THE SECOND REGULARIZED TRACE OF A SECOND ORDER DIFFERENTIAL OPERATOR WITH UNBOUNDED OPERATOR COEFFICIENT

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Abstract: In this work, a formula for the second regularized trace of second order differential operator with unbounded operator coefficient is found.

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1. Introduction

Let $H$ be a infinite dimensional separable Hilbert space. We denote the inner product in $H$ by $(.,.)$ and the norm in $H$ by $\|\|$. Let $H_1 = L^2(H;[0,\pi])$ denote the set of all functions $f$ from $[0,\pi]$ into $H$ which are strongly measurable and satisfy the condition $\int_0^\pi \|f(x)\|^2\,dx < \infty$. If the inner product of arbitrary two elements $f$ and $g$ of the space $H_1$ is defined as

$$(f,g)_{H_1} = \int_0^\pi (f(x),g(x))\,dx,$$

then $H_1$ becomes a infinite dimensional separable Hilbert space [13]. The norm in the space $H_1$ is denoted by $\|\|_{1}$. $\sigma_\infty(H)$ denotes the set of all compact

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operators from $H$ into $H$. If $A \in \sigma_\infty(H)$, then $A^*A$ is a nonnegative self-adjoint operator and $(A^*A)^{1/2} \in \sigma_\infty(H)$ [7]. Let the non-zero eigenvalues of the operator $A^*A$ be $s_1 \geq s_2 \geq \ldots \geq s_k \ (0 \leq k \leq \infty)$. Here, each eigenvalue is repeated according to its own multiplicity number. Since $(A^*A)^{1/2}$ is nonnegative, $s_1, s_2, \ldots, s_k$ are positive numbers. The numbers of $s_1, s_2, \ldots, s_k$ are called $s$-numbers of the operator $A$. If $k < \infty$, then $s_j = 0; j = k+1, k+2, \ldots$ will be accepted. $s$-numbers of the operator $A$ is also denoted by $s_j(A) \ (j = 1, 2, \ldots)$. Here, $s_1(A) = \|A\|$. If $A$ is a normal operator, then $s_j(A) = |\lambda_j(A)| \ (j = 1, 2, \ldots, k)$ [7]. Here, $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots \geq |\lambda_k(A)|$ are the non-zero eigenvalues of the operator $A$. $\sigma_p$ or $\sigma_p(H)$ is the set of all operators $A \in \sigma_\infty(H)$ the $s$-numbers of which satisfy the condition $\sum_{j=1}^{\infty} s_j^p(A) < \infty \ (p \geq 1)$. The set $\sigma_p$ $(p \geq 1)$ is a separable Banach space [7] with respect to the function $\|A\|_{\sigma_p(H)} = [\sum_{j=1}^{\infty} s_j^p(A)]^{1/p} \ (A \in \sigma_p(H))$.

$\sigma_1(H)$ is the set of all the operators $A \in \sigma_\infty(H)$ the $s$-numbers of which satisfy the condition $\sum_{j=1}^{\infty} s_j(A) < \infty$. An operator is called a kernel operator if it belongs to $\sigma_1(H)$. If the operator $A \in \sigma_p(H)$ and $T \in B(H)$ then $AT, TA \in \sigma_p(H)$ and

$$\|AT\|_{\sigma_p(H)} \leq \|T\|\|A\|_{\sigma_p(H)}, \quad \|TA\|_{\sigma_p(H)} \leq \|T\|\|A\|_{\sigma_p(H)}.$$  

If $A$ is a kernel operator and $\{e_j\}_{1}^{\infty} \subset H$ is any orthonormal basis, the series $\sum_{j=1}^{\infty} (Ae_j, e_j)$ is convergent and the sum of the series $\sum_{j=1}^{\infty} (Ae_j, e_j)$ does not depend on the choice of the basis $\{e_j\}_{1}^{\infty}$. The sum of the series $\sum_{j=1}^{\infty} (Ae_j, e_j)$ is said to be matrix trace and is denoted by $\text{tr}A$. It is known that (see [7]):

$$\text{tr} \ A = \sum_{j=1}^{\nu(A)} \lambda_j(A). \quad (1.1)$$

Here, each eigenvalue is added according to its own algebraic multiplicity number. $\nu(A)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator $A$.

