}$, $(f, \phi_n)$ denotes the projection of the function $f$ on the element $\phi_n$ of the orthonormal eigenfunction system.
Theorem 3.3. Let $L$ be an unbounded self-adjoint operator on the infinite dimensional separable Hilbert space $H$. Suppose that $L$ has a compact inverse, and let $\{\mu_n\}$ and $\{\phi_n\}$ be the sets of eigenvalues and eigenfunctions of $L^{-1}$, respectively. Then, the following assertions hold:

(i) The eigenvalues $\{\mu_n\}_{n \in \mathbb{N}}$ of $L^{-1}$ are all non-zero, and $\{\phi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $H$.

(ii) The eigenvalues of $L$ are given as $\lambda_n = \mu_n^{-1}$, for each $n \in \mathbb{N}$, with the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ being infinite and $|\lambda_n| \to \infty$ as $n \to \infty$. The corresponding system of eigenfunctions is given by $\{\phi_n\}_{n \in \mathbb{N}}$. Finally,

$$(\lambda I - L)^{-1} g = \sum_{n=0}^{\infty} (\lambda - \lambda_n)^{-1} (g, \phi_n) \phi_n, \quad \lambda \in \rho(L), g \in H,$$

with $\rho(L)$ being the resolvent of $L$, that is, the set of values $\lambda$ for which $(\lambda I - L)^{-1}$ is a bounded linear operator, where $I$ denotes the identity operator.

From the above results, for each $g \in H$, we have

$$g = \sum_{n=0}^{\infty} (g, \phi_n) \phi_n.$$

In addition, if $p$ is a polynomial, then

$$p(L) g = \sum_{n=0}^{\infty} p(\lambda_n) (g, \phi_n) \phi_n. \quad (8)$$

Equation (8) provides an appropriate definition of an arbitrary continuous function $f$ of $L$. The extension of the above representations to the case of non-compact self-adjoint operators is given in the following theorem.

Theorem 3.4. (The Spectral Theorem) Let $H$ be a Hilbert space, and let $L : H \to H$ be a bounded self-adjoint operator. Then, there exists a unique family $\{P_\lambda\}$ of self-adjoint projections with the following properties:

(i) $P_\lambda$ is a strongly right-continuous function of $\lambda$.

(ii) $P_\lambda$ commute with each other and with any bounded operator which commutes with $L$.

(iii) $P_\lambda$ is null outside the range of the spectrum of $L$.

(iv) $P_\lambda \geq P_\mu$ if $\lambda \geq \mu$.

(v) For any function $f$ defined on $\mathbb{R}$ which is continuous on an open set containing the spectrum, it holds that

$$f(L) = \int_{-\infty}^{\infty} f(\lambda) dP_\lambda,$$
\[ f(L)g = \int_{-\infty}^{\infty} f(\lambda) dP_\lambda g, \quad \forall g \in H, \quad \text{and} \]

\[ (f(L)g,h) = \int_{-\infty}^{\infty} f(\lambda) d(P_\lambda g,h), \quad \forall g,h \in H, \quad (9) \]

where the integrals are understood in the Riemann-Stieltjes sense.

The family of projection operators introduced in Theorem 3.4 is known as the spectral family of \( L \), and each \( P_\lambda \) is called a spectral projection. The properties of functions of operators are described in detail in the following result.

**Theorem 3.5.** (Spectral Calculus) Let \( L \) be a bounded self-adjoint operator defined on the Hilbert space \( H \) and let \( f,g \in H \) be two continuous functions on an open set containing the spectrum of \( L \). Then the following assertions hold:

(i) Equivalences in terms of the usual operations:

\[ (f + g)(L) = f(L) + g(L), \quad (\alpha f)(L) = \alpha f(L), \]
\[ (f \cdot g)(L) = f(L) \cdot g(L). \]

(ii) \( f(L) \) is self-adjoint if \( f \) is real-valued, and positive if \( f \) is non-negative.

(iii) (a) \( \|f(L)\| = \sup \lambda \in \sigma(L) |f(\lambda)| \), with \( \sigma(L) \) being the spectrum of \( L \).

(b) \( \|f(L)h\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d(P_\lambda h,h) \quad \forall h \in H. \)

(iv) The Spectral Mapping Theorem: \( f(\sigma(L)) = \sigma(f(L)) \).

There exists a close relationship between the behaviour of \( P_\lambda \) as a function of \( \lambda \) and the spectrum of a bounded self-adjoint operator \( L \). More specifically, \( P_\lambda \) is constant on the resolvent of \( L, \rho(L) \), and \( P_\lambda \) has a discontinuity at \( \lambda \) if \( \lambda \) is an eigenvalue (the case of unbounded operators is not considered here because our development is based on linear bounded operators).

**3.2. Spectral Representation of Fractional Green Functions**

In this subsection, we apply the spectral results described in the previous section to derive the solutions to spatial and spatiotemporal equations defined in terms of spatial self-adjoint linear operators and driven by fractionally integrated white noise. The elliptic, parabolic and hyperbolic cases corresponding to differential and fractional pseudodifferential spatial operators on bounded and unbounded domains will be addressed.
We use the notation $\tilde{X}_\alpha$ for defining the solution to spatial elliptic models in connection with the preliminary results of Section 2, since we are interested in the definition of pointwise solutions (ordinary case). Then, $\tilde{X}_\alpha$ is defined as the weak-sense solution to

$$\left( [ \mathcal{L}']^* \tilde{X}_\alpha (\cdot) , \mathcal{I}_{-2\alpha} (g) (\cdot) \right)_{U_\alpha} = \left( \mathcal{I}_{-\alpha} (\tilde{\varepsilon}) (\cdot) , \mathcal{I}_{-2\alpha} (g) (\cdot) \right)_{U_\alpha} = \langle \tilde{\varepsilon}_\alpha (\cdot) , g^* \rangle_{U_\alpha}.$$ 

Equivalently,

$$[ \mathcal{L}']^* \tilde{X}_\alpha (\cdot) = \mathcal{I}_{-\alpha} (\tilde{\varepsilon}) (\cdot), \tag{10}$$

$$\mathcal{I}_\alpha [ \mathcal{L}']^* \tilde{X}_\alpha (\cdot) = \tilde{\varepsilon} (\cdot), \tag{11}$$

where $\tilde{\varepsilon}$ is, as before, white noise with reproducing kernel Hilbert space $L^2 (T)$.

