DISTRIBUTION OF SPHEROIDAL FOCAL SINGULARITIES IN STOKES FLOW

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Abstract: Stokes flow for the steady, non-axisymmetric motion of viscous, incompressible fluids in small Reynolds numbers (creeping flow), around small particles embedded within simply connected and bounded flow domains, is described by a pair of partial differential equations, which evolve the vector biharmonic velocity and the scalar harmonic total pressure fields. There exist many representations of the solutions of those kinds of flow, in three-dimensional domains, appearing in the form of differential operators acting on harmonic and biharmonic potentials. On the other hand, the development of Stokes theory for two-dimensional flows has the advantage that uses only one potential function (stream function) for the representation of the flow fields, but refers to axisymmetric flows. The effect of a distribution of sources – singularities, on the surface of a spheroidal particle or marginally on the focal segment, to the basic flow fields, is the goal of the present work. In particular, the proper confrontation of the problem is ensured by the introduction of the well-known Havelock’s Theorem for the presence of singularities, which provides us with the necessary integral representations of the velocity and the pressure. Moreover, the interrelation of the eigenforms of the Papkovich-Neuber differential representation with those that arise from Stokes theory, in two-dimensional
spheroidal geometry, completes the two manners of facing the problem.

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1. Introduction

Stokes equations (Fox et al [3] and Happel et al [4]) consist the basis of the solution of many Newtonian flow problems. Many applications concern flow systems, where, beyond the first spherical approach, the particles are considered to be prolate or oblate spheroids (Moon et al [8]). As a result of the small dimensions of the particles, the consideration of the flow as Stokes flow is sufficiently justified.

As we mentioned previously, our motive is the study of the effect that a uniform distribution of singularities on the surface of spheroidal particles, has to the velocity and to the total pressure fields. More specifically, we are reduced to the existence of singularities on the surface of a prolate spheroidal particle within the main flow and we examine the limited case of the distribution of those anomalies on the focal segment. The corresponding results for the oblate spheroidal geometry can be obtained through a simple transformation (Moon et al [8]). Thus, we consider a prolate spheroidal particle into an external dynamic field, where on its main axis and specific between the two foci are distributed the singularities. In this case, the surface where the singularities lie, is the smallest that can be, since we talk for the focal segment, a situation that leads to the consideration of “infinite” source intensity. Of course, this is an immediate consequence of the fact that our purpose is to study the transition of the intension of the singularities, while we are reduced from a two-dimensional space (surface) to an one-dimensional space (focal segment), keeping the same time the same flow fields around the particle. Our goal is, finally, the construction of an integral representation of the velocity and of the total pressure fields in terms of certain singularities on the focal segment of the prolate spheroidal particle.

The use of spheroidal geometry imposes axial symmetry, meaning that the flow fields are independent of the azimuthal component. Such problems can be confronted with the aim of the theory of generalized eigenfunctions (Dassios et al [1]), according to which, the stream function (Happel et al [4] and Xu et al [10]) is expressed through a full expansion of semiseparable eigenfunctions (Dassios et al [1]). This method provides us, also, with the physical meaning
of every term of the fields. On the contrary, the Papkovich-Neuber differential representation (Neuber [9] and Xu et al [10]) offers a much more practical way of calculating the velocity and the total pressure in terms of harmonic eigenfunctions (Hobson [5]). The interrelation of the flow fields, which ensue from the differential representation \((3-D)\), considering axial symmetry, with those that come up from the Stokes theory \((2-D)\), searching the equivalent potentials, has been extensively examined in Dassios et al [2]. Consequently, we firstly construct the flow fields generated by the appropriate prolate spheroidal harmonic eigenfunctions through the Papkovich-Neuber differential representation, in combination with the theory of Havelock (Kim et al [6] and Miloh [7]), which offers analytical formulae for the integral representation of the pre-mentioned eigenfunctions in terms of singularities on the focal segment of the spheroid. In the end, we face the inverse problem of determining those generalized eigenfunctions, which, through the Stokes theory generate the very same flow fields, making use of the proper interrelation formulae (Dassios et al [2]).