Let us consider the differential expression in the space $H_1 = L_2(H; [0, \pi])$,

$$l_0(y) = -y''(x) + Ay(x).$$
Here, an operator $A$ from $D(A) \subset H$ into $H$ satisfies the conditions

$$A = A^* \geq I, \quad A^{-1} \in \sigma_\infty(H).$$

Let $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \ldots$ be the eigenvalues of the operator $A$ and $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, each eigenvalue is repeated according to its own multiplicity number.

Moreover, $D_0$ denotes the set of the functions $y(x) \in H_1$ satisfying the conditions:

1) $y(x)$ has continuous derivative of the second order with respect to the norm in the space $H$ in the interval $[0, \pi]$.
2) $Ay(x)$ continuous with respect to the norm in the space $H$.
3) $y'(0) = y'(\pi) = 0$.

Here, $D_0 = H_1$ and the operator $L_0' y = l_0(y)$ from $D_0$ into $H_1$ is symmetric. The eigenvalues of $L_0'$ are $k^2 + \gamma_j$ ($k = 0, 1, \ldots; j = 1, 2, \ldots$) and the orthonormal eigenvectors corresponding to these eigenvalues are $M_k \cos kx. \varphi_j (k = 0, 1, \ldots; j = 1, 2, \ldots)$. Here,

$$M_k = \begin{cases} 
\frac{1}{\sqrt{\pi}}, & \text{if } k = 0, \\
\sqrt{\frac{2}{\pi}}, & \text{if } k = 1, 2, \ldots.
\end{cases}$$

As seen, the orthonormal eigenvectors system of the symmetric operator $L_0'$ is an orthonormal basis in the space $H_1$. The operator $L_0 = \overline{L_0'} : D(L_0) \to H_1$ is self-adjoint.

Let $Q(x)$ be an operator function satisfying the following conditions:

1) $Q(x)$ has weak derivative of the fourth order and $Q^{(2k-1)}(0) = Q^{(2k-1)}(\pi) = 0$, $k = 1, 2$.
2) For every $x \in [0, \pi]$, $Q^{(i)}(x)$ ($i = 0, 1, 2, 3, 4$) are self-adjoint operators from $H$ into $H$.
3) For every $x \in [0, \pi]$, $AQ(x), AQ''(x), Q^{IV}(x) \in \sigma_1(H)$ and the functions $\|AQ(x)\|_{\sigma_1(H)}, \|AQ''(x)\|_{\sigma_1(H)}, \|Q^{IV}(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0, \pi]$.
4) For every $f \in H$, $\int_0^\pi (Q(x) f, f) dx = 0$.

Let us consider the self-adjoint operator $L = L_0 + Q$ from $D(L) = D(L_0)$ into $H_1$. The operators $L_0$ and $L$ have purely-discrete spectrum. Let the eigenvalues of the operators $L_0$ and $L$ $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$
\[ \lambda_n \leq \ldots \leq \lambda \text{ respectively.} \]

Let \( R_0^\lambda = (L_0 - \lambda I)^{-1} \) and \( R_\lambda = (L - \lambda I)^{-1} \) be the resolvents of the operators \( L_0 \) and \( L \) respectively.

If \( \gamma_j \sim a_j^\alpha \) as \( j \to \infty \) that is \( \lim_{j \to \infty} \frac{2\alpha}{a_j^\alpha} = 1 \) then as \( n \to \infty \) (see [11])

\[ \lambda_n, \mu_n \sim d n^{\frac{2\alpha}{2+\alpha}}. \tag{1.2} \]

Here \( d > 0 \) is constant. By using this relation, it is seen that the sequence \( \{\mu_n\}_{n=1}^\infty \) has a subsequence \( \{\mu_{n_m}\}_{m=1}^\infty \) such that

\[ \mu_k - \mu_{n_m} \geq d_1 (k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}}) \quad (k = n_m, n_m + 1, \ldots). \]

In [3], the formula in the form

\[ \lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} [\text{tr } Q(0) + \text{tr } Q(\pi)] \]

is found for the regularized trace of the operator \( L \). In this work, we obtain a formula in the following form:

\[ \lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res } \text{tr } \lambda^{j-1} (Q R_{\lambda}^0)^j)] \]

\[ = \frac{1}{2} \left[ \text{tr } A Q(0) + \text{tr } A Q(\pi) \right] - \frac{1}{8} \left[ \text{tr } Q''(0) + \text{tr } Q''(\pi) \right]. \tag{1.3} \]

Here \( \alpha > 2 \) is a constant and \( p = \left\lfloor \frac{5\alpha + 6}{\alpha - 2} \right\rfloor + 1 \). The limit

\[ \lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res } \text{tr } \lambda^{j-1} (Q R_{\lambda}^0)^j)] \]

is called the second regularized trace of operator \( L \).

