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ON THE INVERSE PROBLEM FOR SEMIVALUES OF COOPERATIVE TU GAMES

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Abstract: The semivalues were introduced axiomatically by Dubey et al [7], as weighted values of cooperative games. For transferable utility games (TU games), they obtained a formula for computing the semivalue associated with a given weight vector. Among the semivalues are the well known Shapley value, Banzhaf value, and many other values, for different weight vectors. Let G^N be the space of cooperative TU games with the set of players N and $SE : G^N \longrightarrow R^n$ be a semivalue associated with a given weight vector p^n; $n = |N|$. The inverse problem for this semivalue may be stated as: find out all games $(N, \nu) \in G^N$, such that $SE(N, \nu) = L$, where $L \in R^n$ is an a priori given vector. The inverse problem has been solved for the Shapley value in an earlier paper by Dragan [3]; in the present paper, we solve it for any semivalue. The potential approach by Hart et al [8], [9], has been used in the first case, while now we use the potential due to Calvo et al [2]. An algorithm called a dynamic algorithm is a byproduct of the results.

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1. Introduction

The main motivation for the present work was that of trying to contribute to the unified theory of the two most important one-point linear solutions of the cooperative TU games: the Shapley value [11]and the Banzhaf value [1]. For such a solution, which is a functional Ψ from the space G^N of cooperative TU games with the set of players N, to the space R^n of payoffs, the inverse problem may be stated as follows: given $\Psi_0 \in R^n$, find out the set of games $\{(N,\nu) \in G^N : \Psi(N,\nu) = \Psi_0\}$. In an earlier paper [3], we considered this problem for the Shapley value and the weighted Shapley value, and gave a formula describing this set. The basic tool was a new basis of G^N, we called "the potential basis", which allowed to find a basis for the null space of the weighted Shapley value from which the explicit formula was derived. Recall that the potential of the Shapley value was introduced earlier by. Hart et al [8], [9], even for the weighted case. The unifying work around the Shapley value and the Banzhaf value started with the papers by Dubey et al [7], who introduced the class of semivalues, which contains both values, and Weber [12], who made a wider axiomatic analysis of the linear symmetric values in general. On the other hand, the author has introduced a potential for the Banzhaf value in [4] and a potential for the semivalues is due to Calvo et al [2] (see also Dragan [5]).

In the present paper, we define a basis of G^N relative to a semivalue, we compute the potentials of the subgames of the given game, to show that the basis is a potential basis, from which we get the semivalues of the basic vectors. In this way, we discover a basis of the null space of a semivalue and derive, as in the previous work, a solution of the inverse problem, this time for a semivalue. As the Shapley value was considered in details in the previous work, we give a complete description for the Banzhaf value. As a byproduct of the results on the potential basis relative to a semivalue, we give an algorithm for the computation of a semivalue, called a dynamic algorithm, because the algorithm is building a finite sequence of games, with the same semivalue as the given one, where the last game is providing the semivalue by an easy computation. A similar algorithm for computing the Shapley value was developed by Maschler in [10]. We show also an accelerated algorithm, which solves the problem in n steps. The case of a general three person cooperative TU game is shown for illustrating the concepts introduced, and a particular game is chosen to exhibit the application of the dynamic algorithm.

2. The Semivalues and the Potential Basis

For a fixed finite set of players N, $n = |N|$, a cooperative TU game is a pair (N,ν), where ν is a real function defined on the set of subsets of N, with $\nu(\emptyset) = 0$ The sum (N,ν) of two such games, defined by $\nu(S) = \nu_1(S)+\nu_2(S)$, for all $S \subseteq N$, and all pairs of games (N,ν_1),(N,ν_2), and the scalar multiplication defined by $\nu(S) = \alpha\nu_1(S)$, for all $S \subseteq N$, $\alpha \in R$, and any game (N,ν_1), are building the vector space G^N of all TU games with the set of players N. The space may be identified with R^{2^n-1}, as the functional ν may be conceived as a real $2^n - 1$ dimensional vector; clearly, one can write $\nu \in G^N$, instead of $(N,\nu) \in G^N$.

