International Journal of Pure and Applied Mathematics
Volume 22 No. 4 2005, 449-461

A CHARACTERIZATION OF $PSU(23, q)$

A. Iranmanesh [§]

Department of Mathematics
Tarbiat Modarres University
P.O. Box 14115-137, Tehran, IRAN
and
Institute for Studies in Theoretical Physics and Mathematics
Tehran, IRAN
e-mail: iranmana@modares.ac.ir

Abstract: Let G be a finite group. The order of G is the product of coprime positive integers which is called the order components of G. It was proved that some non-Abelian simple groups are uniquely determined by their order components. As the main result of this paper, we show that the simple groups $PSU(23, q)$ are also uniquely determined by their order components. As corollaries of this result, the validity of a conjecture of J.G. Thompson on $PSU(23, q)$ is obtained.

AMS Subject Classification: 20D06, 20D60
Key Words: simple group, prime graph, order component, finite group

1. Introduction

For an integer n, let $\pi(n)$ be the set of prime divisors of n. If G is a finite group then $\pi(G)$ is defined to be $\pi(|G|)$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two distinct primes p and q are linked by an edge if and only if G contains an element of order pq. Let π_i, $i = 1, 2, \ldots, t(G)$ be the connected components of $\Gamma(G)$. For $|G|$ even, π_1 will be the connected component containing 2. Then $|G|$ can be expressed as a product of some positive integers m_i, $i = 1, 2, \ldots, t(G)$ with $\pi(m_i) = \pi_i$. The integers m_i's are called the order components of G. The set of order

Received: May 10, 2005

[§]Correspondence address: Department of Mathematics, Tarbiat Modarres University, P.O. Box 14115-137, Tehran, IRAN

components of G will be denoted by $OC(G)$. If the order of G is even, we will assume that m_1 is the even order component and $m_2, \ldots, m_{t(G)}$ will be the odd order components of G. The order components of non-Abelian simple groups having at least three prime graph components are obtained by G. Y. Chen [5, Tables 1, 2, 3]. The order components of non-Abelian simple groups with two order components can be obtained according to [17, 18] (see [8]). The following groups are uniquely determined by their order components: sporadic simple groups [2], $PSL(p,q)$ for $p = 2, 3, 5$ [5, 8-10], $C_2(q)$, where $q > 5$ [11], $PSU(p,q)$ for $p \in \{3, 5, 7, 11, 17, 19\}$, see [7, 12-15].

In this paper, we prove that $PSU(23,q)$ are also uniquely determined by their order components, that is we have the following theorem.

Main Theorem. *Let G be a finite group and $M = PSU(23,q)$. Then $OC(G) = OC(M)$ if and only if $G \cong M$.*

2. Preliminary Results

In order to prove Main Theorem, first we bring some definitions and lemmas.

Definition 2.1. (see [6]) A finite group G is called a 2-Frobenius group if it has a normal series $1 < H < K < G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.2. (see [18, Theorem A]) *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) a Frobenius or 2-Frobenius group;

(b) a simple group;

(c) an extension of a π_1-group by a simple group;

(d) an extension of a simple group by a π_1-solvable group;

(e) an extension of a π_1-group by a simple group by a π_1-group.

Lemma 2.3. (see [18, Lemma 3]) *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

The next lemma follows from Theorem 2 in [1].

Lemma 2.4. *Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$ and, G has one of the following structures:*

(a) $2 \in \pi(K)$ and all Sylow subgroups of H are cyclic.

(b) $2 \in \pi(H)$, *K is an Abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the $2-$Sylow subgroups of H are cyclic or generalized quaternion groups.*

(c) $2 \in \pi(H)$, *K is an Abelian group and there exists $H_0 \leq H$ such that* $|H : H_0| \leq 2$, $H_0 = Z \times SL(2, 5)$, $(|Z|, 2.3.5) = 1$ *and the Sylow subgroups of Z are cyclic.*

The next lemma follows from Theorem 2 in [1] and Lemma 2.3.