The regularized trace formulas for scalar differential operators are studied in [8], [10], [12] and in many other works. The list of the works on the subjects is given in [9] and [14], but a small number of these works are on the regularized trace of differential operators with operator coefficient. In [6], the regularized trace of Sturm-Liouville operator with bounded operator coefficient is calculated. In [1], a formula for the regularized trace of the difference of two Sturm-Liouville operators which is given in half-axis with the bounded operator coefficient is found. In [15], a formula for the regularized trace of the Sturm-Liouville operator under Dirichlet boundary conditions with unbounded
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operator coefficient, is found. In [5], the regularized trace of a singular differential operator of second order with bounded operator coefficient is investigated. In [4] and [2], the formulas for the regularized traces of differential operators with bounded operator coefficient are found.

2. Some Relations about the Eigenvalues

If \( \alpha > 2 \) and \( \lambda \neq \lambda_k, \mu_k \) \( (k = 1, 2, \ldots) \), then by (1.2) the serieses \( \sum_{k=1}^{\infty} \frac{1}{|\mu_k - \lambda|} \) and \( \sum_{k=1}^{\infty} \frac{1}{|\lambda_k - \lambda|} \) are convergent. Therefore, \( R^0_\lambda \) and \( R_\lambda \) are kernel operators. In this case we obtain

\[
\text{tr} \left( R_\lambda - R^0_\lambda \right) = \text{tr} R_\lambda - \text{tr} R^0_\lambda = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right),
\]

(2.1)

If the equality (2.1) is multiplied with \( \frac{\lambda^2}{2\pi i} \) and integrated on the circle \( |\lambda| = b_m = 2^{-1}(\mu_n \sigma + \mu_{n+1}) \) then the following is obtained:

\[
\frac{1}{2\pi i} \int_{|\lambda| = b_m} \lambda^2 \text{tr} \left( R_\lambda - R^0_\lambda \right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda^2}{\lambda_k - \lambda} \right) d\lambda - \frac{1}{2\pi i} \int_{|\lambda| = b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda^2}{\mu_k - \lambda} \right) d\lambda.
\]

It is easily seen that for the large values of \( m \)

\[
\{ \lambda_k, \mu_k \}_{k=1}^{n_m} \subset B(0, b_m) = \{ \lambda : |\lambda| < b_m \},
\]

\[
\lambda_k, \mu_k \notin B[0, b_m] = \{ \lambda : |\lambda| \leq b_m \} \ (k \geq n_m + 1).
\]

Therefore by (2.1) we have

\[
\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = -\frac{1}{2\pi i} \int_{|\lambda| = b_m} \lambda^2 \text{tr} \left( R_\lambda - R^0_\lambda \right) d\lambda.
\]

(2.2)

This is a well-known formula for the resolvents of the operators \( L_0 \) and \( L \):

\[
R_\lambda = R^0_\lambda - R_\lambda QR^0_\lambda \quad (\lambda \in \rho(L) \cap \rho(L_0)).
\]
By using this formula we obtain

\[ R_\lambda - R_\lambda^0 = \sum_{j=1}^{p} (-1)^j R_\lambda^0 (QR_\lambda)^j + (-1)^{p+1} R_\lambda (QR_\lambda^0)^{p+1}, \]

for every \( p \) positive integer. By (2.2) and the last equality we have

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ \sum_{j=1}^{p} (-1)^{j+1} R_\lambda^0 (QR_\lambda^0)^j + (-1)^p R_\lambda (QR_\lambda^0)^{p+1} \right] d\lambda, \]

or

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \sum_{j=1}^{p} D_{mj} + D^{(p)}_m. \] (2.3)

Here,

\[ D_{mj} = \frac{(-1)^j+1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^j \right] d\lambda \quad (j = 1, 2, \ldots), \] (2.4)

\[ D^{(p)}_m = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right] d\lambda. \]

**Theorem 2.1.** If \( \gamma_j \sim aj^\alpha \) (\( a > 0, \alpha > 2 \)) as \( j \to \infty \) then

\[ D_{mj} = \frac{(-1)^j}{\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} \left[ (QR_\lambda^0)^j \right] d\lambda. \]

**Proof.** It can be shown that the operator function \((QR_\lambda^0)^j\) is analytic with respect to the norm in the space \( \sigma_1(H_1) \) in the region \( \rho(L_0) \) and

\[ \text{tr} \{ [(QR_\lambda^0)^j]' \} = j. \text{tr} \{ (QR_\lambda^0)' (QR_\lambda^0)^{j-1} \}, \quad (QR_\lambda^0)' = Q(R_\lambda^0)^2. \]

