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THE FUNDAMENTAL GROUP OF THE CONNECTED SUM OF MANIFOLDS AND THEIR FOLDINGS

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Abstract: In this paper, we will introduce the folding on the connected sum of some types of manifolds which are determined by their fundamental group. Also, the fundamental group of the unfolding of the connected sum will be deduced. Some types of conditional foldings restricted on the elements of the fundamental groups are deduced. Theorems governing these relations will be achieved. Some applications in manufacturing are presented.

AMS Subject Classification: 51H10, 57N10
Key Words: manifolds, folding, connected sum, fundamental group

1. Introduction

When a plane sheet of paper is crumpled gently in the hand, and then is crushed flat a against a desk-top. The effect is to criss-cross, the sheet with a pattern of creases, which persist even when the sheet is unfolded and smoothed out again to it is original planar form. At first slight, the pattern may seem random and chaotic. However, a closer inspection will lead to the following observation. First of all, the creases appear to be composed of straight line segments. Secondly, if p is the end-point of such segment, then the total number of crease-segment that end at p is even. Thirdly, the sum of alternate angle between creases at each such point p is equal to π. This physical process can be modelled mathematically as follows. Let us replace both the sheet of paper and the desk-top by Euclidean plane R^2, equipped with its standard flat Rie-

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mannian tensorfield. We model the crumpling and crushing process by a map $f : R^2 \longrightarrow R^2$ that send each piecewise-straight path in R^2 to piecewise-straight path of the same length [12].

More studies on the folding of manifolds are studied in [3-5, 7, 8]. Various folding problems arising in the physics of membrane and polymers reviewed by Francesco [2]. The unfolding of a manifold introduced in [7]. The deformation retracts of a manifolds defined and discussed in [1, 3, 6].

Poincare' [1895] was the first to construct an algebraic group which is a topological invariant of the space Y to which it is associated, called the fundamental group [11], or first homotopy group. The fundamental groups of some types of a manifold are discussed in [9, 10].

Definitions. 1. The set of homotopy classes of loops based at the point $x_\circ$ with the product operation$[f][g] = [f \cdot g]$ is called the fundamental group and denoted by $\pi_1(X, x_\circ)$ [10].

2. A subset A of a topological space X is called a retract of X if there exist a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a, \ \forall \ a \in A$ [11].

3. A subset A of a topological space X is a deformation retract of X if there exist a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that:

$$\left.\begin{array}{r} f(x,0) = x\,, \\ f(x,1) = r(x)\,, \end{array}\right\} \quad \forall \ x \in X\,,$$

and

$$f(a,t) = a \quad \forall \ a \in A, \ t \in [0,1]\,,$$

see [11].

4. Let M and N be two smooth manifolds of dimension m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ. If f does not preserve length it is called topological folding, see [9].

5. Let M and N be two smooth manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [5].

6. Given spaces X and Y with chosen points $x_\circ \in X$ and $y_\circ \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $Y \cup X$ obtained identifying $x_\circ$ and $y_\circ$ to a single point [11].

7. Let X and Y be two compact surfaces, then the connected sum $X \# Y$ is the compact surfaces constructed by deleting a small open disc from each of

them and pasting the remaining surface pieces together along the edge of discs [11].

2. The Main Results

Aiming to our study we will introduce the following theorem.

Theorem 1. *If $\underset{m}{\#}S^n$ is a connected sum of m −spheres. Then the fundamental group of any folding of $\underset{m}{\#}S^n$ into itself is either isomorphic to Z or identity group.*

Proof. Let $\underset{m}{\#}S^n$ be a connected sum of m − spheres, then any folding of $\underset{m}{\#}S^n$ into itself is either a folding without singularity or a folding with singularity.

Thus if $F\left(\underset{m}{\#}S^n\right)$ is a folding without singularity, then $F(\underset{m}{\#}S^n)$ is a manifold homeomorphic to S^n, and so $\pi_1(\underset{m}{\#}S^n) \approx \pi_1(\underset{m}{\#}S^n)$. Hence, $\underset{m}{\#}S^n$ is either isomorphic to Z or identity group. Also, if $F(\underset{m}{\#}S^n)$ is a folding with singularity, then every loop in $F(\underset{m}{\#}S^n)$ is homotopic to the identity loop and so $\pi_1(F(\underset{m}{\#}S^n)) \approx 0$. □

Lemma 1. *Let $\underset{m}{\#}S^n$ be a connected sum of m-spheres and D_n is the disjoint union of n discs. Then $\pi_1(\underset{m}{\#}S^n - D_n)$ is either a free group of rank $n-1$ or identity group.*