In the literature two bases are more popular: the standard basis and the unanimity basis. In an earlier paper [3], we introduced what we called a "potential basis" for the Weighted Shapley value, which for the well known Shapley value becomes $W = \{w_T : T \subseteq N, T \neq \emptyset\}$ with $w_T(S) = t$, $t = |T|$, if $S = T$ then $w_T(S) = -1$, if $S = T\bigcup\{j\}$, with $j \notin T$ then $w_T(S) = 0$ otherwise. This basis has been used to solve what we called "the inverse problem" for the Shapley value and for deriving an algorithm for the computation of the Shapley value. A quite similar program will be followed in the present paper for the semivalues, introduced by Dubey et al [7]. There the semivalues were axiomatically defined for the more general class of cooperative games; however, in the cooperative TU games case, the authors have obtained a general formula which can be used as a definition of semivalues, so that we use here this approach. Let (N,ν) be a TU game in G^N, and $p^n \in R^n$ be a vector of nonnegative weights; p^n_s is the weight of any coalition of size s, a subcoalition of N, for $s = 1, 2, ..., n$. The weight vector is supposed to satisfy the normalization condition

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p^n_s = 1, \tag{1}$$

imposed by the axioms. Then, a semivalue SE is defined on G^N by

$$SE_i(N,\nu) = \sum_{S:i\in S\subseteq N} p^n_s[\nu(S) - \nu(S - \{i\})], \quad \forall i \in N, \tag{2}$$

and the similar definition is valid on G^T for any game in G^T, including the subgames (T,ν), $T \subseteq N$, of (N,ν), except that N and n should be changed into T and t. For a subgame of (N,ν), the notation (T,ν) means that the functional ν of the game (N,ν) is restricted in the subgame to T. Note that in this paper we are assuming that the semivalues are regular, that is all weights are positive.

Clearly, for

$$p_s^t = \frac{(s-1)!(t-s)!}{t!}, \quad \text{and} \quad p_s^t = 2^{1-t}, \quad s = 1, 2, ..., t, \quad t \leq n, \tag{3}$$

we get particular semivalues, precisely the Shapley value and the Banzhaf value. It is easy to see that the normalization condition is satisfied in both cases, and it is obvious that there are many more semivalues.

Following Dubey et al, between the weight vectors for the weights of some semivalues on G^T and $G^{T-\{i\}}$,$i \in T$, there are some recursive relationships

$$p_s^{t-1} = p_s^t + p_{s+1}^t, \quad s = 1, 2, ..., t, \tag{4}$$

that we call the inverse Pascal triangle conditions, because one can set the weight vectors for successive s in a triangle where these relations hold, but they are different of the Pascal triangle relations. It is easy to see that these relationships hold for the weight vectors (3). For what follows it is important to notice that p^t is uniquely determined by p^n and the conditions (4) for $t \leq n$. Moreover, if p^n satisfies (1), then all p^t derived by (4) will also satisfy (1). However, from p^{t-1} to obtain p^t satisfying (1), one should give one new weight and get the other ones from (4). Later, we shall give other relationships between the weight vectors, to be used in the proofs.

Let us introduce a potential basis relative to a semivalue; recall the concept of potential of Hart-Mas Colell type: a functional P_Ψ defined on the union of all spaces G^N is a potential for the value Ψ defined on the same set, if for all games $\nu \in G^N$ we have

$$\begin{gathered} P_\Psi(\{i\}, \nu) = \nu(\{i\}), \quad \forall i \in N, \\ P_\Psi(T, \nu) - P_\Psi(T - \{i\}, \nu) = \Psi_i(T, \nu), \quad \forall i \in T \quad |T| \geq 2, \quad T \subseteq N. \end{gathered} \tag{5}$$

Note again that the notation (T, ν) for a game $\nu \in G^N$ and $T \subseteq N$ is used to mean the game obtained by the restriction of ν to T. Note also that by (5) the potential gives recursively one number for each subgame (T, ν) of (N, ν), the potential of the subgame.

Theorem 1. *Consider a semivalue SE defined by a sequence of positive weight vectors p^1, p^2,..., p^n, satisfying the normalization conditions (1) and the Pascal triangle conditions (4). Then, the functional*

$$P_{SE}(T, \nu) = \sum_{S \subseteq T} p_s^t \nu(S), \qquad \forall T \subseteq N, \tag{6}$$

is a potential function for SE.