We will refer to this process as an ordinary white noise. The operator $\mathcal{L}'$ is assumed to be an elliptic self-adjoint linear operator. Formally, the inverse operator $[ [ \mathcal{L}']^* ]^{-1} \mathcal{I}_{-\alpha}$ of $\mathcal{I}_\alpha [ \mathcal{L}']^*$ is the Green operator defining the solution to the above equations. Under suitable conditions (see, for example, Triebel, 1978, on embedding theorems between fractional Sobolev spaces), $[ [ \mathcal{L}']^* ]^{-1}$ admits a kernel representation on the space $U_\alpha$ given by

$$[ [ \mathcal{L}']^* ]^{-1} (\phi) (z) = \langle l (z, \cdot) , \phi (\cdot) \rangle_{U_\alpha} , \quad \forall \phi \in U_\alpha , \tag{12}$$

where $l (\cdot, \cdot)$ is a function defined on $T \times T$ which is sufficiently regular with respect to both variables. We say that $[ [ \mathcal{L}']^* ]^{-1} = [ [ \mathcal{L}'^{-1} ]^* ]$ is an integral operator on $U_\alpha$. The operator $[ [ \mathcal{L}']^* ]^{-1} \mathcal{I}_{-\alpha}$ can then be represented as

$$[ [ \mathcal{L}']^* ]^{-1} \mathcal{I}_{-\alpha} (f) (z) = \langle l (z, \cdot) , \mathcal{I}_{-\alpha} (f) (\cdot) \rangle_{U_\alpha} \quad \mathcal{I}_{-\alpha} \quad \mathcal{I}_{-\alpha}$$

$$= \int_T \langle l (z, \cdot) , i_{-\alpha} (\cdot, y) \rangle_{U_\alpha} f (y) \, dy, \tag{13}$$

for all $f \in L^2 (T)$, where $i_{-\alpha}$ is the kernel of the trace on $T$ of the Bessel potential $\mathcal{I}_{-\alpha}$ of order $\alpha$ (see Stein, 1970). Thus, from equation (13), the Green function $G_{-\alpha} (\cdot, \cdot)$ associated with the operator $\mathcal{I}_\alpha [ \mathcal{L}']^*$ in equation (10) is defined as

$$G_{-\alpha} (z, y) = \langle l (z, \cdot) , \mathcal{I}_{-\alpha} (\cdot, y) \rangle_{U_\alpha}.$$ 

Note that under the duality condition defined in the previous section, since $\mathcal{L}'$ is involved in the covariance factorization of the dual random field $X_\alpha$ of $\tilde{X}_\alpha$, the mean-square continuity of $X_\alpha$ implies the continuity of $\mathcal{L}'$ on the space
H^{−α}(T). Hence, the operator \( L' \) is bounded and elliptic, and admits a bounded inverse.

In the next sections, we study the following stochastic fractional pseudodifferential models given in terms of the operator \([L']^*\):

\[
[L']^* \tilde{X}_α^e (z) = \mathcal{I}_{-α} [\tilde{e}^e] (z), \quad \forall z ∈ \tilde{T} ⊆ \mathbb{R}^d,
\]  

(14)

\[
\frac{∂}{∂t} \tilde{X}_α^p (t, z) = [L']^* \tilde{X}_α^p (t, z) + \mathcal{I}_{-\tilde{α}} [\tilde{e}^p] (t, z), \quad \forall t ∈ \mathbb{R}_+, \forall z ∈ \tilde{T} ⊆ \mathbb{R}^d,
\]  

(15)

\[
\frac{∂^2}{∂t^2} \tilde{X}_α^h (t, z) = [L']^* \tilde{X}_α^h (t, z) + \mathcal{I}_{-\tilde{α}} [\tilde{e}^h] (t, z), \quad \forall t ∈ [-1, 1], \forall z ∈ \tilde{T} ⊆ \mathbb{R}^d,
\]  

(16)

where \( \tilde{T} \) denotes the interior of the \( C^∞ \)-bounded domain \( T ⊆ \mathbb{R}^d \), and \( \tilde{e}^e, \tilde{e}^p \) and \( \tilde{e}^h \) are spatial and spatiotemporal white noises. The results in the previous section are applied to derive the spectral representation of the Green functions associated with the deterministic problems corresponding to equations (14), (15) and (16), assuming homogeneous boundary conditions. Two cases are considered:

(i) \([L']^*\) is a self-adjoint operator which has a compact inverse.

(ii) \([L']^*\) is a self-adjoint operator which has a bounded inverse.

In the following proposition, the discrete spectral representation of the Green function associated with equation (14) is derived.

**Proposition 3.6.** Let \( G_α^e (\cdot, \cdot) \) be the Green function associated with the deterministic problem corresponding to the stochastic fractional pseudodifferential equation

\[
[L']^* \tilde{X}_α^e (z) = \mathcal{I}_{-α} \tilde{e}^e (z), \quad \forall z ∈ \tilde{T},
\]  

(17)

where \( \mathcal{I}_α [L']^* \) is assumed to be an elliptic self-adjoint fractional pseudodifferential operator from \( U_α \) into \( L^2 (T) \). If \( [[L']^*]^{-1} \) is compact then the Green function \( G_α^e (\cdot, \cdot) \) associated with equation (14) can be represented in the \( U_α \)-topology as

\[
G_α^e (x, y) = \sum_{n=0}^∞ \frac{φ_n (x) φ_n (y)}{λ_n},
\]  

(18)

where \( \{φ_n\}_{n∈\mathbb{N}} ⊆ U_\tilde{α} \) is the eigenfunction system associated with the sequence of eigenvalues \( \{λ_n\}_{n∈\mathbb{N}} \) of the point spectral representation of the operator \([L']^*\) on \( U_α \).
Remark 3.1. Note that the identity (18) on $U_\alpha$ is equivalent to the following identity on $L^2(T)$:

$$G_\alpha^e (x, y) = \mathcal{I}_{-\alpha} \left[ \sum_{n=0}^\infty \frac{\phi_n (x) \phi_n (y)}{\lambda_n} \right].$$

Proof. Applying the operator $[L']^*$ to the function $G_\alpha^e$ defined in equation (18), we obtain

$$[[L']^* G_\alpha^e] (x, y) = \left< l (x, \cdot), \sum_{n=0}^\infty \frac{\phi_n (\cdot)}{\lambda_n} \phi_n (y) \right>_{U_\alpha} = \sum_{n=0}^\infty \frac{1}{\lambda_n} \phi_n (y) \phi_n (x).$$

From equation (19) and the Hilbert-Schmidt Theorem, the following identity holds:

$$\langle [L']^* G_\alpha^e (x, \cdot), \psi (\cdot) \rangle_{U_\alpha} = \sum_{n=0}^\infty \phi_n (x) \langle \phi_n (\cdot), \psi (\cdot) \rangle_{U_\alpha} = \sum_{n=0}^\infty \psi_n \phi_n (x) = \psi (x), \quad \forall \psi \in U_\alpha,$$

where $\{\psi_n\}_{n \in \mathbb{N}}$ are the Fourier coefficients of $\psi$ in the orthonormal basis $\{\phi_n\}_{n \in \mathbb{N}}$ of $U_\alpha$. Therefore, $[L']^* G_\alpha^e (x, y)$ is the $\delta$-function on $U_\alpha$. This is formally expressed as

$$[L']^* G_\alpha^e (x, y) = \delta (x - y).$$

That is, $G_\alpha^e$ admits the series representation (18) on $U_\alpha$. $\square$