Therefore, we have

\[ \text{tr} \{ [(QR_\lambda^0)^j]' \} = j. \text{tr} \{ (QR_\lambda^0)^{j-1} Q(R_\lambda^0)^2 \} \]

\[ = j. \text{tr} \{ (QR_\lambda^0)^j R_\lambda^0 \} = j. \text{tr} \{ R_\lambda^0 (QR_\lambda^0)^j \}. \]
From (2.4) and the last equality we obtain

\[ D_{mj} = \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \{[(QR_0^0)^j]'\} d\lambda. \]

This formula can be written in the following form also:

\[ D_{mj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_m} \lambda \text{tr} (QR_0^0)^j d\lambda + \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_m} \text{tr} [\lambda^2 (QR_0^0)^j]' d\lambda. \quad (2.5) \]

It is easy to see that

\[ \text{tr} \{[\lambda^2 (QR_0^0)^j]'\} = \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' \]

and

\[ \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]'\} d\lambda = \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' d\lambda. \]

We write the right hand side integral of this equality in the following way:

\[ \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' d\lambda = \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' d\lambda + \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' d\lambda. \quad (2.6) \]

Let \( \varepsilon_0 \) be a positive number such that \( b_m + \varepsilon_0 < \mu_{n+1} \). Considering the fact that the function \( \text{tr} [\lambda^2 (QR_0^0)^j] \) is analytic in the simply connected regions

\[ G_1 = \{ \lambda : b_m - \varepsilon_0 < |\lambda| < b_m + \varepsilon_0, \text{Im} \lambda > -\varepsilon_0 \}, \]

and

\[ G_2 = \{ \lambda : b_m - \varepsilon_0 < |\lambda| < b_m + \varepsilon_0, \text{Im} \lambda < \varepsilon_0 \}, \]

using Leibnitz formula, by (2.6), we get

\[ \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR_0^0)^j]\}' d\lambda = \text{tr} [b_m^2 (QR_{b_m}^0)^j] - \text{tr} [b_m^2 (QR_{b_m}^0)^j]. \]
\[ + \text{tr} [b_m^2 (QR_{b_m}^0)^j] - \text{tr} [b_m^2 (QR_{-b_m}^0)^j] = 0. \]

From (2.5) and the last equality we have

\[ D_{mj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda| = b_m} \lambda \text{tr} [(QR_{\lambda})^j] d\lambda. \]

The proof of the theorem is finished. \(
\square \)

**Theorem 2.2.** If the operator function \(Q(x)\) satisfies the condition 1 and for every \(x \in [0, \pi]\), \(AQ''(x), Q IV(x) \in \sigma_1(H)\) and the functions \(\|AQ''(x)\|_1(H), \|Q IV(x)\|_1(H)\) are bounded and measurable in the interval \([0, \pi]\) then

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |(k^2 + \gamma_j) \int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx| < \infty.
\]

**Proof.** Let \(f_i(x) = (Q(x)\varphi_i, \varphi_i)\). Using the partial integration method and considering the condition \(f''_i(0) = f''_i(\pi) = 0\) we have

\[
\int_{0}^{\pi} f_i(x) \cos 2kx dx = \int_{0}^{\pi} f_i(x) \left(\frac{1}{2k} \sin 2kx\right)' dx = \frac{1}{2k} f_i(x) \sin 2kx \bigg|_{0}^{\pi}
\]

\[
- \frac{1}{2k} \int_{0}^{\pi} f'_i(x) \sin 2kx dx = \frac{1}{2k} \int_{0}^{\pi} f'_i(x) \left(\frac{1}{2k} \cos 2kx\right)' dx
\]

\[
= \frac{1}{4k^2} f'_i(x) \cos 2kx \bigg|_{0}^{\pi} - \frac{1}{4k^2} \int_{0}^{\pi} f''_i(x) \cos 2kx dx = - \frac{1}{4k^2} \int_{0}^{\pi} f''_i(x) \cos 2kx dx.
\]

By the last relation we obtain

\[
(k^2 + \gamma_i) \int_{0}^{\pi} f_i(x) \cos 2kx dx = - \frac{1}{4} \int_{0}^{\pi} f''_i(x) \cos 2kx dx - \frac{\gamma_i}{4k^2} \int_{0}^{\pi} f'_i(x) \cos 2kx dx.
\]