Proof. If $n = 1$ then every loop in $\underset{m}{\#}S^n - D_n$ is homotopic to the identity loop, and so $\pi_1(\underset{m}{\#}S^n - D_n) \approx 0$. Also, if $n > 1$, from follows it rose leaved $(n-1)$ is a deformation retract of $\underset{m}{\#}S^n - D_n$, that $\pi_1(\underset{m}{\#}S^n - D_n) \approx \pi_1(\underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{n-1\,\text{term}})$, and so $\pi_1(\underset{m}{\#}S^n - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{n-1\,\text{term}}$. Hence, $\pi_1(\underset{m}{\#}S^2 - D_n)$ is a free group of rank $n-1$. Therefore, $\pi_1(\underset{m}{\#}S^2 - D_n)$ is either a free group of rank $n-1$ or identity group. □

Corollary 1. *Let $\underset{m}{\#}S^2$ be a connected sum of m-spheres, and D_n is the disjoint union of n discs. Then the fundamental group of any folding of $\pi_1(\underset{m}{\#}S^2 - D_n)$ into itself is either a free group of rank $\leq n-1$ or identity group.*

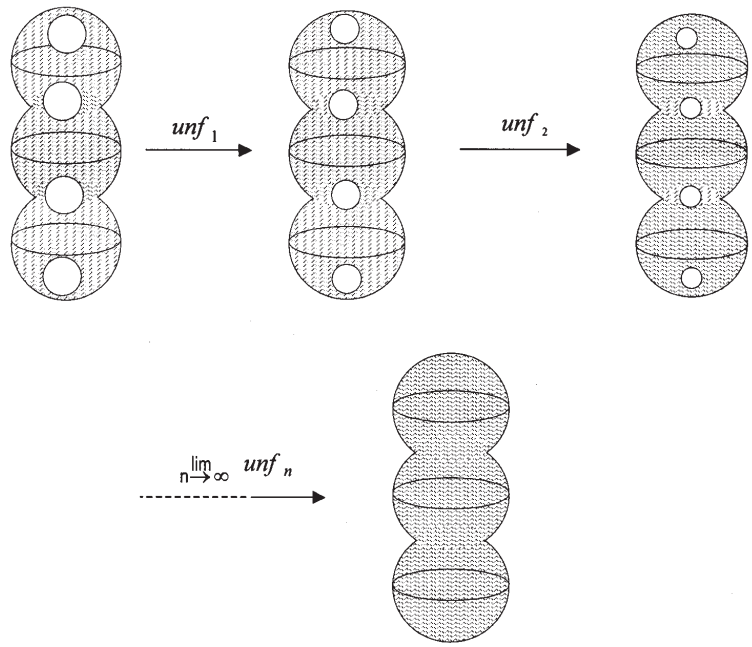

Figure 1:

Theorem 2. *Let* $\underset{m}{\#}S^n$ *be a connected sum of* $m-$*spheres, and* D_n *is the disjoint union of* n *discs. Then there are unfoldings* $unf: \underset{m}{\#}S^2 - D_n \to \underset{m}{\#}S^2 - D_n$ *such that* $\pi_1(\lim_{n\to\infty}(unf_n(\underset{m}{\#}S^2 - D_n))) \approx 0$.

Proof. Let $\underset{m}{\#}S^2$ be a connected sum of $m-$ spheres, and D_n is the disjoint union of n discs. Then, we can define a sequence of unfoldings,

$unf_1 : \underset{m}{\#}S^2 - D_n \to M_1, M_1 \subseteq \underset{m}{\#}S^2$,

$unf_2 : M_1 \to M_2, M_1 \subseteq M_2 \subseteq \underset{m}{\#}S^2$,

$\vdots$

and so $\lim_{n\to\infty} unf_n(\underset{m}{\#}S^2 - D_n) = \underset{m}{\#}S^2$ as in Figure 1 for $m = 3, n = 4$.

Hence, $\pi_1(\lim_{n\to\infty}(unf_n(\underset{m}{\#}S^2 - D_n))) = \pi_1(\underset{m}{\#}S^2)$.

