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BANACH VALUED REAL OR COMPLEX ANALYTIC MAPS AND THEIR ZERO-SETS: AN EXAMPLE WITH A "BAD" BANACH DOMAIN

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1. Banach Valued Analytic Maps

In [1] and [2], Example 1, we used the same non-separable Banach space $B_{\mathbb{K}}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, constructed in [4] (or see [3], Proposition 8) for two different purposes. Here there is a description of $B_{\mathbb{K}}$.

Example 1. Take $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Fix an uncountable set A and let $B_{\mathbb{K}} = C_0(A, \mathbb{K})$ be the Banach space of all $\mathbb{K}$-valued functions f on A which vanish at infinity with the supremum norm, i.e. such that for every $\epsilon > 0$ there is finite set $S \subset A$ such that $|f(i)| < \epsilon$ for every $i \in A \backslash S$. Let U be an open subset of $B_{\mathbb{K}}$. We will say that a function $h : U \to \mathbb{K}$ is analytic if it is continuous and Gatêaux analytic. Every analytic function $h : U \to \mathbb{K}$ depends only on a countable number of variables ([4] or [3], Proposition 8).

The quickest way to explain our extension of [1] and [2] to Banach valued "functions" is to state now the two results we will prove in this note.

Theorem 1. *Let W_i, $i \geq 1$, be countably many complex Banach spaces. Assume that for each index i either the dual $W_i{}'$ is separable or W_i has a*

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Schauder basis. Fix integers $d_i > 0$, $i \geq 1$, and a continuous homogeneous degree d_i and W_i-valued polynomial $f_i : P^{d_i}(B_{\mathbb{C}}) \to W_i$. Let $X \subset \mathbf{P}(B_{\mathbb{C}})$ be the common zero-locus of all the polynomials f_i, $i \geq 1$. Then X is a cone whose vertex has uncountable dimension. Hence if X is not a linear subspace of $\mathbf{P}(B_{\mathbb{C}})$, then X is singular.

Theorem 2. *Let W_i, $i \geq 1$, be countably many real Banach spaces. Assume that for each index i either the dual W_i' is separable or W_i has a Schauder basis. Fix an open subset $U \subseteq B_{\mathbb{R}}$, $P \in U$ and real analytic functions $f_i : U \to W_i$ such that $f_i(P) = 0$ for all i. Then $\bigcap_{i \geq 1} f_i^{-1}(0)$ contains the intersection with U of a linear subspace of $B_{\mathbb{R}}$ with uncountable dimension. In particular, there is no isolated zero-locus of countably many real analytic W_i-valued functions.*

It is obvious that in the statement of Theorem 2 we need to assume some conditions on the target Banach spaces: $0 \in B_{\mathbb{R}}$ is an isolated zero of the indentity map $B_{\mathbb{R}} \to B_{\mathbb{R}}$. Similary, if we take a decomposition $B_{\mathbb{C}} = A \oplus W$ with A finite-dimensional, it is easy to get two Banach valued homogeneous forms whose common zero-locus is a non-linear smooth hypersurface of the finite-dimensional projective space $\mathbf{P}(A)$. In [1] we stated the case of Theorem 1 corresponding to finitely many $\mathbb{C}$-valued polynomials (see the last four lines of [1]).

Proof of Theorem 1. First assume that W_i' is separable. Take a countable and dense subset S of W_i'. For any $\lambda \in S$ set $A(\lambda) := \{a \in A : \lambda \circ f_i(a) = 0\}$. Set $A_i := \bigcap_{\lambda \in S} A(\lambda)$. By [4] or [3], Proposition 8, each set $A \backslash A(\lambda)$ is countable. Hence the set $A \backslash A_i$ is countable. Now assume that W_i has a Schauder basis $\{e_n\}_{n \geq 1}$. Use the continuous functionals $e_n^* : W \to \mathbb{C}$ defined by $e_n^*(\sum_n \alpha_n e_n) := \alpha_n$ ([5], Corollary 4.1.16) to define the set A_i and $B_i := A \backslash A_i$. Set $E := \cap_{i \geq i} A_i$. By construction X is a cone with vertex containing $\mathbf{P}(E)$. □

Proof of Theorem 2. Copy the proof of Theorem 1. □

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