Lemma 2.5. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that:*

(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi(K/H) = \pi_2$;

(b) G/K and K/H are cyclic, $|G/K|$ divides $|Aut(K/H)|$, $(|G/K|, |K/H|) = 1$ *and* $|G/K| < |K/H|$;

(c) H is nilpotent and G is a solvable group.

Lemma 2.6. (see [4, Lemma 8]) *Let G be a finite group with $t(G) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not a π_i-number, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.*

Lemma 2.7. (see [3, Lemma 1.4]) *Suppose G and M are two finite groups satisfying $t(M) \geq 2$, $N(G) = N(M)$, where $N(G)=\{n \mid G$ has a conjugacy class of size $n\}$, and $Z(G) = 1$. Then $|G| = |M|$.*

The next lemma follows from Lemma 1.5 in [3].

Lemma 2.8. *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $OC(G_1) = OC(G_2)$.*

Lemma 2.9. *Let G be a finite group and let M be a non-Abelian simple group with $t(M) = 2$ satisfying $OC(G) = OC(M)$. Let $|M| = m_1 m_2$, $OC(M) = \{m_1, m_2\}$, and $\pi(m_i) = \pi_i$ for i=1 or 2. Then $|G| = m_1 m_2$ and one of the following holds:*

(a) G is a Frobenius or 2-Frobenius group;

(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-Abelian simple group. Moreover $OC(K/H) = \{m'_1, m'_2, \ldots, m'_s, m_2\}$, $|K/H| = m'_1 m'_2 \ldots m'_s m_2$ and $m'_1 m'_2 \ldots m'_s | m_1$, where $\pi(m'_j) = \pi'_j, 1 \leq j \leq s$. Also we have $|G/K| \mid |\mathrm{Out}(K/H)|$.

Proof. The first part of the lemma follows from the above lemmas. Since $t(G) \geq 2$, we have $t(G/H) \geq 2$. Otherwise $t(G/H) = 1$, so that $t(G) = 1$. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H and this automorphism is trivial if and only if $xH \in Z(G/H)$. Therefore, $G/K \leq Out(K/H)$ and since $Z(G/H) = 1$, it follows that $|G/K| \mid |\mathrm{Out}(K/H)|$. □

Lemma 2.10. *Let $M = PSU(23, q)$. Suppose $D(q) = \frac{q^{23}+1}{k(q+1)}$, where $k = (23, q+1)$. Then:*

(a) If $p \in \pi(M)$, then $|S_p| \leq q^{253}$, where $S_p \in Syl_p(M)$;

(b) If $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha - 1 \equiv 0$ (mod $D(q)$) if and only if $p^\alpha = q^{46}$, q^{92}, q^{138}, q^{184} or q^{230};

(c) If $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha + 1 \equiv 0$ (mod $D(q)$) if and only if $p^\alpha = q^{23}$, q^{69}, q^{115}, q^{161}, q^{207} or q^{253}.

Proof. (a) From Table 1 in [8] we have: $|M| = q^{253}(q+1)^{22}(q-1)^{11}(q^2-q+1)^7(q^2+1)^5(q^2+q+1)^3(q^4-q^3+q^2-q+1)^4(q^4-q^2+1)(q^4+q^3+q^2+q+1)^2(q^4+1)^2(1-q+q^2-q^3+q^4-q^5+q^6)^3(q^6-q^3+1)^2(q^6+q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^8+1)(1+q-q^3-q^4-q^5+q^7+q^8)(1-q^2+q^4-q^6+q^8)(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10})^2(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10})^2(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12})(1+q-q^3-q^4+q^6-q^8-q^9+q^{11}+q^{12})(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16})(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16}-q^{17}+q^{18})\frac{(q^{23}+1)}{k(q+1)}$.