Let $G_\alpha^p$ be the Green function on $U_\alpha$ associated with the stochastic fractional pseudodifferential equation

$$\frac{\partial}{\partial t} \tilde{\chi}_\alpha^p (t, z) = [L']^* \tilde{\chi}_\alpha^p (t, z) + \mathcal{I}_{-\alpha} [\tilde{\varepsilon}^p] (t, z).$$

The function $G_\alpha^p (t, z; s, y)$ is then defined on $\mathbb{R}_+ \times T \times \mathbb{R}_+ \times T$, except for $(t, z) = (s, y)$; it is zero if $t < s$, and, as a function of $(t, z)$ for fixed $(s, y) \in \mathbb{R}_+ \times T$, $G_\alpha^p$ satisfies the following equation in $U_\alpha$:

$$\frac{\partial G_\alpha^p (\cdot, \cdot; s, y)}{\partial t} - [L']^* G_\alpha^p (\cdot, \cdot; s, y) = \delta (\cdot - s) \mathcal{I}_{-\alpha} [\delta (\cdot, y)],$$

(22)
for all \( s \in \mathbb{R}_+ \), and \( y \in \hat{T} \). The function \( G^p_\alpha \) is also symmetric with respect to the spatial variable. Under compactness of the inverse operator of \([L']^*\), the spectral representation of \( G^p_\alpha \) is now derived in terms of the eigenfunction system \( \{ \phi_n \}_{n \in \mathbb{N}} \subseteq U_\alpha \) associated with the point spectrum \( \{ \lambda_n \}_{n \in \mathbb{N}} \) of \([L']^*\).

**Proposition 3.7.** Assume that the conditions of Proposition 3.6 hold. Then, the Green function \( G^p_\alpha \) defined in equation (22) admits the following orthogonal representation on \( U_\alpha \):

\[
G^p_\alpha (t, x; s, y) = \sum_{n=0}^{\infty} \phi_n (x) \phi_n (y) \gamma_n (t, s; \lambda_n),
\]

where

\[
[L']^* \phi_n (x) = \lambda_n \phi_n (x), \quad \forall n \in \mathbb{N},
\]

\[
\left[ \frac{\partial}{\partial t} - \lambda_n \right] \gamma_n (t, s; \lambda_n) = \delta (t - s), \quad \forall t, s \in \mathbb{R}_+,
\]

and

\[
C_n (y; t, s) = \phi_n (y) \gamma_n (t, s; \lambda_n) = \langle G^p_\alpha (t, \cdot; s, y), \phi_n (\cdot) \rangle_{U_\alpha}.
\]

**Proof.** From equations (23), (24), (25) and (26), taking the inner product in the space \( U_\alpha \) by the eigenfunction \( \phi_p (\cdot) \) in both sides of equation (22), we obtain

\[
\left\langle \left[ \frac{\partial}{\partial t} - [L']^* G^p_\alpha \right] (t, \cdot; s, y), \phi_p (\cdot) \right\rangle_{U_\alpha} = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} C_n (y; t, s) \langle \phi_n (\cdot), \phi_p (\cdot) \rangle_{U_\alpha} - \sum_{n=0}^{\infty} C_n (y; t, s) \langle \left[ [L']^* \phi_n \right] (\cdot), \phi_p (\cdot) \rangle_{U_\alpha}
\]

\[
= \frac{\partial}{\partial t} C_p (y; t, s) - \sum_{n=0}^{\infty} C_n (y; t, s) \lambda_n \langle \phi_n (\cdot), \phi_p (\cdot) \rangle_{U_\alpha}
\]

\[
= \frac{\partial}{\partial t} C_p (y; t, s) - C_p (y; t, s) \lambda_n = \delta (t - s) \langle \mathbb{I}_\alpha (\cdot - y), \phi_p (\cdot) \rangle_{U_\alpha}
\]

\[
= \delta (t - s) \phi_p (y).
\]

Then, for each \( p \in \mathbb{N} \),

\[
\left[ \frac{\partial}{\partial t} - \lambda_p \right] C_p (y; t, s) = \phi_p (y) \delta (t - s).
\]
Consequently, we can define $C_n(y; t, s)$ as
\begin{equation}
C_n(y; t, s) = \phi_n(y) \gamma_n(t, s; \lambda_n),
\end{equation}
with
\begin{equation}
\left[ \frac{\partial}{\partial t} - \lambda_n \right] \gamma_n(t, s; \lambda_n) = \delta(t - s).
\end{equation}
From equations (27), (28), (29) and (30), the Green function $G^n_\alpha$ given by equation (22) admits an orthogonal expansion on $U_\alpha$ as in equation (23).

We now consider the Green function $G^h_\alpha$ associated with the deterministic problem corresponding to equation (16) on $U_\alpha$, that is, the function satisfying, for all $t, s \in [-1, 1]$ and for all $x, y \in \overset{\circ}{T} \subseteq \mathbb{R}^d$,
\begin{equation}
\left[ \frac{\partial^2}{\partial t^2} - [L']^* \right] G^h_\alpha(t, x; s, y) = \mathcal{I}_{-\alpha} [\delta](x - y) \mathcal{I}_{-\alpha} [\delta](t - s).
\end{equation}

**Proposition 3.8.** Let $G^h_\alpha$ be the Green function associated with equation (16), and defined by equation (31). Assuming that $[[L']^*]^{-1}$ is a compact and self-adjoint operator on $U_\alpha$, the function $G^h_\alpha$ admits the following orthogonal expansion:
\begin{equation}
G^h_\alpha(t, x; s, y) = \sum_{n=0}^{\infty} C_n(y; t, s) \phi_n(x),
\end{equation}
where, for all $n \in \mathbb{N}$, $[L']^* \phi_n = \lambda_n \phi_n$, and
\begin{equation}
C_n(y; t, s) = \langle G^n_\alpha(t, \cdot; s, y), \phi_n(\cdot) \rangle_{U_\alpha}.
\end{equation}
Therefore, for each $n \in \mathbb{N}$, $C_n(y; t, s) = \phi_n(y) c_n(t, s)$, with $c_n(t, s)$ being the solution of the equation
\begin{equation}
\left[ \frac{\partial^2}{\partial t^2} - \lambda_n \right] c_n(t, s) = \mathcal{I}_{-\alpha} [\delta](t - s).
\end{equation}

