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# $\mu$-STATISTICAL CHARACTERIZATION OF COMPLETION OF NORMED AND INNER PRODUCT SPACES

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**Abstract:** In terms of $\mu-$ statistical convergence, the characterization of completion of normed and inner product spaces is given.

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## 1. Introduction

Steinhaus [10] introduced the concept of statistical convergence (see also Fast [4]). If $K$ is a subset of $\mathbb{N}$, the set of natural numbers, then the asymptotic density of $K$, denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, where $|K|$ denotes the cardinality of the set $K$. A sequence $x = (x_k)$ of numbers is statistically convergent to $L$ if

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$ or $x_k \to L\,(stat)$. Note that convergent sequences are statistically convergent, but not conversely (see [1], [5]).

Statistical convergence has been investigated in a number of papers [1], [7], [8] [9]. Some results on characterization of Banach spaces with separable duals via statistical convergence may be found in [6]. This notion of convergence is

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also considered in trigonometric series, see [2].

Connor [2] gave an extension of the notion of statistical convergence where the asymptotic density is replaced by a finitely additive set function. Through the present paper, let $\mu$ be a finitely additive set function taking values in $[0,1]$ defined on a field $\Gamma$ of subsets of $\mathbb{N}$ such that if $|A| < \infty$, then $\mu(A) = 0$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; $\mu(\mathbb{N}) = 1$. Such a set function satisfying the above criteria will be called a measure. Following Connor [2], [3] we say that:

(i) $x$ is $\mu-$density convergent to $L$ if there is an $A \in \Gamma$ such that $(x-L)_{\chi_A}$ is a null sequence and $\mu(A) = 1$, where $\chi_A$ is the characteristic function of $A$.

(ii) $x$ is $\mu-$statistically convergent to $L$, and write $st_\mu - \lim x = L$, provided $\mu(\{k : |x_k - L| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$.

If $T = (t_{nk})$ is a nonnegative regular summability method, then $T$ can be used to generate a measure as follows: for each $n \in \mathbb{N}$, set $\mu_n(A) = \sum\limits_{k=1}^{\infty} t_{nk}\chi_A(k)$ for each $A \subset \mathbb{N}$. Let

$$\Gamma = \left\{ A \subseteq \mathbb{N} : \lim_n \mu_n(A) = 0 \text{ or } \lim_n \mu_n(A) = 1 \right\}.$$

Define $\mu_T : \Gamma \to [0,1]$ by

$$\mu_T(A) = \lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk}\chi_A(k).$$

Then $\mu_T$ and $\Gamma$ satisfy the requirements of the preceding definitions. If $T$ is the Cesaro matrix of order one, then $\mu_T$ statistical convergence is equivalent to statistical convergence.

It is known [2] that (i) implies (ii), but not conversely. These two definitions are equivalent (see [2], [3]) if $\mu$ has the so-called additive property for null sets: if, given a collections of null sets $\{A_j\}_{j\in\mathbb{N}} \subseteq \Gamma$, there exists a collection $\{B_j\}_{j\in\mathbb{N}} \subseteq \Gamma$ with the properties $|A_i \Delta B_i| < \infty$ for each $i \in \mathbb{N}$, $B = \bigcup\limits_{i=1}^{\infty} B_i \in \Gamma$, and $\mu(B) = 0$.

Throughout the present paper, $\mu$ will be a measure with the additive property for null sets.

In this paper we use the notion of $\mu-$statistical convergence in characterization of completion of normed and inner product spaces.

## 2. $\mu-$Statistical Characterization of Completion of Normed Spaces

**Definition 1.** Let $E$ be a normed space. The sequence $x_n \in E$ is called $\mu-$statistically convergent to $x \in E$, and denoted by $st_\mu - \lim x_n = x$, if $\mu(\{k : \|x_k - x\| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$.

**Definition 2.** The sequence $\{x_n\} \subset E$ is called $\mu-$statistically Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0(\varepsilon) = n_0$ such that

$$\mu(\{k : \|x_k - x_{n_0}\| \geq \varepsilon\}) = 0.$$

We note that if $\{x_n\}$ and $\{y_n\}$ are $\mu-$statistically Cauchy sequence, so is $\{x_n + y_n\}$. Similarly, for any scalar $\lambda$, the sequence $\{\lambda x_n\}$ is also a $\mu-$statistically Cauchy sequence. We denote by $C_E$ the set of all $\mu-$statistically Cauchy sequences $\{x_n\}$ in $E$.