Using the partial integration method again and considering the conditions \(f''_i(0) = 0, f''_i(\pi) = 0\)

\[
(k^2 + \gamma_i) \int_{0}^{\pi} f_i(x) \cos 2kx dx = - \frac{1}{4} \int_{0}^{\pi} f''_i(x) \cos 2kx dx
\]

\[
- \frac{\gamma_i}{4k^2} \int_{0}^{\pi} f'_i(x) \cos 2kx dx = - \frac{1}{4} \int_{0}^{\pi} f''_i(x) \sin 2kx dx - \frac{1}{2k} \int_{0}^{\pi} f''_i(x) \sin 2kx dx
\]

\[
- \frac{\gamma_i}{4k^2} \int_{0}^{\pi} f'_i(x) \cos 2kx dx = - \frac{1}{4} \int_{0}^{\pi} f''_i(x) \sin 2kx dx - \frac{1}{2k} \int_{0}^{\pi} f''_i(x) \sin 2kx dx.
\]
\[-\frac{\gamma_i}{4k^2} \int_0^\pi f'_i(x) \cos 2kx \, dx = -\frac{1}{4} \frac{1}{4k^2} f''_i(x) \cos 2kx \bigg|_0^\pi \]

\[-\frac{1}{4k^2} \int_0^\pi f_I^V(x) \cos 2kx \, dx - \frac{\gamma_i}{4k^2} \int_0^\pi f''_i(x) \cos 2kx \, dx \]

\[= \frac{1}{16k^2} \int_0^\pi f_I^V(x) \cos 2kx \, dx - \frac{\gamma_i}{4k^2} \int_0^\pi f''_i(x) \cos 2kx \, dx. \]

Hence we have

\[\sum_{k=1}^\infty \sum_{i=1}^\infty |(k^2 + \gamma_i) \int_0^\pi f_i(x) \cos 2kx \, dx| \]

\[\leq \sum_{k=1}^\infty \sum_{i=1}^\infty \left[ \frac{1}{16k^2} \int_0^\pi |f_I^V(x)| \, dx + \frac{|\gamma_i|}{4k^2} \int_0^\pi |f''_i(x)| \, dx \right] \]

\[\leq \sum_{i=1}^\infty \int_0^\pi |f_I^V(x)| \, dx + \sum_{i=1}^\infty \int_0^\pi |\gamma_i| |f''_i(x)| \, dx \sum_{k=1}^\infty k^{-2}. \quad (2.7)\]

Furthermore

\[\sum_{i=1}^\infty \int_0^\pi |f_I^V(x)| \, dx = \lim_{n \to \infty} \int_0^\pi \left[ \sum_{i=1}^n |f_I^V(x)| \right] \, dx \]

\[\leq \int_0^\pi \sum_{i=1}^\infty |f_I^V(x)| \, dx = \int_0^\pi \sum_{i=1}^\infty |(Q^V(x) \varphi_i, \varphi_i)| \, dx, \quad \text{(2.8)}\]

\[\sum_{i=1}^\infty \int_0^\pi |f''_i(x)| \, dx \]

\[\leq \int_0^\pi \sum_{i=1}^\infty |f''_i(x)| \, dx = \int_0^\pi \sum_{i=1}^\infty |(AQ''(x) \varphi_i, \varphi_i)| \, dx. \quad (2.9)\]

Since for every \( x \in [0, \pi] \) \( Q^V(x) \in \sigma_1(H) \) and \( AQ''(x) \in \sigma_1(H) \), the inequalities

\[\sum_{i=1}^\infty |(Q^V(x) \varphi_i, \varphi_i)| \leq \|Q^V(x)\|_{\sigma_1(H)}, \quad (2.10)\]

\[\sum_{i=1}^\infty |(AQ''(x) \varphi_i, \varphi_i)| \leq \|AQ''(x)\|_{\sigma_1(H)} \quad (2.11)\]
are satisfied. By (2.7), (2.8), (2.9), (2.10) and (2.11) we obtain:

\[ \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |(k^2 + \gamma_i) \int_{0}^{\pi} \cos 2kx f_i(x)dx| < c_0 \left[ \int_{0}^{\pi} \|Q^IV(x)\|_{\sigma_1(H)}dx + \int_{0}^{\pi} \|AQ''(x)\|_{\sigma_1(H)}dx \right], \]

here \( c_0 = \sum_{k=1}^{\infty} k^{-2} \).