Proof. In the difference shown by (5) for the sums explained in (6), any coalition S which contains i appears as $\nu(S)$ in $P_{SE}(T,\nu)$ with the coefficient p_s^t, while a coalition which does not contain appears as $\nu(S-\{i\})$, with the coefficient p_{s-1}^t in $P_{SE}(T,\nu)$ and the coefficient $-p_{s-1}^{t-1}$ in $P_{SE}(T-\{i\},\nu)$, hence the coefficient of $\nu(S-\{i\})$ in the difference is $p_{s-1}^t - p_{s-1}^{t-1} = -p_s^t$. This proves (5) (see also Calvo et al [2], and Dragan [5]). □

Note that for the weights p_s^t given by (3) one gets

$$P_{SE}(T,\nu) = \sum_{S\subseteq T} \frac{(s-1)!(t-s)!}{t!}\nu(S), \qquad \forall T\subseteq N, \tag{7}$$

and

$$P_{SE}(T,\nu) = \frac{1}{2^{t-1}}\sum_{S\subseteq T}\nu(S), \qquad \forall T\subseteq N, \tag{8}$$

where (7) is the explicit expression of the potential for the Shapley value due to Hart -Mas Colell and (8) is the explicit expression of the potential for the Banzhaf value due to the author (Dragan [5]).

A basis of G^N, say $W=\{w_T\in R^n : \forall T\subseteq N, T\neq\emptyset\}$, is called a potential basis relative to the semivalue SE, if for all $\nu\in G^N$, we have the expansion

$$\nu = \sum_{T\subseteq N} c(T,\nu)w_T, \qquad \text{with} \qquad c(T,\nu) = P_{SE}(T,\nu), \quad \forall T\subseteq N. \tag{9}$$

In words, the coordinate T of ν relative to this basis is the potential of the restriction of ν to T relative to SE. Note that we omit in our notation of the basis the fact that this is a basis relative to SE, because we do not believe that some confusion may occur.

Theorem 2. *Let the semivalue SE be defined on G^N and all G^T, $T\subseteq N$, by the positive weight vector $p^n\in R^n$, satisfying the normalization condition (1). Then, the basis W given by*

$$\begin{aligned} w_T(T) &= \frac{1}{p_t^t}, \\ w_T(S) &= \sum_{l=0}^{l=s-t}\frac{(-1)^l\begin{pmatrix} s-t \\ l\end{pmatrix}}{p_{t+l}^{t+l}}, \quad \forall S\supset T, \end{aligned} \tag{10}$$

and $w_T(S)=0$, otherwise, where p_t^t, p_{t+1}^{t+1},..., p_s^s are derived from p^n by means of (4), is a potential basis for G^N relative to the semivalue SE.

The proof of Theorem 2 is given below. Note that W is a basis for G^N, because it consists of $2^n - 1$ linearly independent vectors. Therefore, we have to prove only that this is a potential basis. As shown by (6), P_{SE} is a linear functional, hence in (9), which may be written for any basis, we have to show that for all $S \subseteq N$ we have

$$P_{SE}(T, w_T) = 1, \qquad \text{and} \qquad P_{SE}(S, w_T) = 0, \qquad \forall S \neq T. \tag{11}$$

Indeed, if (11) holds, then due to the linearity of P_{SE} shown by (6), we have

$$P_{SE}(T, \nu) = \sum_{S \subseteq N} c(S, \nu) P_{SE}(S, w_T) = c(T, \nu), \tag{12}$$

that is we got (9), and viceversa, this holds for whatever ν only if (11) holds. So, to prove Theorem 2 we have to compute the potentials of all basic games, to get that (10) implies (11). To become used to formulas (10), and to see that the basic vectors are linearly independent, before proving Theorem 2, we illustrate the formulas (10) by writing the potential basis for a three person TU game.

Example 1. Consider $N = \{1, 2, 3\}$ and let us use the formulas (10) for writing the basic vectors for the space G^N, in which the coalitions are taken in the order $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$. By normalization $p_1^1 = 1$. We obtain