2. Flow Fields / Papkovich-Neuber and Stokes Potentials

Stokes equations for the non-axisymmetric \((3-D)\), incompressible \((\rho=\text{constant})\), viscous \((\mu=\text{constant})\), steady and creeping \((\text{Re} \ll 1)\) flow, within simply connected, bounded flow fields \((\Omega(\mathbb{R}^3))\) are given by the expressions

\[
\mu \Delta \mathbf{v}(\mathbf{r}) = \nabla P(\mathbf{r}) \equiv \nabla p(\mathbf{r}) - \rho \mathbf{g}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3),
\]

\[
\nabla \cdot \mathbf{v}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3),
\]

where \(\mathbf{r}\) is the position vector, \(\mathbf{v}\) is the velocity field, \(P\) is the total pressure field, \(p\) is the thermodynamic pressure and \(\mathbf{g}\) is the acceleration of gravity. Equation (1) states that, in creeping flow, the force caused by the pressure gradient on any material point of the fluid is compensated by the viscous force, while equation (2) secures the incompressibility of the fluid.

Given a fixed positive number \(c > 0\), which specifies the semifocal distance of our system, we define the prolate spheroidal coordinates \((\eta, \theta, \varphi)\), \(0 \leq \eta < +\infty, \, 0 \leq \theta \leq \pi, \, 0 \leq \varphi < 2\pi\), where introducing the simple transformation \(\tau = \cosh \eta, \, 1 \leq \tau < +\infty\) (non-degenerate spheroid for \(\tau > 1\)) and \(\zeta = \cos \theta, \, -1 \leq \zeta \leq 1\) we obtain the following relations

\[
x_1 = c\tau \zeta, \quad x_2 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \cos \varphi, \quad x_3 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \sin \varphi.
\]
The Papkovich-Neuber differential representation of the solutions of Stokes equations, is given in terms of the harmonic potentials \( \Phi \) and \( \Phi_0 \) (\( \Delta \Phi(r) = 0, \Delta \Phi_0(r) = 0, r \in \Omega(\mathbb{R}^3) \)), i.e.

\[
\mathbf{v}(r) = \Phi(r) - \frac{1}{2} \nabla \left( r \cdot \mathbf{\Phi}(r) + \Phi_0(r) \right), \quad r \in \Omega(\mathbb{R}^3),
\]

\[
P(r) = -\mu \nabla \cdot \mathbf{\Phi}(r), \quad r \in \Omega(\mathbb{R}^3),
\]

where the differential operators \( \nabla, \Delta \) in prolate spheroidal coordinates assume the form

\[
\nabla = \frac{1}{c\sqrt{\tau^2 - \zeta^2}} \left[ \sqrt{\tau^2 - 1}\frac{\partial}{\partial \tau} - \sqrt{1 - \zeta^2}\frac{\partial}{\partial \zeta} \right] 
+ \frac{1}{c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}} \hat{\varphi} \frac{\partial}{\partial \varphi},
\]

\[
\Delta = \frac{1}{c^2(\tau^2 - \zeta^2)} \left\{ \frac{\partial}{\partial \tau} \left[ (\tau^2 - 1)\frac{\partial}{\partial \tau} \right] + \frac{\partial}{\partial \zeta} \left[ (1 - \zeta^2)\frac{\partial}{\partial \zeta} \right] \right\}
+ \frac{1}{c^2(\tau^2 - 1)(1 - \zeta^2)} \frac{\partial^2}{\partial \varphi^2},
\]

whereas \( \hat{\tau}, \hat{\zeta}, \hat{\varphi} \) stand for the orthounit vectors of our system.