**Definition 3.** The sequences $\{x_n\}$ and $\{x'_n\}$ of $E$ are called $\mu-$statistically equivalent, and denoted by $x_n \overset{stat_\mu}{\sim} x'_n$, if

$$stat_\mu - \lim_{n\to\infty} \|x_n - x'_n\| = 0.$$

It easily follows from the definition that this an equivalence relation on the set of $\mu-$statistically Cauchy sequences in $E$. Therefore, the set $C_E$ can be divided into equivalence classes. For each such sequence $\{x_n\}$, we denote its equivalence class by $\widehat{x}$, and $\widehat{E}$ is the set of equivalence classes.

The usual operations, summation and multiplication by scalars of $\mu-$statistically Cauchy sequences are well-behaved under this equivalence relation, that is, if $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$ are arbitrarily chosen representatives, then $\{x_n + y_n\} \in \widehat{x} + \widehat{y}$ as well as $\{\lambda x_n\} \in \widehat{\lambda x}$ for any scalar $\lambda$. Therefore, when $E$ is a linear space, we can extend the operations on $E$ to $\widehat{E}$ to make it a linear space. In particular, the null of $\widehat{E}$ corresponds to the equivalence class of zero sequence, which we abbreviate to $\widehat{0}$.

We define a norm $\|.\|_{\widehat{E}}$ on $\widehat{E}$ by setting

$$\left\|\widehat{x}\right\|_{\widehat{E}} := stat_\mu - \lim_{n\to\infty} \|x_n\|_E \tag{2.1}$$

for any $\widehat{x} \in \widehat{E}$ and for an arbitrary representative $\{x_n\} \in \widehat{x}$, whose existence will be shown by the following two lemmas.

**Lemma 1.** *The limit* $stat_\mu - \lim_{n\to\infty} \|x_n\|_E$ *exists and is independent of any particular representative.*

*Proof.* Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is a $\mu-$statistically Cauchy sequence, there exists $n_0 = n_0(\varepsilon)$ such that

$$\mu(\{k : \|x_k - x_{n_0}\| \geq \varepsilon\}) = 0.$$

From the inequality

$$|\|x_k\| - \|x_{n_0}\|| \leq \|x_k - x_{n_0}\|, \tag{2.2}$$

we deduce that

$$\{k : |\|x_k\| - \|x_{n_0}\|| \geq \varepsilon\} \subseteq \{k : \|x_k - x_{n_0}\| \geq \varepsilon\}.$$

Therefore, we must have $\mu(\{k : |\|x_k\| - \|x_{n_0}\|| \geq \varepsilon\}) = 0$, providing that $\|x_k\|_E$ is a $\mu-$statistically Cauchy sequence, so the limit $stat_\mu - \lim_{n\to\infty} \|x_n\|_E$ exists.

Now, assume that $\{x_k\} \overset{stat_\mu}{\sim} \{x'_k\}$ and $\{y_k\} \overset{stat_\mu}{\sim} \{y'_k\}$, then by (2),

$$\left\{k : \left|\|x_k - y_k\| - \left\|x'_k - y'_k\right\|\right| \geq \varepsilon\right\} \\ \subseteq \left\{k : \left\|x_k - x'_k\right\| \geq \varepsilon\right\} \cup \left\{k : \left\|y_k - y'_k\right\| \geq \varepsilon\right\},$$

from which we conclude

$$stat_\mu - \lim_{k\to\infty} \|x_k - y_k\| = stat_\mu - \lim_{k\to\infty} \left\|x'_k - y'_k\right\|. \qquad \square$$

**Lemma 2.** $\|.\|_{\widehat{E}}$ *satisfies the norm axioms.*

*Proof.* Firstly, it can be seen easily from the definitions that

$$\left\|\widehat{x}\right\|_{\widehat{E}} = 0 \Leftrightarrow \{0\} \in \widehat{x} \Leftrightarrow \widehat{x} = \widehat{0}.$$