$$\begin{aligned}
&w_1 = (1, 0, 0, 1 - \tfrac{1}{p_2^2}, 1 - \tfrac{1}{p_2^2}, 0, 1 - \tfrac{2}{p_2^2} + \tfrac{1}{p_3^3}),\\
&w_{12} = (0, 0, 0, \tfrac{1}{p_2^2}, 0, 0, \tfrac{1}{p_2^2} - \tfrac{1}{p_3^3}),\\
&w_2 = (0, 1, 0, 1 - \tfrac{1}{p_2^2}, 0, 1 - \tfrac{1}{p_2^2}, 1 - \tfrac{2}{p_2^2} + \tfrac{1}{p_3^3}),\\
&w_{13} = (0, 0, 0, 0, \tfrac{1}{p_2^2}, 0, \tfrac{1}{p_2^2} - \tfrac{1}{p_3^3}),\\
&w_3 = (0, 0, 1, 0, 1 - \tfrac{1}{p_2^2}, 1 - \tfrac{1}{p_2^2}, 1 - \tfrac{2}{p_2^2} + \tfrac{1}{p_3^3}),\\
&w_{23} = (0, 0, 0, 0, 0, \tfrac{1}{p_2^2}, \tfrac{1}{p_2^2}, -\tfrac{1}{p_3^3}),\\
&w_{123} = (0, 0, 0, 0, 0, 0, \tfrac{1}{p_3^3}).
\end{aligned}$$

Note that the nonzero components for the coalitions of the same size are equal. Therefore, all potentials for the basic games for the coalitions of the same size will be equal; for example for the grand coalition, from (6) and the vectors w_T given in (10), we have

$$\begin{aligned}
P_{SE}(N, w_1) &= P_{SE}(N, w_2) = P_{SE}(N, w_3)\\
&= p_1^3 + 2p_2^3(1 - \frac{1}{p_2^2}) + p_3^3(1 - \frac{7}{p_2^2} + \frac{1}{p_3^3})\\
&= p_1^3 + 2p_2^3 + p_3^3 - \frac{2}{p_2^2}(p_2^3 + p_3^3) + 1 = 0,
\end{aligned}$$

$$P_{SE}(N,w_{12}) = P_{SE}(N,w_{13}) = P_{SE}(N,w_{23}) = p_2^3\frac{1}{p_2^2} + p_3^3(\frac{1}{p_2^2} - \frac{1}{p_3^3})$$
$$= \frac{1}{p_2^2}(p_2^3 + p_3^3) - 1 = 0,$$

$$P_{SE}(N,w_{123}) = 1.$$

This very elementary computation was given only to suggest how potentials could be computed for a game with a larger set of players, by grouping terms multiplying the ratios $\frac{1}{p_{t+l}^{t+l}}$, $l = 0,1,...,n-t$, and using a result that we give here as the following lemma.

Lemma 3. *Let $p^n \in R^n$ be a weight vector satisfying the normalization condition (1), and p^1, p^2,..., p^{n-1}, be the weight vectors derived from p^n by (4). Then, for whatever h and r,$h+1 \le r \le n$, we have*

$$p_h^r + \binom{r-h}{1} p_{h+1}^r + \binom{r-h}{2} p_{h+2}^r + ... + \binom{r-h}{r-h} p_r^r = p_h^h. \qquad (13)$$

Proof. By induction. For $r = h+1$, (13) is (4). Assume that (13) holds for $r-1$ and prove it for r; from (4) we have

$$p_h^{r-1} + \binom{r-h-1}{1} p_{h+1}^{r-1} + \binom{r-h-1}{2} p_{h+2}^{r-1} + ... + \binom{r-h-1}{r-h-1} p_{r-1}^{r-1}$$
$$= p_h^r + \binom{r-h}{1} p_{h+1}^r + \binom{r-h}{2} p_{h+2}^2 + ... + \binom{r-h}{r-h} p_r^r, \quad (14)$$

where we used the well known identity

$$\binom{r-h}{k+1} = \binom{r-h-1}{k} + \binom{r-h-1}{k+1}, \quad k = 0,1,...,r-h-2. \quad (15)$$

This proves (13). □

Proof of Theorem 2. Notice that to compute $P_{SE}(T,w_T)$ we use (6), where there is only one term and (10) where again there is only one term, namely $\frac{1}{p_t^t}$; hence we have $P_{SE}(T,w_T) = 1$. For U such that $U \supset T$ does not hold, we have $P_{SE}(U,w_T) = 0$, because all $w_T(S)$ with $S \subseteq U$ do not contain T, hence they equal zero.