The Papkovich-Neuber potentials are written in terms of the associated Legendre functions of the first and of the second kind (Hobson [5]), for every \( r \in \Omega(\mathbb{R}^3) \)

\[
\Phi(r) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ P_n^m(\tau) \left( e_n^{(i)mc} \cos m\varphi + e_n^{(i)mo} \sin m\varphi \right) 
+ Q_n^m(\tau) \left( e_n^{(e)mc} \cos m\varphi + e_n^{(e)mo} \sin m\varphi \right) \right] P_n^m(\zeta),
\]

and

\[
\Phi_0(r) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ P_n^m(\tau) \left( d_n^{(i)mc} \cos m\varphi + d_n^{(i)mo} \sin m\varphi \right) 
+ Q_n^m(\tau) \left( d_n^{(e)mc} \cos m\varphi + d_n^{(e)mo} \sin m\varphi \right) \right] P_n^m(\zeta),
\]
where if we consider axial symmetry, i.e. \( \partial \mathbf{v} / \partial \varphi = 0 \) and \( \hat{\varphi} \cdot \mathbf{v} = 0 \), the relations (8) and (9) become for every \( \mathbf{r} \in \Omega(\mathbb{R}^2) \)

\[
\Phi(\tau, \zeta) = \sum_{n=0}^{\infty} \left[ \left( e_n^{(i)0e} \cdot \hat{x}_1 \right) P_n(\tau) + \left( e_n^{(e)0e} \cdot \hat{x}_1 \right) Q_n(\tau) \right] \hat{x}_1 P_n(\zeta), \quad \tau > 1, \ |\zeta| \leq 1
\]  

and

\[
\Phi_0(\tau, \zeta) = \sum_{n=0}^{\infty} \left[ d_n^{(i)0e} P_n(\tau) + d_n^{(e)0e} Q_n(\tau) \right] P_n(\zeta), \quad \tau > 1, \ |\zeta| \leq 1.
\]  

On the other hand, the development of Stokes theory for two-dimensional flows needs only one potential function (stream function, \( \psi(\mathbf{r}) \), \( \mathbf{r} \in \Omega(\mathbb{R}^2) \)) for the representation of the flow fields, but refers to axisymmetric domains. The Stokes stream function \( \psi \) satisfy

\[
E^4 \psi(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^2) \equiv \{ (\tau, \zeta) : \tau > 1, -1 \leq \zeta \leq 1 \},
\]

where in prolate spheroidal coordinates we have

\[
E^2 = \frac{1}{c^2(\tau^2 - \zeta^2)} \left[ (\tau^2 - 1) \frac{\partial^2}{\partial \tau^2} + (1 - \zeta^2) \frac{\partial^2}{\partial \zeta^2} \right].
\]

In this system that we examine, for \( \tau > 1, |\zeta| \leq 1 \) the velocity field is provided by the expression

\[
\mathbf{v}(\tau, \zeta) = v_\tau(\tau, \zeta) \hat{\tau} + v_\zeta(\tau, \zeta) \hat{\zeta},
\]

whereas the components are given through the following derivatives of the Stokes stream function \( \psi \), that is

\[
v_\tau(\tau, \zeta) = \frac{1}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{\tau^2 - 1}} \frac{\partial \psi(\tau, \zeta)}{\partial \zeta},
\]

and

\[
v_\zeta(\tau, \zeta) = \frac{1}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{1 - \zeta^2}} \frac{\partial \psi(\tau, \zeta)}{\partial \tau},
\]

while the total pressure field is a function of an arbitrarily chosen constant pressure \( P_0 \), meaning

\[
P(\tau, \zeta) = P_0 + \frac{\mu}{c} \left\{ \int_{\infty}^{\tau} \frac{1}{(\tau' - 1)} \frac{\partial}{\partial \tau'} \left( E^2 \psi(\tau', \zeta) \right) d\tau' \\
- \int_{0}^{\zeta} \frac{1}{(1 - \zeta'^2)} \frac{\partial}{\partial \tau} \left( E^2 \psi(\tau, \zeta') \right) d\zeta' \right\}
\]
and the vorticity field is easily confirmed to be expressed us
\[
\omega(\tau, \zeta) = \nabla \times \mathbf{v}(\tau, \zeta) = \hat{\omega} \frac{E^2 \psi(\tau, \zeta)}{c \sqrt{\tau^2 - 1} \sqrt{1 - \zeta^2} E^2 \psi(\tau, \zeta)}, \quad (18)
\]
a fact which declares that irrotational fields are described by a stream function \(\psi\), for which it is valid that \(E^2 \psi(\tau, \zeta) = 0\). Thus, every problem of axisymmetric flow is, initially, reduced to the finding of the stream function \(\psi(\mathbf{r})\), \(\mathbf{r} \in \Omega(\mathbb{R}^2)\).