For convenience, let $c_1 = q+1$, $c_2 = q-1$, $c_3 = q^2+1$, $c_4 := q^2-q+1$, $c_5 := q^2+q+1$, $c_6 := q^4-q^2+1$, $c_7 = q^4+1$, $c_8 = q^4-q^3+q^2-q+1$, $c_9 = q^4+q^3+q^2+q+1$, $c_{10} = 1-q+q^2-q^3+q^4-q^5+q^6$, $c_{11} = q^6-q^3+1$, $c_{12} = q^6+q^3+1$, $c_{13} = q^6+q^5+q^4+q^3+q^2+q+1$, $c_{14} = q^8+1$, $c_{15} = (1-q^2+q^4-q^6+q^8)$, $c_{16} = 1+q-q^3-q^4-q^5+q^7+q^8$, $c_{17} = 1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}$, $c_{18} = (1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10})$, $c_{19} = 1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}$, $C_{20} = (1+q-q^3-q^4+q^6-q^8-q^9+q^{11}+q^{12})$, $c_{21} = 1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16}$, $c_{14} = q^8+1$, $C_{22} = (1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16}-q^{17}+q^{18})$.

Now by easy calculations, we can compute the greatest common divisors of every pair of $\{c_i,\ c_j\}$. Obviously q is coprime with respect to another factors of $|M|$. In Tables (I) we present these results.

By easy calculations we determine the greatest common divisors of any two factors of $|M|$.

Now let $p^\alpha \mid |M|$ and $p \in \pi_1$. As we mentioned above we can claim that one of the following occurs: p^α is a divisor of q^{253}, $2^{19} \times 3^9 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19 \times (q+1)^{22}$, $2^{30} \times 3^4 \times 5^2 \times 7 \times 11 \times (q-1)^{11}$, $3^{24} \times 7(q^2-q+1)^6$, $2^{36} \times 5(q^2+1)^5$, $3^{12}(q^2+q+1)^3$, (q^4-q^2+1), $2^{39}(q^4+1)^2$, $5^{22}(q^4-q^3+q^2-q+1)^3$, $5^{11}(q^4+q^3+q^2+q+1)$, $7^{22}(1-q+q^2-q^3+q^4-q^5+q^6)^2$, $3^{29}(q^6-q^3+1)^2$, $3^{14}(q^6+q^3+1)$, $7^{11}(q^6+q^5+q^4+q^3+q^2+q+1)$, $2^{40}(q^8+1)$, $5^{22}(q^8-q^6+q^4-q^2+1)$,

GCD	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
c_1	c_1	1 or 2	1 or 2	1 or 3	1	1	1 or 2	1 or 5	1	1 or 7	1 or 3	1
c_2	1 or 2	c_2	1 or 2	1	1 or 3	1	1 or 2	1	1 or 5	1	1	1 or 3
c_3	1 or 2	1 or 2	c_3	1	1	1	1 or 2	1	1	1	1	1
c_4	1 or 3	1	1	c_4	1	1	1	1	1	1	1 or 3	1
c_5	1	1 or 3	1	1	c_5	1	1	1	1	1	1	1 or 3
c_6	1	1	1	1	1	c_6	1	1	1	1	1	1
c_7	1 or 2	1 or 2	1 or 2	1	1	1	c_7	1	1	1	1	1
c_8	1 or 5	1	1	1	1	1	1	c_8	1	1	1	1
c_9	1	1 or 5	1	1	1	1	1	1	c_9	1	1	1
c_{10}	1 or 7	1	1	1	1	1	1	1	1	c_{10}	1	1
c_{11}	1 or 3	1	1	1 or 3	1	1	1	1	1	1	c_{11}	1
c_{12}	1	1 or 3	1	1	1 or 3	1	1	1	1	1	1	c_{12}
c_{13}	1	1 or 7	1	1	1	1	1	1	1	1	1	1
c_{14}	1 or 2	1 or 2	1 or 2	1	1	1	1 or 2	1	1	1	1	1
c_{15}	1	1	1 or 5	1	1	1	1	1	1	1	1	1
c_{16}	1	1	1	1	1	1	1	1	1	1	1	1
c_{17}	1 or 11	1	1	1	1	1	1	1	1	1	1	1
c_{18}	1	1 or 11	1	1	1	1	1	1	1	1	1	1
c_{19}	1 or 13	1	1	1	1	1	1	1	1	1	1	1
c_{20}	1	1	1	1 or 7	1	1	1	1	1	1	1	1
c_{21}	1 or 17	1	1	1	1	1	1	1	1	1	1	1
c_{22}	1 or 19	1	1	1	1	1	1	1	1	1	1	1