**Proof.** As in the parabolic case, taking the inner product in the space $U_\alpha$ by the eigenfunction $\phi_p(\cdot)$ in both sides of equation (31), the coefficients $C_n(y; t, s)$, for $n \in \mathbb{N}$, in equation (33) satisfy the following equation:
\begin{equation}
\frac{\partial^2}{\partial t^2} C_n(y; t, s) - \lambda_n C_n(y; t, s) = \phi_n(y) \mathcal{I}_{-\alpha} [\delta](t - s).
\end{equation}
Therefore, \( C_n (y; t, s) = c_n (t, s) \phi_n (y) \), and
\[
\begin{bmatrix}
\partial^2 / \partial t^2 - \lambda_n
\end{bmatrix} c_n (t, s) = \mathcal{I}_{-\alpha} [\delta] (t - s).
\]
Then,
\[
G_p^\alpha (t, x ; s, y) = \sum_{n=0}^{\infty} \phi_n (x) \phi_n (y) c_n (t, s).
\] (35)

The results derived have been obtained assuming that \([L]^* \) is a compact self-adjoint operator, and has pure point spectrum. In the non-compact case, Theorem 3.4 provides the spectral representation of the Green functions respectively associated with equations (14), (15) and (16), in terms of the self-adjoint projection family \( \{ P_\lambda \} \) associated with \([L]^* \). That is,
\[
\begin{align*}
G^\alpha_e (z, y) &= \int_{\Lambda} \frac{1}{\lambda} dP_\lambda (z, y), \\
G^\alpha_p (t, z ; s, y) &= \int_{\Lambda} \gamma_\lambda dP_\lambda (z, y), \\
G^h (t, z ; s, y) &= \int_{\tilde{\Lambda}} \frac{1}{\lambda} d\tilde{P}_\lambda (t, z ; s, y),
\end{align*}
\] (36-38)

where \( \Lambda \) and \( \tilde{\Lambda} \) respectively denote the continuous spectrum of the operators \([L]^* \) and \( \partial^2 / \partial t^2 + [L]^* \), and \( \{ P_\lambda \}_{\lambda \in \Lambda} \) and \( \{ \tilde{P}_\lambda \}_{\lambda \in \tilde{\Lambda}} \) are the respective spectral families of the self-adjoint projection operators associated with \([L]^* \) and \( \partial^2 / \partial t^2 + [L]^* \).

4. Generalized Approach Using Fractional Sobolev Spaces

The construction of generalized versions of stochastic partial differential equations through the geometry of fractional Sobolev spaces leads to the definition of generalized solutions on \( U_\alpha, \alpha \in \mathbb{Q} \), from fractionally integrated white noise. In this section, we consider the generalized version of equations (14), (15) and (16) on \( U_\alpha \), and construct their solutions in terms of the geometry of this space.

4.1. Elliptic Case

We consider the stochastic fractional pseudodifferential equation
\[
[L]^* \tilde{X}_\alpha^\varepsilon (z) = \mathcal{I}_{-\alpha} [\tilde{\varepsilon}^\varepsilon] (z).
\] (39)
We formulate its generalized version in $U_\alpha$, $\alpha \in \mathbb{Q}$, as well as its dual equation in $[U_\alpha]^*$, jointly with the corresponding formulation in $L^2(T)$.

The generalized version in $U_\alpha$ of equation (39) is obtained by taking the inner product in this space by a test function $\phi \in U_\alpha$ in both sides of equation (39), that is,

$$
\left\langle [L']^* \tilde{X}_\alpha^e (\cdot) , \phi (\cdot) \right\rangle_{U_\alpha} = \left\langle I_{-\alpha} (\tilde{\varepsilon}^e) (\cdot) , \phi (\cdot) \right\rangle_{U_\alpha} .
$$

(40)

Its dual equation is then given by

$$
\left\langle I_{2\alpha} \left( \tilde{X}_\alpha^e \right) (\cdot) , L' [I_{2\alpha} (\phi)] (\cdot) \right\rangle_{[U_\alpha]^*} = \left\langle I_{\alpha} (\tilde{\varepsilon}^e) (\cdot) , I_{2\alpha} (\phi) (\cdot) \right\rangle_{[U_\alpha]^*} ,
$$

(41)

for all $\phi \in U_\alpha$, or, equivalently,

$$
\left\langle I_{2\alpha} \left( \tilde{X}_\alpha^e \right) (\cdot) , L' f (\cdot) \right\rangle_{[U_\alpha]} = \left\langle I_{\alpha} (\tilde{\varepsilon}^e) (\cdot) , f (\cdot) \right\rangle_{[U_\alpha]^*} ,
$$

(42)

for all $f \in [U_\alpha]^*$. The $L^2(T)$-version of the above equation is

$$
\left\langle \tilde{X}_\alpha^e (\cdot) , L' f (\cdot) \right\rangle_{L^2(T)} = \left\langle \tilde{\varepsilon}^e (\cdot) , I_{-\alpha} (f) (\cdot) \right\rangle_{L^2(T)} ,
$$

(43)

which can also be written as

$$
\tilde{X}_\alpha^e (L' f) = \tilde{\varepsilon}^e (I_{-\alpha} f) , \quad \forall f \in [U_\alpha]^* .
$$

(44)

The following proposition provides the definition of a generalized solution to the above equation.

**Proposition 4.1.** The generalized solution of equation (44) in $U_\alpha$ is defined as

$$
\tilde{X}_\alpha^e (g) = \left( G_\alpha^e (g , \cdot) , I_{-\alpha} (\tilde{\varepsilon}^e) (\cdot) \right)_{U_\alpha} , \quad \forall g \in [U_\alpha]^* ,
$$

(45)

where, for all $g \in [U_\alpha]^*$,

$$
G_\alpha^e (g , z) = \left\langle g (\cdot) , I_{2\alpha} \left( G_\alpha^e \left( \cdot , z \right) \right) \right\rangle_{[U_\alpha]^*} = \left\langle g^* (\cdot) , G_\alpha^e (\cdot , z) \right\rangle_{U_\alpha} ,
$$

with $g^* = I_{-2\alpha} (g)$ being the dual element of $g$ with respect to the $U_\alpha$-topology, and $G_\alpha^e$ being the Green function corresponding to the operator $[L']^*$.

**Proof.** From equation (45), for all $g \in [U_\alpha]^*$,

$$
\tilde{X}_\alpha^e (L' g) = \left\langle G_\alpha^e (L' g , \cdot) , I_{-\alpha} (\tilde{\varepsilon}^e) (\cdot) \right\rangle_{U_\alpha}
= \left\langle \left\langle g^* (\cdot) , [L']^* G_\alpha^e (\cdot , z) \right\rangle_{U_\alpha} , I_{-\alpha} (\tilde{\varepsilon}^e) (z) \right\rangle_{U_\alpha}
= \left\langle g^* (z) , I_{-\alpha} (\tilde{\varepsilon}^e) (z) \right\rangle_{U_\alpha} = \left\langle \tilde{\varepsilon}^e (\cdot) , I_{-\alpha} (g) (\cdot) \right\rangle_{L^2(T)} = \tilde{\varepsilon}^e (I_{-\alpha} (g)) .
$$

(46)

Thus, the fractional generalised random field defined by equation (45) is the generalised solution in the space $U_\alpha$ of equation (44).
4.2. Parabolic Case

We next derive the generalized version of equation (16) in $U_\alpha$, and its corresponding generalized solution. The construction of this generalized solution is based on the Green operator $G_\alpha^p$, which satisfies the generalized version of equation (22) in $U_\alpha$, and on the fractionally spatially integrated white noise $	ilde{\varepsilon}_\alpha(t, z) = \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^p)(t, z)$, for each fixed $t \in \mathbb{R}_+$.