For the remainder of proof, we note that the notion of $stat_\mu - \lim$ is linear with respect to the norm on $E$, so claim follows. $\square$

**Theorem 1.** *For any normed space* $E$, *the space* $\widehat{E}$ *is a Banach space containing* $E$ *as a dense subspace.*

*Proof.* We first show that we $E$ can be embeded into $\widehat{E}$ as a linear subspace. We associate with each $x \in E$ the class $\widehat{\mathbf{x}}$ corresponding to the constant $\mu-$statistically Cauchy sequence $\{\mathbf{x}, \mathbf{x}, ...\}$. Then it is an easy argument to show that with this correspondence, $E$ be comes an subspace of $\widehat{E}$.

To show that $E$ is dense in $\widehat{E}$, we note that for each $\mathbf{x} \in E$, since $\|\mathbf{x}\|_{\widehat{E}} = stat_\mu - \lim_{n\to\infty} \|\mathbf{x}\|_E$, then $\|\mathbf{x}\|_{\widehat{E}} = \|\mathbf{x}\|$, where $\|.\|$ is norm of $E$. For any given $\widehat{x} \in \widehat{E}$, we will show that there exists a sequence $\left\{\widehat{\mathbf{x}}_n\right\} \subset E$ satisfying $\left\|\widehat{x} - \widehat{\mathbf{x}}_n\right\|_{\widehat{E}} \to 0$ as $n \to \infty$. So, $\{x_k\} \in \widehat{x}$ be given. Since $\{x_k\}$ is a $\mu-$statistically Cauchy sequence, for each $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that

$$\mu\left(\left\{n : \|x_k - x_{n_0}\| \geq \frac{\varepsilon}{2}\right\}\right) = 0.$$

In other words, for each $\varepsilon > 0$,

$$\|x_k - x_{n_0}\|_E < \frac{\varepsilon}{2} \quad \text{a.a. } k \quad (k > n_0). \tag{2.3}$$

Then, by recalling the fact that $\{\mathbf{x}_{n_0}, \mathbf{x}_{n_0}, ...\} \in \widehat{\mathbf{x}}_{n_0}$, we have

$$\left\|\widehat{x} - \widehat{\mathbf{x}}_{n_0}\right\|_{\widehat{E}} = stat_\mu - \lim_{n_0\to\infty} \|x_k - \mathbf{x}_{n_0}\|_E \leq \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, every $\varepsilon-$neighborhood of $\widehat{x} \in \widehat{E}$ intersects with $E$, that is, $E$ is dense in $\widehat{E}$.

Finally, to show that $\widehat{E}$ is a Banach space, we let $\left\{\widehat{x}_n\right\}$ be any $\mu-$statistically Cauchy sequence in $\widehat{E}$. Then there is a subset $K \subseteq \mathbb{N}$ such that $\mu(K) = 1$ and $\left\{\widehat{x}_{n_k}\right\}$ is Cauchy sequence in $\widehat{E}$. Since $E$ is dense in $\widehat{E}$, for every $\widehat{x}_{n_k}$there is an $\mathbf{x}_{n_k} \in E$ such that

$$\left\|\widehat{x}_{n_k} - \mathbf{x}_{n_k}\right\| < \frac{1}{n_k}. \tag{2.4}$$

Thus, by the triangle inequality, we have that

$$\begin{aligned}\|\mathbf{x}_{n_k} - \mathbf{x}_{m_k}\|_{\widehat{E}} &\leq \left\|\mathbf{x}_{n_k} - \widehat{x}_{n_k}\right\|_{\widehat{E}} + \left\|\widehat{x}_{n_k} - \widehat{x}_{m_k}\right\|_{\widehat{E}} + \left\|\widehat{x}_{m_k} - \mathbf{x}_{m_k}\right\|_{\widehat{E}} \\ &< \frac{1}{n_k} + \left\|\widehat{x}_{n_k} - \widehat{x}_{m_k}\right\|_{\widehat{E}} + \frac{1}{m_k} \to 0 \quad (n_k, m_k \to \infty),\end{aligned}$$

showing that $\{\mathbf{x}_{n_k}\}$ is a Cauchy sequence in $\widehat{E}$. On the other hand, since we consider $E$ as a linear subspace of $\widehat{E}$, we deduce that

$$\|\mathbf{x}_{n_k} - \mathbf{x}_{m_k}\|_E = \|\mathbf{x}_{n_k} - \mathbf{x}_{m_k}\|_{\widehat{E}}$$

is satisfied; hence, $\{\mathbf{x}_{n_k}\}$ is a Cauchy sequence in $E$ as well. We now denote by $\widehat{\mathbf{x}}$ the class which corresponded to the equivalence class of the sequence $\{\mathbf{x}_{n_k}\}$.