Table 1

$(1+q-q^3-q^4-q^5+q^7+q^8)$, $11^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10})$, $11^{11}(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10})$, $13^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12})$, $7^7(1+q-q^3-q^4+q^6-q^8-q^9+q^{11}+q^{12})$, $17^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16})$, $19^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16}-q^{17}+q^{18})$.

Therefore the order of every Sylow subgroup of G is less than or equal to q^{253}, and hence (a) follows.

(b) Let there exists $p \in \pi_1(M)$, $p^\alpha \mid |M|$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$. It is obvious that $p^\alpha > D(q)$. Similar to the proof of (a) we must consider different cases. For $q \leq 29$, numerical calculations show that there is no p^α such that (b) holds. Hence we can let $q > 29$. But it is straightforward to see that if $q \geq 31$, then $D(q) > (q+1)^{22}/46$.

We consider the following cases:

(1) If $p^\alpha \mid 2^{19} \times 3^9 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19 \times (q+1)^{22}$, then we consider the following subcases:

(1.1) Let $p \neq 2$, 3, 5, 7, 11, 13,17, 19 and $p^\alpha \mid (q+1)^{22}$ and $p^\alpha - 1 \equiv 0$ (mod $D(q)$), then $p^\alpha - 1 = sD(q)$ for some $s > 0$. But $(q+1)^{22}/46 < D(q)$, which implies that $p^\alpha = (q+1)^{22}/t$, where $st \leq 46$. Now, numerical calculations show that these equations have no a solution in $\mathbb{Z}$ and hence, there can not exist any p and α such that the above relations holds.

(1.2) If $p = 2$, then $2^\alpha | 2^{19}(q+1)^{22}$. Hence $2^{19}(q+1)^{22}/t - 1 = sD(q)$, where

GCD	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}	c_{22}
c_1 c_1	1	1 or 2	1	1	1 or 11	1	1 or 13	1	1 or 17	1 or 19
c_2	1 or 7	1 or 2	1	1	1	1 or 11	1	1	1	1
c_3	1	1 or 2	1 or 5	1	1	1	1	1	1	1
c_4	1	1	1	1	1	1	1	1 or 7	1	1
c_5	1	1	1	1	1	1	1	1	1	1
c_6	1	1	1	1	1	1	1	1	1	1
c_7	1	1 or 2	1	1	1	1	1	1	1	1
c_8	1	1	1	1	1	1	1	1	1	1
c_9	1	1	1	1	1	1	1	1	1	1
c_{10}	1	1	1	1	1	1	1	1	1	1
c_{11}	1	1	1	1	1	1	1	1	1	1
c_{12}	1	1	1	1	1	1	1	1	1	1
c_{13}	c_{13}	1	1	1	1	1	1	1	1	1
c_{14}	1	c_{14}	1	1	1	1	1	1	1	1
c_{15}	1	1	c_{15}	1	1	1	1	1	1	1
c_{16}	1	1	1	c_{16}	1	1	1	1	1	1
c_{17}	1	1	1	1	c_{17}	1	1	1	1	1
c_{18}	1	1	1	1	1	c_{18}	1	1	1	1
c_{19}	1	1	1	1	1	1	c_{19}	1	1	1
c_{20}	1	1	1	1	1	1	1	c_{20}	1	1
c_{21}	1	1	1	1	1	1	1	1	c_{21}	1
c_{22}	1	1	1	1	1	1	1	1	1	c_{22}

Table 2: Continuation of Table 1

$st \leq 2^{20} \times 23$, since $2^{19}(q+1)^{22} < 2^{20} \times 23D(q)$ for $q > 29$. By expanding the above equation we can get a diophantine equation and by solving this equation we see that there exist no α such that (b) holds.