If $U \supset T$, to use formula (6), recall that $w_T(S) \neq 0$ only for $T \subseteq S \subseteq U$; in this case, denote $S = T \cup M$ and $m = |M| = s-t$,and notice that for all

$\binom{u-t}{m}$ sets $S = T \cup M$, the worth $w_T(T \cup M)$ is the same, given by (10). Then, we have

$$P_{SE}(U, w_T) = \sum_{m=0}^{u-t} p^u_{t+m} \binom{u-t}{m} [\sum_{l=0}^{m} \frac{(-1)^l}{p^{t+l}_{t+l}} \binom{m}{l}]. \tag{16}$$

We want to show that this sum makes zero for all $U \supset T$. To compute the sum we interchange the summations

$$P_{SE}(U, w_T) = \sum_{l=0}^{u-t} \frac{(-1)^l}{p^{t+l}_{t+l}} [\sum_{m=l}^{u-t} \binom{u-t}{m} \binom{m}{l} p^u_{t+m}], \tag{17}$$

and use the combinatorial identity

$$\binom{u-t}{m} \binom{m}{l} = \binom{u-t}{l} \binom{u-t-l}{m-l}, \quad l \leq m \leq u-t, \tag{18}$$

to take the factor $\binom{u-t}{l}$ in front of the second sum, that is

$$P_{SE}(U, w_T) = \sum_{l=0}^{u-t} \frac{(-1)^l}{p^{t+l}_{t+l}} \binom{u-t}{l} [\sum_{u-t-l}^{k=0} \binom{u-t-l}{k} p^u_{t+l+k}]. \tag{19}$$

Now, the sum in the bracket makes p^{t+l}_{t+l} by the equality given in Lemma 3 for $h = t + l$ and $r = u, \quad u \geq t + l + 1$. We obtain

$$P_{SE}(U, w_T) = \sum_{l=0}^{u-t} (-1)^l \binom{u-t}{l}, \tag{20}$$

and the last sum makes zero by a well known combinatorial identity.

3. The Inverse Problem for Semivalues

Let $SE : G^N \rightarrow R^n$ be a semivalue defined by a positive weight vector p^n satisfying the normalization condition (1). As noticed above, from p^n one can derive the weight vectors p^1, p^2,..., p^{n-1}, by means of the inverse Pascal triangle conditions (4). Let $L \in R^n$ be a given payoff vector. The inverse problem for the semivalue is the problem of finding the set of games

$$V_L = \{\nu \in G^N : SE(N, \nu) = L\}. \tag{21}$$

In order to solve this problem, we need the semivalues of the basic games w_T, $T \subseteq N$, introduced by (10). In the previous section, we computed in the proof of Theorem 2, the potentials of these games, shown in (11). Then, we get the semivalues of the basic games from the definition (5).

Theorem 4. *Consider the semivalue SE associated with the weight vector p^n, and the basis W of G^N introduced by (10). For each $i \in N$ we have*

$$SE_i(N, w_T) = 1, \quad \text{for} \quad T = N; \quad SE_i(N, w_T) = -1, \quad \text{for} \quad T = N - \{i\};$$

$$SE_i(N, w_T) = 0, \quad \text{otherwise}. \tag{22}$$

Proof. For a fixed $i \in N$, from (5) we can obtain

$$SE_i(N, w_T) = P_{SE}(N, w_T) - P_{SE}(N - \{i\}, w_T). \tag{23}$$

Hence, if $T \neq N$, or $T \neq N - \{i\}$, we have $SE_i(N, w_T) = 0$. If $T = N$, then the first term of the difference makes 1, while the second makes 0. If $T = N - \{i\}$, then the first term makes 0, while the second term makes 1. These prove (22). □

Theorem 5. *For a given semivalue SE and a given vector $L \in R^n$, the set of games $\nu \in G^N$ with $SE(N, \nu) = L$ is obtained from*

$$\nu = \sum_{|T| \leq n-2} c(T, \nu) w_T + c(N, \nu)(w_N + \sum_{i \in N} w_{N-\{i\}}) - \sum_{i \in N} L_i w_{N-\{i\}}, \tag{24}$$

where $c(N, \nu)$ and $c(T, \nu)$ with $|T| \leq n-2$, are arbitrary constants.