The weak-sense identity in $U_\alpha$ corresponding to equation (22) is derived by taking the inner product in $U_\alpha$ by a test function $\phi \in U_\alpha$ in both sides of equation (22). Thus, the following generalized equation is obtained:

$$G_\alpha^p(t, \phi; s, z) = U_\alpha\phi(z) + \int_s^t G_\alpha^p(u, L'\phi^*; s, z) \, du, \quad \forall \phi \in U_\alpha, \quad (47)$$

where $G_\alpha^p(t, \phi; s, z) = \langle \phi(\cdot), G_\alpha^p(t, \cdot; s, z) \rangle_{U_\alpha}$, for all $\phi \in U_\alpha$.

In addition, the generalized version of equation (16) in $U_\alpha$ can be derived by taking again the inner product in $U_\alpha$ by $\phi \in U_\alpha$ in both sides of the equation

$$\tilde{X}_\alpha^p(t, z) = \int_0^t [L]^* \tilde{X}_\alpha^p(u, z) \, du + \int_0^t \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^p)(u, z) \, du, \quad (48)$$

for all $t \in \mathbb{R}_+$ and $z \in T$. Thus, for each $\phi \in U_\alpha$,

$$\langle \tilde{X}_\alpha^p(t, \cdot), \phi(\cdot) \rangle_{U_\alpha} = \int_0^t \langle [L]^* \tilde{X}_\alpha^p(u, \cdot), \phi(\cdot) \rangle_{U_\alpha} \, du$$

$$+ \int_0^t \langle \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^p)(u, \cdot), \phi(\cdot) \rangle_{U_\alpha} \, du. \quad (49)$$

The corresponding dual equation in $[U_\alpha]^*$ is then given by

$$\langle \mathcal{I}_{2\alpha}(\tilde{X}_\alpha^p)(t, \cdot), f(\cdot) \rangle_{[U_\alpha]^*} = \int_0^t \langle \mathcal{I}_{2\alpha}(\tilde{X}_\alpha^p)(u, \cdot), L'f(\cdot) \rangle_{[U_\alpha]^*} \, du$$

$$+ \int_0^t \langle \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^p)(u, \cdot), f(\cdot) \rangle_{[U_\alpha]^*} \, du, \quad \forall f \in [U_\alpha]^*, \quad (50)$$

and its expression in the $L^2(T)$-topology is formulated as

$$\langle \tilde{X}_\alpha^p(t, \cdot), f(\cdot) \rangle_{L^2(T)} = \int_0^t \langle \tilde{X}_\alpha^p(u, \cdot), L'f(\cdot) \rangle_{L^2(T)} \, du$$

$$+ \int_0^t \langle \tilde{\varepsilon}^p(u, \cdot), \mathcal{I}_{-\alpha}(f)(\cdot) \rangle_{L^2(T)} \, du, \quad \forall f \in [U_\alpha]^* \quad (51)$$
Equation (51) can also be written as
\[ \tilde{X}_p^\alpha (t, f) = \int_0^t \tilde{X}_p^\beta [u, L'f] \, du + \int_0^t \tilde{\varepsilon}_p^\alpha [u, I_{-\alpha} (f)] \, du, \quad \forall f \in [U_\alpha]^*. \] (52)

In the proposition below, the generalized solution to equation (52) is defined from the Green operator \( G_p^\alpha \) and the fractionally integrated white noise \( \tilde{\varepsilon}_p^\alpha (t, z) = I_{-\alpha} (\tilde{\varepsilon}_p^\beta) (t, z) \).

**Proposition 4.2.** Let \( G_p^\alpha \) be the Green operator associated with the parabolic fractional pseudodifferential model (16), defined in equation (47). The generalized solution of equation (52) is given by
\[ \tilde{X}_p^\alpha (t, f) = \int_0^t \langle G_p^\alpha (t, f; s, \cdot), I_{-\alpha} (\tilde{\varepsilon}_p^\beta) (s, \cdot) \rangle_U \, ds, \quad \forall f \in [U_\alpha]^*, \] (53)

where \( I_{-\alpha} (\tilde{\varepsilon}_p^\beta) \) is the fractionally integrated white noise appearing in equation (16).

**Proof.** From equations equation (47), (52) and (53),
\[ \tilde{X}_p^\alpha (t, f) - \int_0^t \tilde{X}_p^\alpha [v, L'f] \, dv - \int_0^t \tilde{\varepsilon}_p^\alpha [s, I_{-\alpha} (f)] \, ds = \int_0^t G_p^\alpha (t, I_{-2\alpha} (f) ; s, \cdot) \]
\[ - \int_0^t G_p^\alpha [v, L'f ; s, \cdot] \, dv - \langle I_{-2\alpha} (f) (\cdot), I_{-\alpha} (\tilde{\varepsilon}_p^\beta) (s, \cdot) \rangle_U \, ds = 0, \]
\[ \forall f \in [U_\alpha]^*, \] (54)

where \( G_p^\alpha \) is defined as
\[ G_p^\alpha (t, \phi; s, z) = \langle \phi (\cdot), G_p^\alpha (t, \cdot; s, z) \rangle_U, \quad \forall \phi \in U_\alpha, \]
or, equivalently,
\[ G_p^\alpha (t, f; s, z) = \langle f (\cdot), I_{2\alpha} (G_p^\alpha (t, \cdot; s, z)) \rangle_{[U_\alpha]^*}, \quad \forall f \in [U_\alpha]^*. \]

\[ \square \]

**4.3. Hyperbolic Case**

We now consider the weak-sense version of equation (31) in \( U_\alpha \), and its solution defined in terms of the Green operator \( G_h^\alpha \) with kernel \( G_h^\alpha \) in this space, that is,
\[ G_h^\alpha \left[ \left[ \frac{\partial^2}{\partial t^2} - [L]' \right]^* \right] (g ; s, z) = g^* (s, z), \quad \forall g \in I_\alpha (L^2 [-1, 1]) \times [U_\alpha]^*, \] (55)
where

\[ G_\alpha^h (g; s, z) = \langle g^* (\cdot, \cdot), G_\alpha^h (\cdot, \cdot; s, z) \rangle_{J_\alpha} = \langle g (\cdot, \cdot), \left[ G_\alpha^h \right]^* (\cdot, \cdot; s, z) \rangle_{J_\alpha}, \]  

(56)

where \( \langle \cdot, \cdot \rangle_{J_\alpha} \) and \( \langle \cdot, \cdot \rangle_{J_\alpha} \) denote respectively the inner products in the spaces \( \mathcal{I}_\alpha L^2 [-1, 1] \times [U_\alpha]^* \) and \( \mathcal{I}_\alpha (L^2 [-1, 1]) \times U_\alpha \), and \( g^* \) and \( \left[ G_\alpha^h \right]^* \) denote the dual elements corresponding to \( g \) and \( G_\alpha^h \) in the \( \mathcal{I}_\alpha (L^2 [-1, 1]) \times U_\alpha \)-geometry, respectively.