Since the functions \( \|AQ''(x)\|_{\sigma_1(H)}, \|Q^IV(x)\|_{\sigma_1(H)} \) are bounded and measurable in the interval \( x \in [0, \pi] \), by the last inequality

\[ \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |(k^2 + \gamma_i) \int_{0}^{\pi} \cos 2kx (Q(x)\varphi_i \varphi_i)dx| < \infty \]

is obtained. \( \square \)

Let \( \{\psi_q\}_{q=1}^{\infty} \) be the orthonormal eigenvectors system corresponding to the eigenvalues \( \{\mu_q\}_{q=1}^{\infty} \) of the operator \( L_0 \). Since the orthonormal eigenvectors according to the eigenvalues \( k^2 + \gamma_j \) \( (k = 0, 1, 2..., j = 1, 2,...) \) of the operator \( L_0 \) are \( M_k \cos kx \phi_j \) \( (k = 0, 1, 2..., j = 1, 2,...) \) respectively then

\[ \psi_q = M_{kj} \cos k_{q}x \phi_{j_{q}} \quad (q = 1, 2,...). \quad (2.12) \]

### 3. Calculating of the Second Regularized Trace

From (2.3) and (2.1) we have

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \sum_{j=1}^{p} D_{mj} + D_{m}^{(p)}. \quad (3.1) \]

Here

\[ D_{mj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda|=b_m} \lambda \text{tr} [(QR_{\lambda}^0)^j]d\lambda, \quad (3.2) \]

\[ D_{m}^{(p)} = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} [R_{\lambda}(QR_{\lambda}^0)^{p+1}]d\lambda. \quad (3.3) \]

By using the equality
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\[ \frac{2(-1)^j}{j} \cdot \frac{1}{2\pi i} \int_{|\lambda|=b_m} \text{tr} \left[ \lambda (QR_\lambda^0)^j \right] d\lambda \]

\[ = 2(-1)^j j^{-1} \sum_{k=1}^{n_m} \text{Res}_{\lambda=\mu_k} \text{tr} \left[ \lambda (QR_\lambda^0)^j \right], \]

the formula (3.1) can be written in the following form:

\[ \sum_{k=1}^{n_m} \left( \lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^{p} (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} \left[ \lambda (QR_\lambda^0)^j \right] \right) = D_{m1} + D_{m1}^{(p)}. \tag{3.4} \]

**Theorem 3.1.** If \( \gamma_j \sim a j^\alpha \) (\( a > 0, \alpha > 2 \)) as \( j \to \infty \) and \( Q(x) \) satisfies the conditions 1), 2), 3), 4) then

\[ \lim_{m \to \infty} D_{m1} = \frac{1}{2} \left[ \text{tr} AQ(0) + \text{tr} AQ(\pi) \right] - \frac{1}{8} \left[ \text{tr} Q''(0) + \text{tr} Q''(\pi) \right]. \]

**Proof.** According the formula (3.2)

\[ D_{m1} = -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} \left( QR_\lambda^0 \right) d\lambda. \tag{3.5} \]

Since \( QR_\lambda^0 \) is a kernel operator for every \( \lambda \in \rho(L_0) \) and \( \{\psi_q\}_1^\infty \) is an orthonormal basis in the space \( H_1 \), we have

\[ \text{tr} \left( QR_\lambda^0 \right) = \sum_{q=1}^{\infty} (QR_\lambda^0 \psi_q, \psi_q)_1. \]

If this equality is written into the equality (3.5) and the equality

\[ R_\lambda^0 \psi_q = (L_0 - \lambda I)^{-1} \psi_q = (\mu_q - \lambda I)^{-1} \psi_q \]

is considered, then we obtain

\[ D_{m1} = -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \sum_{q=1}^{\infty} (QR_\lambda^0 \psi_q, \psi_q)_{H_1} d\lambda \]

\[ = -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \sum_{q=1}^{\infty} \frac{1}{\mu_q - \lambda} (Q \psi_q, \psi_q)_{H_1} d\lambda \]

\[ = \frac{1}{\pi i} \sum_{q=1}^{\infty} (Q \psi_q, \psi_q)_{H_1} \int_{|\lambda|=b_m} \frac{\lambda}{\mu_q - \lambda} d\lambda. \]