The complete representation of the vector space of the solutions of the Stokes stream function that belongs to the kernel of the differential operator \(E^4\), has the following full expansion representation for \(\tau > 1\) and \(|\zeta| \leq 1\)
\[
\psi(\tau, \zeta) = \sum_{n=2}^{\infty} \left[ A_n^i \Theta_n^{(i)}(\tau, \zeta) + B_n^i \Omega_n^{(i)}(\tau, \zeta) \right] + \sum_{n=2}^{\infty} \left[ A_n^e \Theta_n^{(e)}(\tau, \zeta) + B_n^e \Omega_n^{(e)}(\tau, \zeta) \right],
\]
(19)
where the sum with the \(\Theta\)-functions represents an element of the space \(\text{ker } E^2\) (separable solutions) and the sum with the \(\Omega\)-functions represents an element of the space \(\text{ker } E^4\) (semiseparable solutions (Dassios et al [1])). The eigenfunctions \(\Theta_n\) and \(\Omega_n\) (see Appendix) are provided in terms of Gegenbauer functions of the first and of the second kind (Hobson [5]), using the theory of the generalized eigenfunctions, developed in Dassios et al [1].

The corresponding results and all the information for the oblate geometry can be obtained through the simple transformation
\[
\tau \rightarrow i\lambda \quad \text{and} \quad c \rightarrow -i\bar{c},
\]
(20)
where \(0 \leq \lambda \equiv \sinh \eta < +\infty\) and \(\bar{c} > 0\) are the new characteristic variables of the oblate spheroidal geometry and comprises the imaginary complex unit.

3. Havelock’s Theorem – The Singular Flow Fields

Our aim is to examine the effect of the presence of a certain distribution of singularities (on the surface of a prolate spheroidal particle) to the main flow fields. In particular, we are interested to the very important case where the singularities lie on the focal segment of the spheroid. There, the prolate spheroid has been degenerated to the one-dimensional focal distance and our purpose is to calculate the velocity and the total pressure fields as integral representations
of the singularities, in terms of the generalized eigenfunctions $\Theta_n$ and $\Omega_n$. In the presence of such kind of singularities on the focal segment between the two foci of the prolate spheroidal particle, Havelock (Kim et al [6] and Miloh [7]) has developed the next important formulae for every $\tau > 1$ and $|\zeta| \leq 1$,

$$Q_n(\tau)P_n(\zeta) = \frac{1}{2} \int_{-c}^{c} \frac{P_n(x'_1)}{\sqrt{(x_1 - x'_1)^2 + x_2^2 + x_3^2}} dx'_1, \quad (21)$$

where

$$|r - r'| = \sqrt{(x_1 - x'_1)^2 + x_2^2 + x_3^2}. \quad (22)$$

Consequently, if we use the Papkovich-Neuber differential representation, we import relations (10) and (11) to the velocity field (4) and to the total pressure field (5), we eliminate the interior terms by setting the corresponding constant coefficients to nil, i.e.

$$\left( e^{(i)0e} \cdot \hat{x}_1 \right) = d_n^{(i)0e} = 0, \quad n \geq 0, \quad (23)$$

since we solve the external flow problem and making use of the expressions (21) and (22), we conclude to the velocity field

$$v(r) = \int_{-c}^{c} \sum_{n=0}^{\infty} \left[ \left( e^{(e)0e} \cdot \hat{x}_1 \right) \left( x_1 r + |r - r'|^2 \hat{x}_1 - x_1 x'_1 \hat{x}_1 \right) + d_n^{(e)0e} (r - x'_1 \hat{x}_1) \right] \frac{P_n(x'_1)}{4|r - r'|^3} dx'_1, \quad (24)$$

for every $r \in \Omega(\mathbb{R}^2)$ and to the total pressure field

$$P(r) = \mu \int_{-c}^{c} \sum_{n=0}^{\infty} \left[ \left( e^{(e)0e} \cdot \hat{x}_1 \right) \left( x_1 - x'_1 \right) \right] \frac{P_n(x'_1)}{2|r - r'|^3} dx'_1, \quad r \in \Omega(\mathbb{R}^2). \quad (25)$$

Eventually, we have an integral representation of the flow fields containing the mentioned singularities on the focal segment of a prolate spheroidal particle, in terms of the external constant coefficients of the corresponding Papkovich-Neuber potentials.

