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A NOTE ON THE SOLUTIONS OF DISTRIBUTIONAL EQUATIONS

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Abstract: In this study we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then we attempt to solve the distributional equation in the form of

$$p \cdot f = g,$$

where p is a polynomial which has several zeros and g is a given distribution in $\mathcal{D}'$ and further we prove that $p \cdot f = g$ has distributional solution in $\mathcal{D}'$.

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Although now known as the Dirac delta function, the "delta" function $\delta(x)$ can be said to have been first introduced by Kirchhoff, see [6]. He defined $\delta(x)$ by

$$\delta(x) = \lim_{\mu \to \infty} \pi^{-1/2} \mu \exp(-\mu^2 x^2).$$

It is easily seen that $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$.

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The delta function, also known as the unit impulse function, is important in the study of phenomena of an impulsive nature, such as the action of heat flow over a very short time interval or over a very small region. As $\delta(x)$ is undefined at $x = 0$ thus $\delta(x)$ is not a quantity which can be generally used in mathematical analysis like an ordinary function, instead it is called a generalised function or distribution.

This distribution cannot be obtained from a locally summable function, the most important property of $\delta(x)$ is exemplified by the following equation,

$$\phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx \tag{1}$$

for all $\phi(x)$ in $\mathcal{D}$. By making a change of origin in (1), we obtain the formula

$$\phi(a) = \int_{-\infty}^{\infty} \delta(x-a)\phi(x)dx\,,$$

where a is any real number. Thus the process of multiplying a function of x by $\delta(x-a)$ and integrating over all x is equivalent to the process of substituting a for x.

There are a number of elementary equations which one can write down about δ functions. These equations are essentially rules of manipulation for algebraic work involving δ functions. In the following, some properties of $\delta(x)$ are given (see Dirac [1]):

$$\begin{aligned}
&\delta(-x) = \delta(x),\\
&x\delta(x) = x^2\delta(x) = x^3\delta(x) = \ldots = x^n\delta(x) = 0, \quad n \in \mathbb{N},\\
&\delta(ax) = a^{-1}\delta(x) \quad (a > 0),\\
&\delta(x^2 - a^2) = \frac{1}{2}a^{-1}\left\{\delta(x-a) + \delta(x+a)\right\} \quad (a > 0),\\
&\int \delta(x-a)\delta(x-b)dx = \delta(a-b),\\
&f(a)\delta(x-a) = f(a)\delta(x-a).
\end{aligned}$$

In order to define the derivative of the distribution, first of all consider, a continuous function f of a single variable, having a continuous first derivative f' will also define linear functional, namely

$$\langle f', \phi\rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = \Big[f(x)\phi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx$$

$$= -\langle f, \phi' \rangle, \quad (2)$$

for all ϕ in $\mathcal{D}$. This suggests that if f is an arbitrary distribution, we define its derivative by the equation

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle.$$

Let H be Heaviside's function, it is defined to be equal zero for every negative value of x and to unity for every positive value of x:

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and the associated distribution is given by

$$\langle H, \phi \rangle = \int_0^\infty \phi(x) dx.$$

It has a jump discontinuity at $x = 0$ and also called the unit step function. One can easily show that

$$H(-x) = 1 - H(x)\,, \qquad H(x-a) = 1 - H(x-a).$$

The function $H(x)$ will prove very useful in the study of the generalized function (distributions theory), especially in the discussion of the functions with jump discontinuities. For instance, let $F(x)$ be a function that is continuous everywhere except for the point $x = \xi$, at which point it has a jump discontinuity,

$$F(x) = \begin{cases} F_1(x), & x < \xi, \\ F_2(x), & x > \xi. \end{cases}$$

Then it can be written that

$$F(x) = F_1(x) H(\xi - x) + F_2(x) H(x - \xi).$$

This concept can be extended to enable one to write a function that has jump discontinuities at several points.

Then its derivative H' is defined by

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_{-\infty}^{\infty} H(x) \phi'(x) dx = -\int_0^\infty \phi'(x)\, dx = \phi(0) = \langle \delta, \phi \rangle$$

for all ϕ in $\mathcal{D}$ and so H' is the Dirac delta function. In particular, the n-th derivative of δ is defined by

$$\langle H^{(n+1)}, \phi \rangle = \langle \delta^{(n)}, \phi \rangle = (-1)^n \phi^{(n)}(0).$$

We may now use the notation $\delta = H'$. The δ function appears whenever one differentiates a function with jump discontinuous.