Therefore, $\pi_1(\lim_{n\to\infty}(unf_n(\underset{m}{\#}S^2 - D_n))) = 0$. □

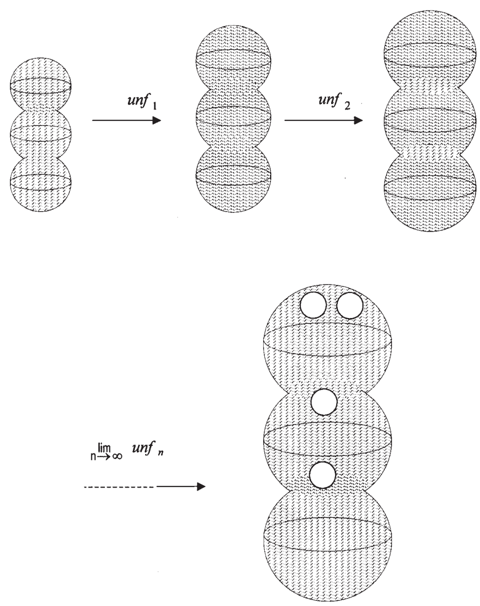

Figure 2:

Theorem 3. *Let* $\underset{m}{\#}S^2$ *be a connected sum of* m *- spheres, and* D_n *is the disjoint union of* n *discs. Then there are unfoldings* $unf : \underset{m}{\#}S^2 \to \underset{m}{\#}S^2$ *such that* $\pi_1(\lim_{k\to\infty} unf_k(\underset{m}{\#}S^2))$ *is either a free group of rank* $n-1$ *or identity group.*

Proof. Let $\underset{m}{\#}S^2$ be a connected sum of $m-$ spheres, and D_n is the disjoint union of n discs. Then, we can define a sequence of unfoldings

$unf_2 : \underset{m}{\#}S_1^2 \to \underset{m}{\#}S_2^2,$

$unf_2 : \underset{m}{\#}S_1^2 \to \underset{m}{\#}S_2^2,$

$\vdots$

$unf_k : \underset{m}{\#}S_{k-1}^2 \to \underset{m}{\#}S_k^2,$

so and $\lim_{k\to\infty} unf_k(\underset{m}{\#}S^2) = \underset{m}{\#}S^2 - D_n$, as in Figure 2 for $m = 3, n = 4$.

Hence $\pi_1(\lim_{k\to\infty} unf_k(\underset{m}{\#}S^2)) = \pi_1(\underset{m}{\#}S^2 - D_n)$.

Therefore, $\pi_1(\lim_{k\to\infty} unf_k(\underset{m}{\#}S^2))$ is either a free group of rank $n-1$ or identity group. □

Lemma 2. *Let* $\underset{m}{\#}T^1$ *be a connected sum of* $m-$*tori, and* D_n *is the disjoint* *union of n discs. Then $\pi_1(\underset{m}{\#}T^1 - D_n)$ is a free group of rank$= 2m + n - 1$.*

Proof. Since (2m+n-1) leaved rose is a deformation retract of $\#T^1 - D_n$, it follows that $\pi_1(\underset{m}{\#}T^n - D_n) \approx \pi_1(\underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{2m+n-1\,\text{term}})$,

and so $\pi_1(\underset{m}{\#}T^n - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n-1\,\text{term}}$.

Therefore, $\pi_1(\underset{m}{\#}T^n - D_n)$ is a free group of rank$= 2m + n - 1$. □

Corollary 2. *Let $\underset{m}{\#}T^1$ be a connected sum of $m-$ tori, and D_n is the disjoint union of n discs. Then the fundamental group of any folding of $\underset{m}{\#}T^1 - D_n$ into itself is either a free group of rank $\leq 2m + n - 1$ or identity group.*

Theorem 4. *Let $\underset{m}{\#}T^1$ be a connected sum of m-tori, andD_n is the disjoint union of n discs. Then there are unfoldings $unf : \underset{m}{\#}T^1 - D_n \to \underset{m}{\#}T^1 - D_n$ such that $\pi_1(\lim_{n\to\infty}(unf_n(\underset{m}{\#}T^1 - D_n)))$ is a free Abelian group of rank $2m$.*

Proof. Let $\underset{m}{\#}T^1$ be a connected sum of, $m-$tori and D_n is the disjoint union of n discs. Then, we can define a sequence of unfoldings

$unf_1 : \underset{m}{\#}T^1 - D_n \to M_1, M_1 \subseteq \underset{m}{\#}T^1,$

$unf_2 : M_1 \to M_2, M_1 \subseteq M_2 \subseteq \underset{m}{\#}T^1,$

$\vdots$

$unf_n : M_{n-1} \to M_n, M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n \subseteq \underset{m}{\#}T^1,$

$\lim_{n\to\infty} unf_n(\underset{m}{\#}T^1 - D_n) = \underset{m}{\#}T^1$as in Figure 3 for $m = 4, n = 4$.

Hence $\pi_1(\lim_{n\to\infty} unf_n(\underset{m}{\#}T^1 - D_n) = \pi_1(\underset{m}{\#}T^1)$.