Thus, by utilizing the known interrelation formulae between the Papkovich-Neuber and the Stokes representations, which connect the constant coefficients of the corresponding potentials $\Phi, \Phi_0$ and $\psi$ (Dassios et al [2]),
\[
\left( e_n^{(e)0e} \cdot \hat{x}_1 \right) = \frac{1}{c^2} \left[ \frac{1}{(2n+3)(n+1)(n+2)} B_{n+2}^e - \frac{1}{(2n-1)(n-1)n} B_n^e \right] \quad (26)
\]

and

\[
d_n^{(e)0e} = -\frac{1}{c} \left\{ \frac{2}{n(n+1)} A_{n+1}^e + \frac{1}{(2n-1)^2(2n+3)^2} \left[ \frac{(n+1)(2n-1)^2}{(2n+5)(n+2)} B_{n+3}^e \right. \\
- \left. \left( 4 + \frac{3}{n(n+1)} \right) B_{n+1}^e \right. - \frac{n(2n+3)^2}{(2n-3)(n-1)} B_{n-1}^e \right\}, \quad (27)
\]

for \( n \geq 2 \) and \( c > 0 \), while

\[
d_1^{(e)0e} = -\frac{1}{c} \left\{ A_2^e - c^2 \left( e_0^{(e)0e} \cdot \hat{x}_1 \right) + \frac{2}{25} \left( \frac{1}{21} B_1^e - \frac{2}{3} B_2^e \right) \right\}, \quad c > 0, \quad (28)
\]

\[
3d_0^{(e)0e} = -c \left( e_1^{(e)0e} \cdot \hat{x}_1 \right), \quad c > 0, \quad (29)
\]

\[
B_2^e = 6c^2 \left( e_0^{(e)0e} \cdot \hat{x}_1 \right), \quad B_3^e = 30c^2 \left( e_1^{(e)0e} \cdot \hat{x}_1 \right), \quad c > 0 \quad (30)
\]

and

\[
\left( e_0^{(e)0e} \cdot \hat{x}_1 \right), \quad \left( e_1^{(e)0e} \cdot \hat{x}_1 \right) \in \mathbb{R}, \quad c > 0, \quad (31)
\]

we arrive, from equations (24) and (25), at the required expressions of the flow fields (velocity and total pressure) in terms of the eigenfunctions \( \Theta_n \) and in terms of the generalized eigenfunctions \( \Omega_n \) for every \( n \geq 0 \).

References


Appendix

For convenience, we summarize here the internal and external eigenfunctions $\Theta_n$ and $\Omega_n$ for $n \geq 0$ and $\tau > 1$, $|\zeta| \leq 1$. More specifically we have the following formulae as a function of the Gegenbauer functions of the first $G_n$ and of the second $H_n$ kind (Hobson [5]), i.e.

$$\Theta^{(i)}_n(\tau, \zeta) = G_n(\tau)G_n(\zeta), \quad (A.1)$$

$$\Theta^{(e)}_n(\tau, \zeta) = H_n(\tau)G_n(\zeta), \quad (A.2)$$

for the eigenfunctions which belong to the kernel space of the differential operator $E^2$, that is

$$E^2\Theta^{(i)}_n(\tau, \zeta) = E^2\Theta^{(e)}_n(\tau, \zeta) = 0, \quad n = 0, 1, 2, ..., \tau > 1, \ |\zeta| \leq 1. \quad (A.3)$$