The δ derivative of the Heaviside function $H(x)$ has no meaning as a point function, in applications to certain physical problems the Dirac δ represents the contraction of unit charge at a single point, the origin and δ', represent a dipole of unit electric moment at the origin since

$$\int_{-\infty}^{\infty} x\delta'(x)\,dx = \lim_{\mu\to\infty} \pi^{-1/2}\mu \int_{-\infty}^{\infty} x\left[\exp(-\mu^2x^2)\right]'\,dx = -1.$$

Similarly higher derivatives of δ represent more complicated multiple layers, and had been used in the physical and engineering sciences long before the advent of distribution theory, see [2].

Since theory of distributions is a linear theory. We can extend some operations which are valid for ordinary functions to D' such operations are called regular operations such as addition, multiplication by scalars. Other operations can be defined only for particular distributions or for certain restricted subclasses of distributions; these are called irregular operations such as convolution, multiplication and change of variables.

Now we consider the equation $x\ f(x) = 0$ for an unknown function f. If $x \neq 0$ then it is obvious that $f(x) = 0$. Now if $x = 0$ we do not have solution in the classical sense. However by using the delta function we give the generalized solution for $x\ f(x) = 0$ as

$$f(x) = c_1\ \delta(x),$$

where c_1 is arbitrary constant. As an extension to this result we state the following theorem, see [2].

Theorem 1. *If f is an arbitrary distribution, there is an infinite set of distributions g satisfying the equation*

$$x \cdot f(x) = g(x),$$

and two distributions of the set are distinguished from each other by an arbitrary multiple of δ. In other words general solution is given by

$$f(x) = \frac{g(x)}{x} + c\ \delta(x),$$

where c is an arbitrary "division" constant. In the same way one obtains the general solution of the equation $x^n f(x) = g(x)$ as

$$f(x) = \frac{g(x)}{x^n} + \sum_{i=0}^{n-1} c_i\delta^{(i)}(x),$$

where the c_i are arbitrary division constants.

In fact this idea can also be extended further as follows. Consider that $p(x)$ a polynomial having the zeros at $x = a_1$, $x = a_2$, $\dots x = a_n$ that is

$$p(x) = (x - a_1)\ (x - a_2)\ (x - a_3)\ \dots\ (x - a_n) = \prod_{i=1}^{n}(x - a_i).$$

Then the equation $p(x)\ f(x) = g(x)$ has the distributional solutions

$$f(x) \ = \ \frac{g(x)}{p(x)} + \sum_{i=0}^{n-1} c_i \delta^{(i)}(x \ - \ a_i) \ = \ \frac{g(x)}{\prod_{i=1}^{n}(x - a_i)} + \sum_{i=0}^{n-1} c_i \delta^{(i)}(x \ - \ a_i), \quad (3)$$

for any constants $c_1, c_2, c_3, \dots, c_n$ and $a_1 \neq a_2 \neq a_3 \neq \dots \neq a_n$.

Example 1. Consider the equation $x(x-1)(x-2)(x-3)f(x) = \delta$. Then the solutions is

$$f(x) = c_1\delta(x) + c_2\delta(x-1) + c_2\delta(x-2) + \delta(x-3) + \frac{\delta(x)}{x\ (x-1)\ (x-2)\ (x-3)}.$$

Example 2. Consider the equation $x^2\,(x-\pi)^2\ \left(x - \frac{\pi}{2}\right)\ \left(x - \frac{3\pi}{2}\right) f(x) = \cos x$ then has the solutions

$$f(x) = c_1\delta(x) + c_2\delta'(x) + c_3\delta(x-\pi) + c_4\delta'\,(x-\pi) + c_5\delta\left(x - \frac{\pi}{2}\right) + c_3\delta\left(x - \frac{3\pi}{2}\right) + \frac{\cos x}{x^2\ (x-\pi)^2\ \left(x-\frac{\pi}{2}\right)\ \left(x - \frac{3\pi}{2}\right)}.$$

Now we note that in the equation 3 the term $\frac{g(x)}{p(x)}$ might have no meaning in classical sense. For example the term $\frac{\delta}{x}$ in Example 1 has no meaning as $x \to 0$. In fact it is impossible to solve these equations in classical sense, see [8].