Proof. From Theorem 4, we get

$$SE(N, w_N + \sum_{i \in N} w_{N-\{i\}}) = 0. \tag{25}$$

Hence, if ν is given by formula (24), then by the linearity of SE, taking into account Theorem 4 and the result (25), we get $SE(N, \nu) = L$. Conversely, any game $\nu \in G^N$ can be written in basis W as shown in (9), where $c(T, \nu)$, $T \subseteq N$, are the potentials of the restriction of ν to T, relative to SE, as proved in Theorem 2. If we assume that $SE(N, \nu) = L$, then by (5) we have

$$c(N - \{i\}, \nu) = P_{SE}(N - \{i\}, \nu) = P_{SE}(N, \nu) - L_i = c(N, \nu) - L_i, \quad \forall i \in N, \tag{26}$$

and substituting in the expansion (9) we get (24) □

Example 2. For a three person TU game, considering the basic games shown in the Example 1, if L is given, all games with this semivalue are obtained from (24), where

$$w_N + \sum_{i \in N} w_{N-\{i\}} = (0, 0, 0, \frac{1}{p_2^2}, \frac{1}{p_2^2}, \frac{3}{p_2^2} - \frac{2}{p_3^3}).$$

We get

$$\begin{aligned}
&\nu(1) = c_1, \qquad \nu(2) = c_2, \qquad \nu(3) = c_3,\\
&\nu(1,2) = (1 - \frac{1}{p_2^2})(c_1 + c_2) + (c_{123} - L_3)\frac{1}{p_2^2},\\
&\nu(1,3) = (1 - \frac{1}{p_2^2})(c_1 + c_3) + (c_{123} - L_2)\frac{1}{p_2^2},\\
&\nu(2,3) = (1 - \frac{1}{p_2^2})(c_2 + c_3) + (c_{123} - L_1)\frac{1}{p_2^2},\\
&\nu(1,2,3) = (1 - \frac{2}{p_2^2} + \frac{1}{p_3^3})(c_1 + c_2 + c_3)\\
&\qquad + c_{123}(\frac{3}{p_2^2} - \frac{2}{p_3^3}) - (\frac{1}{p_2^2} - \frac{1}{p_3^3})(L_1 + L_2 + L_3),
\end{aligned}$$

where c_1, c_2, c_3 and c_{123}, are arbitrary constants.

Corollary 6. *The null space of a semivalue SE defined by the weight vector p^n has the basis*

$$B = \{w_T \in R^n : T \subseteq N, \quad 1 \leq |T| \leq n-2\} \cup \{w_N + \sum_{i \in N} w_{N-\{i\}}\}, \tag{27}$$

hence the nullity is $2^n - n - 1$.

Before closing this section, let us return to the Shapley value and the Banzhaf value, which are the semivalues given by (3). For the Shapley value, we have $p_k^k = k^{-1}$, so that we obtain, beside $w_T(S) = 0$ for the case when $S \supset T$ does not hold, that

$$w_T(T) = t, \qquad w_T(S) = \sum_{l=0}^{s-t} (-1)^l \left(\begin{array}{c} s-t \\ l \end{array} \right) (t+l), \quad \forall S \supset T. \tag{28}$$

As expected, from the second formula, for $S = T \cup \{j\}$, $j \notin T$, we get $w_T(T \cup \{j\}) = -1$, while for any other $S \supset T$, with $s \geq t+2$, we get $w_T(S) = 0$, i.e. we have the potential basis obtained earlier by the author for the Shapley value (see [3]). For the Banzhaf value, we have $p_k^k = 2^{1-k}$, so that we obtain, beside $w_T(S) = 0$ for the case when $S \supset T$ does not hold, that

$$w_T(T) = 2^{t-1}, \quad w_T(S) = \sum_{l=0}^{s-t} (-1)^l \left(\begin{array}{c} s-t \\ l \end{array} \right) 2^{t+l-1} = (-1)^{s-t} 2^{t-1}, \quad \forall S \supset T, \tag{29}$$

where the second result was obtained as follows: the factor 2^{t-1} can be written in front of the sum and we have

$$\sum_{l=0}^{s-t}(-1)^l \binom{s-t}{l} 2^l = (-1)^{s-t}, \tag{30}$$

as it could be easily seen for example by taking $x = 1$ in the expansion of $(x-2)^{s-t}$.

Example 3. Returning to the basic games from Example 1, for the Banzhaf value we have

$$\begin{aligned} w_1 &= (1,0,0,-1,-1,0,1), & w_{12} &= (0,0,0,2,0,0,-2),\\ w_2 &= (0,1,0,-1,0,-1,1), & w_{13} &= (0,0,0,0,2,0,-2),\\ w_3 &= (0,0,1,0,-1,-1,1), & w_{23} &= (0,0,0,0,0,2,-2),\\ w_{123} &= (0,0,0,0,0,0,4). \end{aligned}$$