We show that $\widehat{\mathbf{x}}$ is the limit of $\left\{\widehat{x}_{n_k}\right\}$. By (4), we have

$$\left\|\widehat{x}_{n_k} - \widehat{\mathbf{x}}\right\|_{\widehat{E}} \leq \left\|\widehat{x}_{n_k} - \mathbf{x}_{n_k}\right\|_{\widehat{E}} + \left\|\widehat{\mathbf{x}} - \mathbf{x}_{n_k}\right\|_{\widehat{E}} < \frac{1}{n_k} + \left\|\widehat{\mathbf{x}} - \mathbf{x}_{n_k}\right\|_{\widehat{E}}.$$

Hence, it follows that $\left\|\widehat{x}_{n_k} - \widehat{\mathbf{x}}\right\|_{\widehat{E}} \to 0$ as $k \to \infty$, which completes the proof. □

## 3. $\mu-$Statistical Characterization of Completion of Inner Product Spaces

In this section, we carry out a similar programme in order to provide a characterization of a $\mu-$statistically completion of the normed space obtained from an inner product space.

Let $E$ be any inner product space with an inner product denoted by $\langle , \rangle$. So, by setting

$$\|x\| := \sqrt{\langle x, x\rangle}$$

for each $x \in E$, we may turn $E$ into a normed space.

**Lemma 3.** *If $\{x_k\}$ and $\{y_k\}$ are any two $\mu-$statistically Cauchy sequence in $E$, and satisfy $stat_\mu - \lim_k x_k = x$ and $stat_\mu - \lim_k y_k = y$, then $stat_\mu - \lim_k \langle x_k, y_k\rangle = \langle x, y\rangle$.*

*Proof.* In such case, it is well known that the inequality

$$|(x_k, y_k) - (x, y)| \leq \|x_k\| \, \|y_k - y\| + \|x_k - x\| \, \|y\|$$

is valid. So, we have,

$$\{k : |(x_k, y_k) - (x, y)| \geq \varepsilon\} \subseteq \{k : \|x_k\| \, \|y_k - y\| \geq \varepsilon\} \cup \{k : \|x_k - x\| \, \|y\| \geq \varepsilon\}.$$

In this containment, the $\mu-$density of the right-hand side is zero, since $stat_\mu - \lim_k x_k = x$ and $stat_\mu - \lim_k y_k = y$. Thus, $stat_\mu - \lim_k \langle x_k, y_k\rangle = \langle x, y\rangle$. □

As a result of the previous lemma, we will now prove that every inner product space $\mu-$statistically completed with the norm induced by its inner product. To achieve this, let $\widehat{E}$ be the set of equivalence classes defined as earlier for the normed space $E$. For any $\widehat{x}, \widehat{y} \in \widehat{E}$, from the previous lemma, we may define that

$$\left\langle \widehat{x}, \widehat{y} \right\rangle_{\widehat{E}} := stat_{\mu} - \lim \left\langle x_n, y_n \right\rangle \ , \qquad (3.1)$$

where $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. Then, it may be readily proved that (5) defines an inner product on $\widehat{E}$.

**Corollary 1.** *If $(E, \langle , \rangle)$ is an inner product space, then $\left(\widehat{E}, \langle , \rangle_{\widehat{E}}\right)$ is a Hilbert space.*

## 4. An Application: $\mu-$Statistically Completion of $\widetilde{\mathcal{L}}_1 [a, b]$

In this section we give an application of $\mu-$statistical convergence in case of concrete normed space, namely we shall investigate the space $\widetilde{\mathcal{L}}_1 [a, b]$.