Therefore, $\pi_1(\lim_{n\to\infty} unf_n(\underset{m}{\#}T^1 - D_n)))$, is a free Abelian group of rank $2m$. □

Theorem 5. *Let $\underset{m}{\#}T^1$ be a connected sum of $m-$ tori, and D_n is the disjoint union of n discs. Then there are unfoldings $unf : \underset{m}{\#}T^1 \to \underset{m}{\#}T^1$ such that*

$\pi_1(\lim_{k\to\infty}(unf_k(\underset{m}{\#}T^1))$ is a free group of rank $2m + n - 1$.

Proof. Let $\underset{m}{\#}T^1$ be a connected sum of $m-$ tori, and D_n is the disjoint union of n discs. Then, we can define a sequence of unfoldings

$unf_1 : \underset{m}{\#}T^1 \to \underset{m}{\#}T^1_1,$

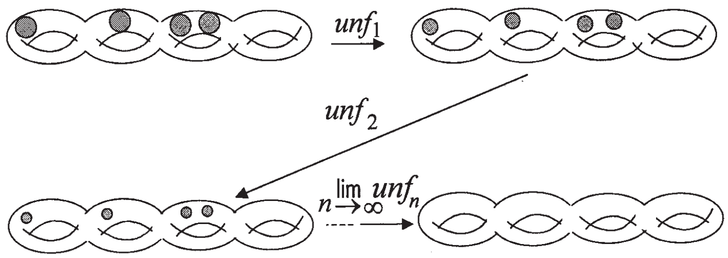

Figure 3:

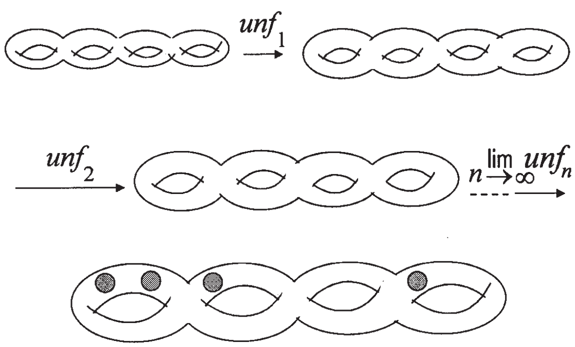

Figure 4:

$unf_2 : \underset{m}{\#}T_1^1 \to \underset{m}{\#}T_2^1,$

$\vdots$

$unf_k : \underset{m}{\#}T_{k-1}^1 \to \underset{m}{\#}T_k^1,$

$\lim\limits_{k\to\infty} unf_k(\underset{m}{\#}T^1) = \underset{m}{\#}T^1 - D_n$ as in Figure 4 for $m = 4, n = 4$.

Hence$\pi_1(\lim\limits_{k\to\infty} unf_k(\underset{m}{\#}T^1)) = \pi_1(\underset{m}{\#}T^1 - D_n)$.

Therefore, $\pi_1(\lim\limits_{k\to\infty} unf_k(\underset{m}{\#}T^1))$ is a free group of rank $2m + n - 1$. □

Theorem 6. *Let* $\underset{m}{\#}T^1$ *be a connected sum of* $m-$*tori and the circles* S_i^1 *are generators of* $T_i^1, i = 1, 2, \ldots, n, n \leq m$. *Then* $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})$*, is a free group of rank* $2m - n$.

Proof. Since $(2m - n)$ leaved rose is a deformation retract of $\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}$, it follows that $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})$

$\approx \pi_1(\underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{2m+n\,\text{term}})$,

and so $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n\,\text{term}}$.

Therefore, $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})$ is a free group of rank $2m - n$. □

Corollary 3. *Let $\underset{m}{\#}T^1$ be a connected sum of $m-$tori and the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then the fundamental group of any folding of $T_n - \{S_1^1, S_2^1, \ldots, S_n^1\}$ into itself is either a free group of rank $\leq 2m - n$ or identity group.*

Theorem 7. *Let $\underset{m}{\#}T^1$ be a connected sum of $m-$tori and the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, m$. Then there are unfoldings $unf : \underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \to \underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\}$ such that*

$$\pi_1(\lim_{n\to\infty} unf_n(F(\underset{m}{\#}T^1)))$$

is a free Abelian group of rank 2m.

Proof. Let $\underset{m}{\#}T^1$ be a connected sum of $m-$tori and the circles are generators of $T_i^1, i = 1, 2, \ldots, m$. Then, we can define a sequence of unfoldings

$unf_1 : \underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \to M_1, \underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \subseteq M_1 \subseteq \underset{m}{\#}T^1$,

$unf_2 : M_1 \to M_2\ , M_1 \subseteq M_2 \subseteq \underset{m}{\#}T^1$,

$\vdots$

$unf_n : M_{n-1} \to M_n, M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n \subseteq \underset{m}{\#}T^1$.