Returning to the solution of the inverse problem for any semivalue, shown in Example 2, the three person TU games with a given Banzhaf value are explicitly given by

$$\begin{gathered} \nu(1) = c_1, \qquad \nu(2) = c_2, \qquad \nu(3) = c_3,\\ \nu(1,2) = -(c_1+c_2) + 2(c_{123} - B_3),\\ \nu(1,3) = -(c_1+c_3) + 2(c_{123} - B_2),\\ \nu(2,3) = -(c_2+c_3) + 2(c_{123} - B_1),\\ \nu(1,2,3) = c_1 + c_2 + c_3 + 2(c_{123} - B_1 - B_2 - B_3). \end{gathered}$$

4. A Dynamic Algorithm for Computing a Semivalue

The main idea of the algorithm is that of subtracting from the given game some linear combination of the games in the basis of the null space of the semivalue, such that the semivalue does not change. This will be done in $2^n - n - 2$ steps, such that the worth of each coalition of size at most $n-2$ becomes zero: in each step another game in the null space multiplied by some factor is subtracted, such that in the new game a new worth equal zero. The game $w_N + \sum_{j\in N} w_{N-\{j\}}$ multiplied by some factor may be added in some cases to the last game obtained, in order to make the worth of the grand coalition equal to zero. In this case, we are left with a game which is a linear combination of games $w_{N-\{j\}}$; anyway, the semivalue is easily computed whether or not, we make the value of the grand coalition equal to zero, as shown by the following lemma.

Lemma 7. *Consider an n person game ν^* with $\nu^*(S) = 0$ for all coalitions of size $|S| \leq n-2$, and let SE be a semivalue associated with a weight vector p^n satisfying the normalization condition (1). Then, we get for $\forall i \in N$:*

$$SE_i(N, \nu^*) = x - x_i \text{ with}$$
$$x = p_{n-1}^n \sum_{j \in N} \nu^*(N - \{j\}) + p_n^n \nu^*(N), \quad x_i = p_{n-1}^{n-1} \nu^*(N - \{i\}). \tag{31}$$

The proof is so easy that we omit it. The theorem which justifies the algorithm may be stated as follows.

Theorem 8. *Let ν be a TU game in G^N such that for some t, $t \leq n-2$, we have $\nu(S) = 0$ for all coalitions $S \subset N$ with sizes smaller than t, or $t = 1$. Let T be a coalition with $|T| = t$, for which $\nu(T) \neq 0$. Then, the game ν' defined by*

$$\nu' = \nu - p_t^t \nu(T) w_T, \tag{32}$$

where w_T is the corresponding game defined by (10), has the properties:
(a) $SE(N, \nu') = SE(N, \nu)$; (b) $\nu'(T) = 0$; (c) $\nu'(S) = 0 \quad \forall S \subset N$, $|S| < t$;
(d) $\nu'(S) = \nu(S)$ for all $S \subset N$ of size t, but $S \neq T$.

Proof. We have (a) as w_T is in the null space of SE which is a linear operator; (b) follows from $w_T(T) = \dfrac{1}{p_t^t}$; we get (c) as $w_T(S) = \nu(S) = 0$ for all $S \subset N$ with sizes smaller than t; (d) follows from $w_T(S) = 0$ for all $S \subset N$ with size t, but $S \neq T$. □

In words, Theorem 8 says that by (32) in each step we get a game with a new coalition of worth zero, while the other coalitions of the same or smaller sizes keep their worth. In this way, after $2^n - n - 2$ steps all coalitions of sizes at most $n-2$ have the worth zero, while the last game obtained has the same semivalue as the given game.

As shown by formula (32), which can be used for all $\binom{n}{t}$ coalitions of the same size t, the total vector to be subtracted for all these coalitions is the vector obtained by multiplying p_t^t with the weighted sum of vectors w_T, $|T| = t$, the weights being $\nu(T)$.

Each term of the sum does not change the other components for the coalitions of size t. Therefore, the algorithm can be accelerated, as it is fully described by the formula

$$\nu^{t+1} = \nu^t - p_t^t \sum_{|T|=t} \nu^t(T) w_T, \quad t = 1, 2, ..., n-2, \tag{33}$$

where ν^t is the game obtained after $t-1$ steps, or it is the given game for $t = 1$. The game ν^{n-1} satisfies the conditions of Lemma 7, hence the semivalue can be computed easily, and by Theorem 8 equals the semivalue of the starting game.