The generalized eigenfunctions $\Omega_n$ (Dassios et al [1]) represent the preimage, under the map $E^2$, of an arbitrary element of the space $\text{ker } E^2$. Thus, $\Omega_n$ solve the equations

$$c^2 E^2\Omega^{(i)}_n(\tau, \zeta) = \Theta^{(i)}_n(\tau, \zeta) \quad \text{and} \quad c^2 E^2\Omega^{(e)}_n(\tau, \zeta) = \Theta^{(e)}_n(\tau, \zeta), \quad (A.4)$$
for \( n = 0,1,2,\ldots \) and \( \tau > 1, \) \( |\zeta| \leq 1, \) where the constant \( c^2 \) enters equation (A.4) for notational convenience. The last relation effects a complete spectral decomposition of the solution space \( \text{ker } E^4. \) These eigenfunctions are expressed as (Dassios et al [1])

\[
\Omega_0^{(i)}(\tau, \zeta) = -G_0(\tau)G_2(\zeta) - G_2(\tau)G_0(\zeta), \quad (A.5)
\]

\[
\Omega_0^{(e)}(\tau, \zeta) = \frac{1}{3}G_3(\tau)G_0(\zeta) - G_1(\tau)G_2(\zeta), \quad (A.6)
\]

\[
\Omega_1^{(i)}(\tau, \zeta) = \frac{1}{3}G_1(\tau)G_3(\zeta) + \frac{1}{3}G_3(\tau)G_1(\zeta), \quad (A.7)
\]

\[
\Omega_1^{(e)}(\tau, \zeta) = -\frac{1}{3}G_0(\tau)G_3(\zeta) + G_2(\tau)G_1(\zeta), \quad (A.8)
\]

\[
\Omega_2^{(i)}(\tau, \zeta) = \frac{2}{25}G_2(\tau)G_4(\zeta) + \frac{3}{25}G_4(\tau)G_2(\zeta), \quad (A.9)
\]

\[
\Omega_2^{(e)}(\tau, \zeta) = \frac{2}{25}H_2(\tau)G_4(\zeta) + \frac{2}{25}H_4(\tau)G_2(\zeta) + \frac{1}{5}G_1(\tau)G_2(\zeta), \quad (A.10)
\]

\[
\Omega_3^{(i)}(\tau, \zeta) = \frac{2}{39}G_3(\tau)G_5(\zeta) + \frac{2}{39}G_5(\tau)G_3(\zeta), \quad (A.11)
\]

\[
\Omega_3^{(e)}(\tau, \zeta) = \frac{2}{39}H_3(\tau)G_5(\zeta) + \frac{2}{39}H_5(\tau)G_3(\zeta) - \frac{1}{39}G_0(\tau)G_3(\zeta) \quad (A.12)
\]

and for \( n = 4,5,6,\ldots \) they are given by

\[
\Omega_n^{(i)}(\tau, \zeta) = -\frac{\alpha_n}{2(2n-3)}\left[G_{n-2}(\tau)G_n(\zeta) + G_n(\tau)G_{n-2}(\zeta)\right] \\
+ \frac{\beta_n}{2(2n+1)}\left[G_{n+2}(\tau)G_n(\zeta) + G_n(\tau)G_{n+2}(\zeta)\right], \quad (A.13)
\]

\[
\Omega_n^{(e)}(\tau, \zeta) = -\frac{\alpha_n}{2(2n-3)}\left[H_{n-2}(\tau)G_n(\zeta) + H_n(\tau)G_{n-2}(\zeta)\right] \\
+ \frac{\beta_n}{2(2n+1)}\left[H_{n+2}(\tau)G_n(\zeta) + H_n(\tau)G_{n+2}(\zeta)\right], \quad (A.14)
\]

where \( \alpha_n \) and \( \beta_n \) are provided by the expressions

\[
\alpha_n = \frac{(n-3)(n-2)}{2(2n-3)(2n-1)} \quad \text{and} \quad \beta_n = \frac{(n+1)(n+2)}{(2n-1)(2n+1)}, \quad n \geq 4. \quad (A.15)
\]

The proper information for the associated Legendre functions of the first \( P_n \) and of the second \( Q_n \) kind, as well as for the Gegenbauer functions of the first \( G_n \) and of the second \( H_n \) kind can be found in Hobson [5].