Now if we have a simple expression as follows

$$x\,f(x) = \delta(x) \quad \Longrightarrow \quad f(x) = x^{-1}\,\delta(x)\,,$$

or more general form of

$$x^s\,f(x) = \delta^{(r)}(x) \quad \Longrightarrow \quad f(x) = x^{-s}\ \delta^{(r)}(x), \quad (4)$$

for $r, s = 0, 1, 2, 3, 4, \dots$, then we do not have classical solution and we offer solution in distributional sense. In order to solve these equations, first of all we need to have the following definition.

Definition. A sequence $\delta_n : \mathbb{R} \to \mathbb{R}$ is a delta sequence of ordinary functions which converges to the singular distribution $\delta(x)$ and satisfy the following conditions:

(i) $\delta_n(x) \geq 0$ for all $x \in \mathbb{R}$,

(ii) δ_n is continuous and integrable over $\mathbb{R}$ with $\displaystyle\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1$,

(iii) Given any $\epsilon > 0$,

$$\lim_{n\to\infty} \int_{-\infty}^{-\epsilon} \delta_n(x)\, d\, x + \int_{\epsilon}^{\infty} \delta_n(x)\, d\, x = 0\,.$$

Example 3. Let $\delta_n(x)$ be the Cauchy density that is

$$\delta_n(x) = \frac{n}{\pi(n^2t^2+1)},$$

then

$$\int_a^b \delta_n = \frac{1}{\pi}\left[\arctan(nb) - \arctan(an)\right],$$

then it follows that δ_n is a delta sequence.

Example 4. Let ϕ be a continuous, nonnegative, $\phi(x) = 0$ for all $|x| \geq 1$ and $\displaystyle\int_{-1}^{1} \phi = 1$ then set $\delta_n(x) = n\phi(nx)$. Then δ_n is a delta sequence.

As in the above two examples there are many ways to construct a delta sequence. In this work we let ρ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$, (ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$, (iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Define, the function δ_n by putting

$$\delta_n(x) = n\rho(nx) \quad \text{for } n = 1, 2, \dots,$$

it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

On using the delta sequences for the expression $x^{-s}\ \delta^{(r)}(x)$ there are four possibilities to give a meaning:

$$\begin{aligned}
x^{-s} \cdot (\delta_n^{(r)}(x)) &= \lim_{n\to\infty} x^{-s}(\delta_n^{(r)})\,, \\
(x^{-s})_n \cdot (\delta^{(r)}(x)) &= \lim_{n\to\infty} (x^{-s} * \delta_n)(\delta^{(r)}(x))\,, \\
(x^{-s})_n \cdot (\delta_n^{(r)}(x)) &= \lim_{n\to\infty} (x^{-s} * \epsilon_n(x))(\delta_n^{(r)}(x))\,, \\
(x^{-s} \cdot \delta^{(r)})_n(x) &= \lim_{n\to\infty} (x^{-s} * \delta_n)(\delta_n^{(r)})\,,
\end{aligned}$$

provided that all limits exist in $\mathcal{D}'$, see details in [4].

By using the above regular sequential approach one can offer solution to the distributional equation $x^r \cdot f = \delta^{(r-1)}(x)$ and the solution as follows:

$$f(x) = \frac{\delta^{(r-1)}(x)}{x^r} + \sum_{i=0}^{r-1} c_i \delta^{(i)}(x) = \frac{(-1)^r r!}{(2r)!}\delta^{(2r-1)}(x) + \sum_{i=0}^{r-1} c_i \delta^{(i)}(x) \qquad (5)$$

for $r = 1, 2, \ldots$. In particular for $x \cdot f = \delta(x)$ we have

$$f(x) = \frac{\delta(x)}{x} + c_0\delta(x) = -\frac{1}{2}\delta'(x) + c_0\delta(x) \qquad (6)$$

for $r = 1$, see [3]. Then we have the following theorem.

Theorem 2. *The equation $p(x)\ f(x) = g(x)$ has the distributional solution and the solution is given by*

$$f(x) = \lim_{n\to\infty} \frac{(g * \delta_n)(x)}{(p * \delta_n)(x)} + \sum_{i=0}^{n-1} c_i \delta^{(i)}(x - a_i)\,. \qquad (7)$$

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