The generalized version of equation (16) in \( U_\alpha \) is obtained as

\[
\left\langle \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right] \lambda^h_\alpha \left( \cdot, \cdot \right), g^* \left( \cdot, \cdot \right) \right\rangle_{J_\alpha} = \left\langle \mathcal{I}_{2\alpha} \lambda^h_\alpha \left( \cdot, \cdot \right), \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right] g \left( \cdot, \cdot \right) \right\rangle_{J_\alpha}, \quad \forall g \in \mathcal{I}_\alpha (L^2 [-1, 1]) \times [U_\alpha]^*, \tag{57}
\]

where, as before, \( \mathcal{I}_{2\alpha} \lambda^h_\alpha \) denotes fractionally integrated white noise (see equation (16)). The dual equation of (57) is given in \([U_\alpha]^*\) by

\[
\left\langle \mathcal{I}_\alpha \left( \tilde{\varepsilon}^h \right) \left( \cdot, \cdot \right), \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right]^* g \left( \cdot, \cdot \right) \right\rangle_{J_\alpha} = \left\langle \mathcal{I}_\alpha \left( \tilde{\varepsilon}^h \right) \left( \cdot, \cdot \right), \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right]^* g \left( \cdot, \cdot \right) \right\rangle_{J_\alpha}, \quad \forall g \in \mathcal{I}_\alpha (L^2 [-1, 1]) \times [U_\alpha]^*, \tag{58}
\]

where, as before, \( \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right]^* \) denotes the adjoint operator of \( \frac{\partial^2}{\partial t^2} - [L]^* \), and the corresponding equation in the \( L^2 \)-topology is defined as

\[
\left\langle \tilde{\lambda}^h_\alpha \left( \cdot, \cdot \right), \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right]^* g \left( \cdot, \cdot \right) \right\rangle_{L^2 [-1, 1] \times L^2 (T)} = \left\langle \tilde{\varepsilon}^h \left( \cdot, \cdot \right), \mathcal{I}_\alpha (g) \left( \cdot, \cdot \right) \right\rangle_{L^2 [-1, 1] \times L^2 (T)}, \quad \forall g \in \mathcal{I}_\alpha (L^2 [-1, 1]) \times [U_\alpha]^*. \tag{59}
\]

The last equation can also be expressed as

\[
\tilde{\lambda}^h_\alpha \left[ \left[ \frac{\partial^2}{\partial t^2} - [L]^* \right]^* (g) \right] = \tilde{\varepsilon}^h \left[ \mathcal{I}_\alpha (g) \right], \quad \forall g \in \mathcal{I}_\alpha (L^2 [-1, 1]) \times [U_\alpha]^*. \tag{60}
\]

The generalized solution to equation (60) is defined in the next proposition.
Proposition 4.3. Consider the Green operator $G^h_\alpha$ defined by equations (55) and (56). Then, the fractional generalized random field $\tilde{X}^h_\alpha$ defined by
\[
\tilde{X}^h_\alpha(g) = \left\langle G^h_\alpha(g; \cdot, \cdot), \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^h)(\cdot, \cdot) \right\rangle_{J_{-\alpha}},
\]
for all $g \in \mathcal{I}_\alpha(L^2[-1,1]) \times [U_\alpha]^*$, satisfies equation (60).

Proof. Considering the definition of $\tilde{X}^h_\alpha$ given in equations (60) and (61), we obtain, from equations (55) and (56), for all $g \in \mathcal{I}_\alpha(L^2[-1,1]) \times [U_\alpha]^*$,
\[
\tilde{X}^h_\alpha\left[\left[\frac{\partial^2}{\partial t^2} - [L']^*\right]^*g \right] - \tilde{\varepsilon}^h[\mathcal{I}_{-\alpha}(g)]
= \left\langle G^h_\alpha\left[\left[\frac{\partial^2}{\partial t^2} - [L']^*\right]^*g; \cdot, \cdot\right], \mathcal{I}_{-\alpha}(\tilde{\varepsilon}^h)(\cdot, \cdot) \right\rangle_{J_{-\alpha}} = 0. \tag{62}
\]

5. Spectral and Weak-Sense Relationships between the Green Function and Covariance Function

The spectral properties of the (not necessarily stationary) solutions to equations (14), (15) and (16) are now studied from the results derived in the previous sections. In particular, the spectral relationship between the Green function and the covariance function is obtained.

According to the approach followed in the derivation of the generalized versions of equations (14), (15) and (16), we define the generalized ordinary case in terms of the geometry of fractional Sobolev spaces. That is, the generalized ordinary field $\tilde{X}^e_\alpha$ corresponding to the ordinary random field $\tilde{X}_\alpha$ is defined in the elliptic, parabolic and hyperbolic cases, respectively, as follows:

• Elliptic case
\[
\tilde{X}^e_\alpha(f) = \left\langle \tilde{X}^e_\alpha(\cdot), f^*(\cdot) \right\rangle_{U_\alpha}
= \left\langle \mathcal{I}_{2\alpha}(\tilde{X}^e_\alpha)(\cdot), f(\cdot) \right\rangle_{[U_\alpha]^*} = \left\langle \tilde{X}^e_\alpha(\cdot), f(\cdot) \right\rangle_{L^2(T)}, \tag{63}
\]
for all $f \in [U_\alpha]^*$. 

\begin{itemize}
  \item **Parabolic case**
  \[
  \tilde{X}_\alpha^p (t,f) = \left< \tilde{X}_\alpha^p (t,\cdot), f^* (\cdot) \right>_{U_\alpha}
  = \left< \mathcal{I}_{2\alpha} \left( \tilde{X}_\alpha^p \right) (t,\cdot), f (\cdot) \right>_U = \left< \tilde{X}_\alpha^p (t,\cdot), f (\cdot) \right>_{L^2(T)},
  \]
  for all $f \in [U_\alpha]^*$. 

  \item **Hyperbolic case**
  \[
  \tilde{X}_\alpha^h (g) = \left< \tilde{X}_\alpha^h (\cdot), g^* (\cdot) \right>_{J_{-\alpha}}
  = \left< \mathcal{I}_{2\alpha} \left( \tilde{X}_\alpha^h \right) (\cdot), g (\cdot) \right>_{J_\alpha} = \left< \tilde{X}_\alpha^h (\cdot), g (\cdot) \right>_{L^2([-1,1])},
  \]
  for all $g \in \mathcal{I}_{\alpha} (L^2([-1,1]))$. 