By using the formula
\[ \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda \frac{d\lambda}{\lambda - \mu_q} = \begin{cases} \mu_q, & \text{if } q \leq n_m, \\ 0, & \text{if } q > n_m, \end{cases} \]
and the equality (2.12), we obtain
\[ D_{m1} = 2 \sum_{q=1}^{n_m} \mu_q (Q\psi_q, \psi_q) H_1 = 2 \sum_{q=1}^{n_m} \mu_q \int_0^\pi (Q(x)\psi_q(x), \psi_q(x)) \, dx \]
\[ = 2 \sum_{q=1}^{n_m} \mu_q \int_0^\pi (Q(x)M_{kq} \cos k_q x, \varphi_{j_q}, M_{kq} \cos k_q x, \varphi_{j_q}) \, dx \]
\[ = 2 \sum_{q=1}^{n_m} M_{kq}^2 \mu_q \int_0^\pi \cos^2 k_q x (Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx \]
\[ = \sum_{q=1}^{n_m} M_{kq}^2 \mu_q \int_0^\pi (1 + \cos 2k_q x, (Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx. \]

Since \( Q(x) \) satisfies the condition 4) and \( M_k = \sqrt{2\pi - 1} (k = 1, 2, ...), \) by the last equality we have
\[ D_{m1} = \frac{2}{\pi} \sum_{q=1}^{n_m} \mu_q \int_0^\pi \cos 2k_q x (Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx. \] (3.6)

In accordance with Theorem 2.1, the multiple series
\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx.(Q(x)\varphi_j, \varphi_j) \, dx \]
is absolute convergent. In this case as known
\[ \lim_{j \to \infty} \sum_{q=1}^{n_m} (k^2 + \gamma_j) \int_0^\pi \cos 2k_q x, (Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx \]
\[ = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx.(Q(x)\varphi_j, \varphi_j) \, dx. \]

By using (3.6) and the last equality we obtain
\[ \lim_{m \to \infty} D_{m1} = \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx.(Q(x)\varphi_j, \varphi_j) \, dx, \]
or

\[
\lim_{m \to \infty} D_{m1} = \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k^2 \int_{0}^{\pi} \cos 2kx.(Q(x)\varphi_j, \varphi_j)dx \\
+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \gamma_j \int_{0}^{\pi} \cos 2kx.(Q(x)\varphi_j, \varphi_j)dx.
\]

If we use the equality

\[
\int_{0}^{\pi} \cos 2kx.(Q(x)\varphi_j, \varphi_j)dx = -\frac{1}{4k^2} \int_{0}^{\pi} \cos 2kx.(Q''(x)\varphi_j, \varphi_j)dx,
\]

then we have

\[
\lim_{m \to \infty} D_{m1} = -\frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx.(Q''(x)\varphi_j, \varphi_j)dx \\
+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx.(Q(x)\varphi_j, \gamma_j)dx.
\]

Hence, we obtain

\[
\lim_{m \to \infty} D_{m1} = -\frac{1}{4\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{0}^{\pi} \cos kx.(Q''(x)\varphi_j, \varphi_j)dx \\
+ (-1)^b \int_{0}^{\pi} \cos kx.(Q''(x)\varphi_j, \varphi_j)dx] \\
+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx.(Q(x)\varphi_j, A\varphi_j)dx \\
= -\frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx.(Q''(x)\varphi_j, \varphi_j)dx \right) \cos k0 \\
+ \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx.(Q''(x)\varphi_j, \varphi_j)dx \right) \cos k\pi \\
+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx.(AQ(x)\varphi_j, \varphi_j)dx \\
= -\frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx.(Q''(x)\varphi_j, \varphi_j)dx \right) \cos k0
\]
\[ + \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx. Q''(x) \varphi_j, \varphi_j \right) dx \cos k \pi \]
\[ + \frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \int_{0}^{\pi} \cos kx. AQ(x) \varphi_j, \varphi_j \right) dx \]
\[ + (-1)^k \int_{0}^{\pi} \cos kx. AQ(x) \varphi_j, \varphi_j \] \]
\[ = - \frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx. Q''(x) \varphi_j, \varphi_j \right) dx \cos k0 \]
\[ + \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_{0}^{\pi} \cos kx. Q''(x) \varphi_j, \varphi_j \right) dx \cos k \pi \]
\[ + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2}{\pi} \left( \int_{0}^{\pi} \cos kx. AQ(x) \varphi_j, \varphi_j \right) dx \cos k0 \]
\[ + \sum_{k=1}^{\infty} \frac{2}{\pi} \left( \int_{0}^{\pi} \cos kx. AQ(x) \varphi_j, \varphi_j \right) dx \cos k \pi \].