We recall that $\widetilde{\mathcal{L}}_1 [a, b]$ is a normed space of continuous functions $x(t)$ on the closed interval $[a, b]$ with the norm

$$\|x(t)\| = \int_a^b |x(t)| \, dt.$$

By $\mathcal{L} [a, b]$, we denote a completion of $\widetilde{\mathcal{L}}_1 [a, b]$. The space $\mathcal{L} [a, b]$ is called Lebesque space.

**Definition 4.** Let $\{x_n(t)\}$ and $\{x_n^*(t)\}$ be any two sequences of functions on $[a, b]$. If

$$stat_{\mu} - \lim \|x_n - x_n^*\| = stat_{\mu} - \lim \int_a^b |x_n(t) - x_n^*(t)| \, dt = 0,$$

then the sequences $\{x_n\}$ and $\{x_n^*\}$ are said to be $\mu-$statistically equivalent in the sense of mean valued on $\widetilde{\mathcal{L}}_1 [a, b]$. For brevity, in such case, we write $\{x_n(t)\}$ and $\{x_n^*(t)\}$ are e.m.v.

**Definition 5.** Let $\{x_k(t)\} \in \widetilde{\mathcal{L}}_1[a,b]$ be given. If for each $\varepsilon > 0$, there exists positive integers $k = k(\varepsilon)$ and $p$ such that

$$\mu\left(\left\{k : \|x_{k+p} - x_k\| = \int_a^b |x_{k+p}(t) - x_k(t)|\, dt \geq \varepsilon\right\}\right) = 0,$$

then the sequence $\{x_k(t)\}$ is called $\mu-$statistically Cauchy in the sense of mean valued on $\widetilde{\mathcal{L}}_1[a,b]$, or for short, $\mu - stat - mv$.

According to Theorem 1, Lebesque space $\mathcal{L}[a,b]$ is the set of elements $\widehat{x}(t)$, which are equivalence classes of continuous $\mu-$statistically Cauchy sequences in the sense of mean valued. On this set,

$$\left\|\widehat{x}\right\|_{\mathcal{L}[a,b]} = stat_\mu - \lim \int_a^b |x_n(t)|\, dt = stat_\mu - \lim \|x_n\|_{\widetilde{\mathcal{L}}_1[a,b]}$$

defines a norm.

**Theorem 2.** *If $x(t)$ is a discontinuous of the first kind (or jump discontinuity) on $[a,b]$, then there exists a $\mu-$statistically Cauchy sequence of continuous functions in the sense of mean valued converging to $x(t)$ in sense of mean valued.*

We now provide an example showing that there exists a $\mu-$statistically Cauchy sequence in the sense of mean valued which is not a Cauchy sequence in sense of mean valued.

**Example 1.** Let $x(t)$ be a function on $[a,b]$ that has discontinuity of the first kind on the points $a < t_1 < t_2 < ... < t_l < b$. We assume that $x(t_k) = \frac{x_+(t_k)+x_-(t_k)}{2}$, where $x_+(t_k)$ and $x_-(t_k)$ are the right and left limits of $x$ in $t_k$, respectively. Moreover, for any small enough $\delta > 0$, we suppose that $a < t_l - \delta, t_{l+\delta} < b$ and $(t_k - \delta, t_k + \delta)$ are disjoint. Then, by taking

$$x_n(t) =$$
$$\begin{cases} x(t), & [a,b] \setminus \bigcup_{k=1}^{l} \left(t_k - \frac{\delta}{n}, t_k + \frac{\delta}{n}\right), \\ \frac{x\left(t_k+\frac{\delta}{n}\right) - x\left(t_k-\frac{\delta}{n}\right)}{2\frac{\delta}{n}} \left(t - t_k + \frac{\delta}{n}\right) + x\left(t_k - \frac{\delta}{n}\right), & \text{for each } t \in \left(t_k - \frac{\delta}{n}, t_k + \frac{\delta}{n}\right), \\ & k = 1, 2, ...l, \end{cases}$$

we define

$$y_n(t) = \begin{cases} x_n(t), & n \neq k^2 \\ n, & n = k^2 \end{cases} .$$

It can be easily shown that the sequence $y_n(t)$ satisfies the required property.

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