  The $\alpha$-GRF $\tilde{X}_\alpha$ satisfies (see also Section 2)
  \[
  \tilde{X}_\alpha (L'f) = \left< \tilde{X}_\alpha (\cdot), L'f (\cdot) \right>_{L^2(T)} = \left< \tilde{X}_\alpha (\cdot), \mathcal{I}_{-\alpha} (f) (\cdot) \right>_{L^2(T)} = \tilde{\varepsilon} (\mathcal{I}_{-\alpha} (f)),
  \]
  for all $f \in [U_\alpha]^*$, which can be equivalently expressed as
  \[
  \tilde{X}_\alpha (f) = \tilde{\varepsilon} \left[ \mathcal{I}_{-\alpha} \left( (L')^{-1} f \right) \right], \quad \forall f \in [U_\alpha]^*.
  \]

  The generalized covariance function $\tilde{B}_\alpha$ of $\tilde{X}_\alpha$ is then given by
  \[
  \tilde{B}_\alpha (f,g) = E \left[ \tilde{X}_\alpha (f) \tilde{X}_\alpha (g) \right]
  = \left< \mathcal{I}_{-\alpha} \left( L' \right)^{-1} (f) (\cdot), \mathcal{I}_{-\alpha} \left( L' \right)^{-1} (g) (\cdot) \right>_{L^2(T)}
  = \left< g^* (\cdot), \left[ \left( L' \right)^{-1} \right]^* \mathcal{I}_{-\alpha} \left[ \left( L' \right)^{-1} \right] \left( f \right) \right>_U = \left< g^* (\cdot), \tilde{R}_\alpha (f) \right>_{U_\alpha},
  \]
  for all $f,g \in [U_\alpha]^*$; thus, the covariance operator $\tilde{R}_\alpha$ of $\tilde{X}_\alpha$ is given by
  \[
  \tilde{R}_\alpha = \left[ \mathcal{I}_{-\alpha} \left( L' \right)^{-1} \right]^* \left[ \mathcal{I}_{-\alpha} \left( L' \right)^{-1} \right].
  \]
  From equations (12) and (68), the covariance function $\tilde{B}_\alpha$ of the ordinary random field $\tilde{X}_\alpha$ corresponding to $\tilde{X}_\alpha$ is defined in the weak sense in $U_\alpha$ as
  \[
  \tilde{B}_\alpha (z,y) = E \left[ \tilde{X}_\alpha (z) \tilde{X}_\alpha (y) \right].
  \]
\end{itemize}
\[ = \int_{T} \langle l (z, \cdot), i_{-\alpha} (\cdot, x) \rangle_{L^2(T)} \langle i_{-\alpha} (x, \cdot), l (\cdot, y) \rangle_{L^2(T)} dx, \quad (70) \]

where \( l (\cdot, \cdot) \) is, as before, the kernel of the inverse operator \([L^{-1}]^* = [(L')^*]^{-1}\) of \([L']^*\) or, equivalently, the Green function associated with \([L']^*\), and \( i_{-\alpha} \) denotes the kernel of the trace on \( T \) of the Bessel potential of order \( \alpha \).

The relationship shown in equation (70) holds in the stochastic fractional models of elliptic, parabolic and hyperbolic types defined in equations (14), (15) and (16). We next calculate the covariance function of the fractional generalized solution obtained in Propositions 4.1, 4.2 and 4.3 to derive such a relationship.

5.1. Elliptic Case

Let \( G_{\alpha}^e \) be the Green operator associated with the generalized solution of equation (14), defined in Proposition 4.1. Then, the solution of equation (44) is expressed in terms of \( G_{\alpha}^e \) as

\[ \tilde{X}_{\alpha}^e (f) = \langle G_{\alpha}^e (f, \cdot), I_{\alpha} (\tilde{\varepsilon}^e) (\cdot) \rangle_{U_{\alpha}}, \quad \forall f \in U_{\alpha}. \]

In the following result, using Proposition 4.1 and equation (63), the covariance function of the ordinary field \( \tilde{X}_{\alpha}^e \) corresponding to the fractional generalized random field \( \tilde{X}_{\alpha}^e \) is obtained in terms of the Green function \( G_{\alpha}^e \) associated with \([L']^*\).

**Proposition 5.1.** The covariance function \( \tilde{B}_{\alpha}^e \) of the ordinary random field \( \tilde{X}_{\alpha}^e \), which is the weak-sense solution in \( U_{\alpha} \) to equation (14), is given by

\[ \tilde{B}_{\alpha}^e (z, y) = \langle G_{\alpha}^e (z, \cdot), G_{\alpha}^e (y, \cdot) \rangle_{U_{\alpha}}, \quad (71) \]

where \( G_{\alpha}^e \) is defined as in Proposition 4.1.

**Proof.** The covariance functional \( \tilde{B}_{\alpha}^e \) of the fractional generalised random field \( \tilde{X}_{\alpha}^e \) defined in equation (45) is given by the following identities:

\[
\begin{align*}
\tilde{B}_{\alpha}^e (f, g) & = E \left[ \tilde{X}_{\alpha}^e (f) \tilde{X}_{\alpha}^e (g) \right] \\
& = E \left[ \int_{T} \mathcal{I}_{\alpha} (G_{\alpha}^e (f, z) \tilde{\varepsilon}^e (z)) dz \int_{T} \mathcal{I}_{\alpha} (G_{\alpha}^e (g, y) \tilde{\varepsilon}^e (y)) dy \right] \\
& = \langle G_{\alpha}^e (f, \cdot), G_{\alpha}^e (g, \cdot) \rangle_{U_{\alpha}} \\
& = \langle f^* (z), (G_{\alpha}^e (z, \cdot), G_{\alpha}^e (y, \cdot))_{U_{\alpha}}, g^* (y) \rangle_{U_{\alpha}}.
\end{align*}
\]
\[
\left\langle f^*(z), \tilde{B}_\alpha^e (z,y) \right\rangle_{U_\alpha}, \quad \forall f, g \in [U_\alpha]^*.
\]

(72)

Thus, \( \tilde{B}_\alpha^e \) is defined in the weak sense in \( U_\alpha \) as in equation (71).

In the case where \( [[L]^*]^{-1} \) is a compact self-adjoint operator in \( U_\alpha \), from Proposition 3.6 and equation (71), \( \tilde{B}_\alpha^e \) admits the following series representation:

\[
\tilde{B}_\alpha^e (z,y) = \sum_{n=0}^{\infty} \frac{\phi_n (z) \phi_n (y)}{\lambda_n^2},
\]

(73)

where \( \{ \phi \}_n \in \mathbb{N} \) is the system of eigenfunctions associated with the sequence of eigenvalues \( \{ \lambda_n \}_n \in \mathbb{N} \) of the point spectrum of \( [L]^* \).