Let us clarify the remark made before Lemma 7. As shown in the previous section, the game $w_N + \sum_{j\in N} w_{N-\{j\}}$ is also in the null space of the semivalue, so that we may try in a last step to make the worth of the grand coalition equal to zero. However, this is possible only if

$$\alpha = (w_N + \sum_{j\in N} w_{N-\{j\}})(N) = \frac{n}{p_n^n} - \frac{n-1}{p_n^n} \neq 0. \tag{34}$$

Indeed, in this case we can use the transformation

$$\nu^n = \nu^{n-1} - \alpha^{-1}\nu^{n-1}(N)(w_N + \sum_{j\in N} w_{N-\{j\}}), \tag{35}$$

where ν^{n-1} is the game obtained by the $2^n - n - 2$ steps. We get the worth of the grand coalition equal to zero, while keeping the worth of all coalitions of size at most $n-2$ equal to zero and the same value for the semivalue.

Example 4. Returning to Example 1, for $|N| = 3$, we can change the three person game by means of the formula

$$\begin{aligned}\nu^2 &= \nu - [\nu(1)w_1 + \nu(2)w_2 + \nu(3)w_3]\\ &= \nu - (\nu(1), \nu(2), \nu(3), [\nu(1)+\nu(2)](1-\frac{1}{p_2^2}), [\nu(1)+\nu(3)](1-\frac{1}{p_2^2}),\\ &\quad [\nu(2)+\nu(3)]\,(1-\frac{1}{p_2^2}), \quad [\nu(1)+\nu(2)+\nu(3)](1-\frac{2}{p_2^2}+\frac{1}{p_3^3})),\end{aligned}$$

as $p_1^1 = 1$. Now, if $\alpha = \frac{3}{p_2^2} - \frac{2}{p_3^3} \neq 0$, then we make the worth of the grand coalition equal to zero by subtracting

$$\left(\frac{3}{p_2^2} - \frac{2}{p_3^3}\right)^{-1} \nu^2(N)[w_N + \sum_{j\in N} w_{N-\{j\}}].$$

Then, we use (31); some computations and the inverse Pascal triangle property will give the well known formula for semivalues.

Example 5. We close with a numerical example. Consider the semivalue SE defined on G^N with $|N| = 3$ by the weight vector $p^3 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{2})$, which obviously satisfies the normalization condition (1). Then, $p^2 = (\frac{1}{3}, \frac{2}{3})$ and $p^1 = (1)$. Take $\nu(1) = 100$, $\nu(2) = 200$, $\nu(3) = 300$, $\nu(1,2) = 400$, $\nu(1,3) = 500$, $\nu(1,2,3) = 900$.

Use the accelerated algorithm to compute the semivalue for this game, taking into account that for our weights we obtain from Example 1

$$w_1 = (1, 0, 0, -\frac{1}{2}, -\frac{1}{2}, 0, 0), \quad w_{12} = (0, 0, 0, \frac{3}{2}, 0, 0, -\frac{1}{2}),$$

$$w_2 = (0, 1, 0, -\frac{1}{2}, 0, -\frac{1}{2}, 0), \quad w_{13} = (0, 0, 0, 0, \frac{3}{2}, 0, -\frac{1}{2}),$$

$$w_3 = (0, 0, 1, 0, -\frac{1}{2}, -\frac{1}{2}, 0), \quad w_{23} = (0, 0, 0, 0, 0, \frac{3}{2}, -\frac{1}{2}),$$

$$w_{123} = (0, 0, 0, 0, 0, 0, 2).$$

We have

$$\nu^2 = \nu - 100w_1 - 200w_2 - 300w_3 = (0, 0, 0, 550, 700, 850, 900).$$

Notice that $\alpha = \frac{1}{2} \neq 0$, hence we may make the worth of the grand coalition equal zero, by subtracting

$$\nu^* = \nu^2 - 1800(w_{123} + w_1 + w_2 + w_3) = (0, 0, 0, -2150, -2000, -1850, 0).$$

Now, we get from (31)

$$x = -1000, \quad x_1 = -\frac{3700}{3}, \quad x_2 = -\frac{4000}{3}, \quad x_3 = -\frac{4300}{3}.$$

Hence the semivalue is

$$SE(N, \nu) = (\frac{700}{3}, \frac{1000}{3}, \frac{1300}{3}),$$

Of course, the semivalue may be computed by using the formula (2) given as the definition, or by means of what was called the Shapley blueprint property (see [5]), or via other methods to be discussed in subsequent papers.

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