$\lim_{n\to\infty} unf_n(F(\underset{m}{\#}T^1)) = \underset{m}{\#}T^1$. Hence $\pi_1(\lim_{n\to\infty} unf_n(F(\underset{m}{\#}T^1))) \approx \underset{m}{\#}T^1$.

Therefore $\pi_1(\lim_{n\to\infty} unf_n(F(\underset{m}{\#}T^1)))$, is a free Abelian group of rank $2m$. □

Theorem 8. *If $\underset{m}{\#}T^1$ is a connected sum of $m-$tori, $\underset{k}{\#}S^2$ is a connected sum of k spheres, and D_n is the disjoint union of n discs $D_n \in \underset{m}{\#}T^1$ or $\underset{k}{\#}S^2$. Then $\pi_1(((\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2 - D_n)$ is a free group of rank$= 2m + n - 1$.*

Proof. Since $(2m + n - 1)$ leaved rose is a deformation retract of

$(\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2) - D_n$, it follows that

$$\pi_1(((\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2 - D_n) \approx \pi_1\underbrace{(S^1 \vee S^1 \vee \cdots \vee S^1)}_{2m+n-1 \text{ term}},$$

and so $\pi_1(((\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2 - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n-1 \text{ term}}$.

Therefore, $\pi_1(((\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2 - D_n)$ is a free group of rank $= 2m + n - 1$. ☐

Corollary 4. *If $\underset{m}{\#}T^1$ is a connected sum of $m-$ tori, and $\underset{k}{\#}S^2$ is a connected sum of k spheres, and D_n is the disjoint union of n discs $D_n \in \underset{m}{\#}T^1$ or $\underset{k}{\#}S^2$. Then the fundamental group of any folding of $\pi_1(((\underset{m}{\#}T^1)\#(\underset{k}{\#}S^2 - D_n)$ into itself is a free group of rank $\leq 2m + n - 1$ or identity group.*

Theorem 9. *Let $\underset{m}{\#}T^1$ be a connected sum of tori - m, $\underset{k}{\#}S^2$ is a connected sum of k spheres, and the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}) \# (\underset{k}{\#}S^2)$, is a free group of rank $2m - n$.*

Proof. Since $(2m - n)$ leaved rose deformation retract of $(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}) \# (\underset{k}{\#}S^2)$ it follows that $\pi_1(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\underset{k}{\#}S^2) \approx$
$\pi_1\underbrace{(S^1 \vee S^1 \vee \cdots \vee S^1)}_{2m+n \text{ term}}$ so we have
$\pi_1((\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\underset{k}{\#}S^2)) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n \text{ term}},$. Therefore,
$\pi_1((\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\underset{k}{\#}S^2))$, is a free group of rank $2m - n$.

Corollary 5. *Let $\underset{m}{\#}T^1$ be a connected sum of $m-$tori, $\underset{k}{\#}S^2$ is a connected sum of k spheres, and the circles S_i^1 are generators of T_i^1, $i = 1, 2, \ldots, n$, $n \leq m$. Then the fundamental group of any folding of $(\underset{m}{\#}T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\underset{k}{\#}S^2)$ into itself, is a free group of rank$\leq 2m - n$ or identity group.*

3. Applications

1. According to the elasticity of wall of S^2 the energy inside S^2, if it is increasing, the sphere will be unfolded until some quantity of the internal

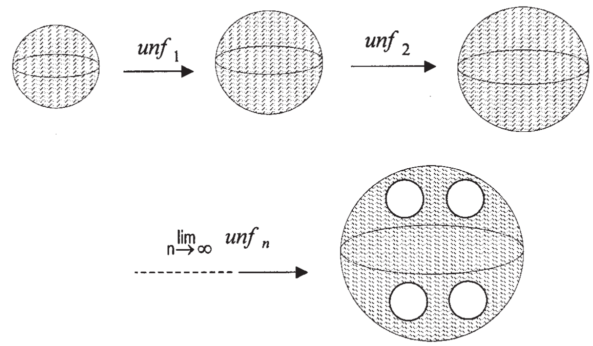

Figure 5:

energy, after that some explosion will happen causing some holes on the wall as in Figure 5.

2. During the form of cream in the milk, it has a small holes on its surface, these holes gradually decreasing by the time to arrive to a similar shape of a section of sphere.

3. In industry, mechanics use some kinds of hit and heat to close the holes in the cars bodies surface. Also, in wheels which has holes the man closes these holes by iron.

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