In the unbounded domain case, from equations (36) and (71), \( \tilde{B}_\alpha^e \) admits the following spectral representation:

\[
\tilde{B}_\alpha^e (z,y) = \int \frac{1}{\lambda^2} dP_\lambda (z,y),
\]

(74)

where, as before, \( \{ P_\lambda : \lambda \in \Lambda \} \) denotes the family of self-adjoint projection operators associated with \( [L]^* \), and \( \Lambda \) the continuous spectrum of \( [L]^* \).

5.2. Parabolic Case

Let \( G_\alpha^p \) be the Green operator satisfying equation (47) and defining the generalized solution to equation (15) derived in Proposition 4.2. The following proposition establishes the weak-sense relationship between the Green function \( G_\alpha^p \), defined by equation (22), and the covariance function \( \tilde{B}_\alpha^p \) of \( \tilde{X}_\alpha^p \).

**Proposition 5.2.** Let \( \tilde{X}_\alpha^p \) be a weak-sense solution in \( U_\alpha \) to equation (15). Then, the covariance function \( \tilde{B}_\alpha^p \) of \( \tilde{X}_\alpha^p \) satisfies the following weak-sense identity in \( U_\alpha \):

\[
\tilde{B}_\alpha^p (t,z,y) = \int_0^t \langle G_\alpha^p (t,z,s,\cdot), G_\alpha^p (t,y,s,\cdot) \rangle_{U_\alpha} ds,
\]

(75)

where \( G_\alpha^p \) is defined as in Proposition 4.2.

**Proof.** From equation (53), the generalised covariance function \( \tilde{B}_\alpha^p \) of \( \tilde{X}_\alpha^p \) is given by, for all \( f, g \in [U_\alpha]^* \),

\[
\tilde{B}_\alpha^p (t,f,g) = \int_0^t \langle G_\alpha^p (t,f,s,\cdot), G_\alpha^p (t,g,s,\cdot) \rangle_{U_\alpha} ds
\]
\[
\left\langle f^* (z), \left[ \int_0^t \langle G_p^\alpha (t, z; s, \cdot) , G_p^\alpha (t, y; s, \cdot) \rangle_{U_\alpha} ds \right]_{U_\alpha}, g^* (y) \right\rangle_{U_\alpha} = \left\langle f^* (z), \tilde{B}_p^\alpha (t, z; y) \right\rangle_{U_\alpha}, g^* (y) \right\rangle_{U_\alpha}, f, g \in [U_\alpha]^* \quad (76)
\]

From Proposition 3.7 in the compact case, \( \tilde{B}_p^\alpha \) admits the following spectral representation:

\[
\tilde{B}_p^\alpha (z, y) = \sum_{n=0}^{\infty} \phi_n (z) \phi_n (y) \int_0^t \gamma_n^2 (t, s) ds, \quad (77)
\]

where \( \{ \phi_n \}_{n \in \mathbb{N}} \) is the system of eigenfunctions associated with the sequence of eigenvalues \( \{ \lambda_n \}_{n \in \mathbb{N}} \) of the point spectrum of \([L]^*\). In the non-compact case, \( \tilde{B}_p^\alpha \) is represented in the spatial spectral domain as

\[
\tilde{B}_p^\alpha (t; z, y) = \int_{\Lambda} \left[ \int_0^t \gamma_\lambda^2 (t, s) ds \right] dP_\lambda (z, y), \quad (78)
\]

where \( \{ P_\lambda \}_{\lambda \in \Lambda} \) denotes the spectral family of projection operators corresponding to \([L]^*\), and \( \gamma_\lambda (t, s), \lambda \in \Lambda \), is defined as in equation (37).

### 5.3. Hyperbolic Case

The following proposition provides the relationship between the covariance function and the Green function in the hyperbolic case.

**Proposition 5.3.** Let \( \tilde{X}_p^h \) be the ordinary random field corresponding to the generalized random field \( \tilde{X}_h^\alpha \) given by equation (60). Then, the covariance function \( \tilde{B}_p^h \) of \( \tilde{X}_h^\alpha \) satisfies the following weak-sense identity in \( U_\alpha \):

\[
\tilde{B}_p^h (t; z, u, y) = \left\langle G_p^h (t, z; \cdot, \cdot), G_p^h (u, y; \cdot, \cdot) \right\rangle_{J^-}, \quad (79)
\]

where \( G_p^h \) is the Green function defined in equation (31).

**Proof.** From equation (61), we can calculate the generalised covariance function \( \tilde{B}_p^h \) of the generalised solution \( \tilde{X}_h^\alpha \) in equation (16), that is, for all \( f, g \in \mathcal{I}_\alpha (L^2 [-1, 1]) \times [U_\alpha]^* \),

\[
\tilde{B}_p^h (f, g) = E \left[ \tilde{X}_h^\alpha (f) \tilde{X}_h^\alpha (g) \right] = \left\langle G_p^h (f; \cdot, \cdot), G_p^h (g; \cdot, \cdot) \right\rangle_{J^-}
\]
\begin{align}
&= \left\langle f^* (t, z), \left\langle G_\alpha^h (t, z; u, y) \right\rangle_{J_{-\alpha}} \right\rangle_{J_{-\alpha}}, g^* (u, y) \right\rangle_{J_{-\alpha}} \\
&= \left\langle f^* (t, z), \tilde{B}_\alpha^h (t; z; u, y) \right\rangle_{J_{-\alpha}}, g^* (u, y) \right\rangle_{J_{-\alpha}}. \quad (80)
\end{align}

In the compact case, from Proposition 3.8 and equation (79), \( \tilde{B}_\alpha^h \) admits the following series representation:

\begin{align}
\tilde{B}_\alpha^h (t, z; u, y) &= \sum_{n=0}^{\infty} \phi_n (z) \phi_n (y) \left\langle c_n (t, \cdot), c_n (u, \cdot) \right\rangle_{L^2([-1,1])}, \quad (81)
\end{align}

where \( \{\phi_n\}_{n \in \mathbb{N}} \) is the system of eigenfunctions associated with the point spectrum \( \{\lambda_n\}_{n \in \mathbb{N}} \) of \( [L']^* \), and the sequence \( \{c_n\}_{n \in \mathbb{N}} \) is defined as in Proposition 3.8.

Furthermore, in the non-compact case, from Proposition 5.3 and equation (38), \( \tilde{B}_\alpha^h \) admits the following spatiotemporal spectral representation:

\begin{align}
\tilde{B}_\alpha^h (t, z; s, y) &= \int_{\tilde{\Lambda}} \frac{1}{\lambda^2} d\tilde{P}_\lambda (t, z; s, y), \quad (82)
\end{align}

with \( \{\tilde{P}_\lambda\}_{\lambda \in \tilde{\Lambda}} \) being the family of the spectral projection operators corresponding to the operator \( \frac{\partial^2}{\partial t^2} - [L']^* \), and \( \tilde{\Lambda} \) denotes its continuous spectrum.

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\section*{References}