If we consider that \( Q(x) \) satisfies the conditions 1) and 4), then the sums according to \( k \) on the right hand side of the last relation are the values at 0 and \( \pi \) of the Fourier series of the functions \( (Q''(x) \varphi_j, \varphi_j) \) and \( (AQ(x) \varphi_j, \varphi_j) \) according to the functions \( \{\cos kx\}_{k=0}^{\infty} \) in the interval \([0, \pi]\) respectively. Therefore

\[ \lim_{m \to \infty} D_{m1} = - \frac{1}{8} \sum_{j=1}^{\infty} [(Q''(0) \varphi_j, \varphi_j) + (Q''(\pi) \varphi_j, \varphi_j)] \]
\[ + \frac{1}{2} \sum_{j=1}^{\infty} [(AQ(0) \varphi_j, \varphi_j) + (AQ(\pi) \varphi_j, \varphi_j)], \]

or

\[ \lim_{m \to \infty} D_{m1} = \frac{1}{2} \text{tr} AQ(0) + \text{tr} AQ(\pi) - \frac{1}{8} [\text{tr} Q''(0) + \text{tr} Q''(\pi)] \]

is obtained. \( \Box \)

**Theorem 3.2.** If \( \gamma_j \sim a_j \alpha \) \( (0 < a < \infty, 2 < \alpha < \infty) \) as \( j \to \infty \) and \( Q(x) \) satisfies the conditions 1), 2), 3), 4) then

\[ \lim_{m \to \infty} \sum_{k=1}^{n_{m}} (\lambda_{k}^{2} - \mu_{k}^{2} - 2 \sum_{j=2}^{p} (-1)^{j} j^{-1} \text{Res} \left[ \lambda(QR_{\lambda}^{j} \varphi_{j}) \right]) \]
\[
\frac{1}{2} \left[ \text{tr} \: AQ(0) + \text{tr} \: AQ(\pi) \right] - \frac{1}{8} \left[ \text{tr} \: Q''(0) + \text{tr} \: Q''(\pi) \right].
\]

Here \( p = \left\lceil \frac{5\alpha + \delta}{\alpha} \right\rceil + 1. \)

**Proof.** By using the formula (3.3) we have

\[
|D_m^{(p)}| \leq \int_{|\lambda|=b_m} |\lambda|^2 \left| \text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right] \right| d\lambda
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \| R_\lambda (QR_\lambda^0)^{p+1} \|_{\sigma_1(H)} |d\lambda|
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| (QR_\lambda^0)^{p+1} \|_{\sigma_1(H)} |d\lambda|
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| (QR_\lambda^0)^p \|_1 \| (QR_\lambda^0) \|_{\sigma_1(H)} |d\lambda|
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| Q \|_1 \| R_\lambda^0 \|_1 \| (QR_\lambda^0) \|_{\sigma_1(H)} |d\lambda|. \quad (3.7)
\]

Using the inequalities

\[
\| R_\lambda \|_{\sigma_1(H)} \leq \text{const} \, n^{-\delta} \quad \text{and} \quad \| R_\lambda \|_1 \leq \text{const} \, n^{-\delta} \quad \left( \delta = \frac{\alpha - 2}{\alpha + 2} \right)
\]

(see [3]), and the inequality (3.7), we obtain

\[
|D_m^{(p)}| \leq \text{const} b_m^3 \, n_m^{(1+p)\delta} \, n_m^{-\delta} \quad \text{or}
\]

\[
|D_m^{(p)}| \leq \text{const} n_m^{3(1+\delta)} \, n_m^{-(1+p)\delta} \, n_m^{-\delta} = \text{const} n_m^{4-(p-1)\delta},
\]

when \( b_m \leq \text{const} n_m^{1+\delta} \).

Therefore, if \( p = \left\lceil \frac{4}{\delta} + 1 \right\rceil + 1 \) or \( p = \left\lceil \frac{5\alpha + \delta}{\alpha} \right\rceil + 1 \) then we obtain

\[
\lim_{m \to \infty} D_m^{(p)} = 0. \quad (3.8)
\]

By Theorem 3.1 and the formulas (3.4) and (3.8), the formula in the form

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^{p} (-1)^j j^{-1} \text{Res tr} \left[ \lambda (QR_\lambda^0)^j \right])
\]

\[
= \frac{1}{2} \left[ \text{tr} \: AQ(0) + \text{tr} \: AQ(\pi) \right] - \frac{1}{8} \left[ \text{tr} \: Q''(0) + \text{tr} \: Q''(\pi) \right]
\]

is obtained for the second regularized trace of the operator \( L \). \qed
References