(1.3) If $p = 3, 5, 7, 11, 13, 17$ or 19, then $3^\alpha | 3^9(q+1)^{22}$, $5^\alpha | 5^4(q+1)^{22}$, $7^\alpha | 7^3(q+1)^{22}$, $11^\alpha | 11^2(q+1)^{22}$, $13^\alpha | 13(q+1)^{22}$, $17^\alpha | 17(q+1)^{22}$, $19^\alpha | 19(q+1)^{22}$, respectively. We get a contradiction similar to (1.2).

(2) If $p^\alpha \mid 2^{30} \times 3^4 \times 5^2 \times 7 \times 11 \times (q-1)^{11}$, then p^α divides $2^{30}(q-1)^{11}$, $3^4(q-1)^{11}$, $5^2(q-1)^{11}$, $7(q-1)^{11}$ or $11(q-1)^{11}$. But in each case $p^\alpha < D(q)$ which implies that $p^\alpha - 1 \not\equiv 0 \pmod{D(q)}$.

(3) If p^α is a divisor of $3^{24} \times 7(q^2-q+1)^6$, $2^{36} \times 5(q^2+1)^5$, $3^{12}(q^2+q+1)^3$, (q^4-q^2+1), $2^{39}(q^4+1)^2$, $5^{22}(q^4-q^3+q^2-q+1)^3$, $5^{11}(q^4+q^3+q^2+q+1)$, $7^{22}(1-q+q^2-q^3+q^4-q^5+q^6)^2$, $3^{29}(q^6-q^3+1)^2$, $3^{14}(q^6+q^3+1)$, $7^{11}(q^6+q^5+q^4+q^3+q^2+q+1)$, $2^{40}(q^8+1)$, $5^{22}q^8-q^6+q^4-q^2+1$, $(1+q-q^3-q^4-q^5+q^7+q^8)$, $11^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10})$, $11^{11}(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10})$, $13^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12})$, $7^7(1+q-q^3-q^4+q^6-q^8-q^9+q^{11}+q^{12})$, $17^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16})$, $19^{22}(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{16}-q^{17}+q^{18})$, then in each case $p^\alpha < D(q)$ which implies that $p^\alpha - 1 \not\equiv 0$

(mod $D(q)$).

(4) At last, let $p^\alpha | q^{253}$. Then we consider two subcases, namely $k=1$ and $k=23$. Since the proofs are similar we state only one of them, namely $k=1$ and the other case is similar.

We know that $q=p^n$ for some $n>0$.

First, we prove that if $p^\beta | q^{23}$ and $p^\beta + 1 \equiv 0 \pmod{D(q)}$, then $p^\beta = q^{23}$. We have

$$p^\beta + 1 = s.D(q) = s.\frac{q^{23}+1}{q+1} = s(q^{22} - q^{21} + q^{20} - \cdots + q^2 - q + 1),$$

which implies that $1 \le s \le q+1$, since $p^\beta \le q^{23}$. Also since $q|p^\beta$ we have $q|s-1$ which implies that $q \le s-1$. Therefore $q=s$ and hence $p^\beta = q^{23}$.

Now we prove that if $p^\alpha | q^{46}$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$, then $p^\alpha = q^{46}$. Now we consider two cases. First let $p^\alpha \le q^{23}$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$. In this case $p^\alpha - 1 = s.D(q)$, where $s < q+1$. Since $q|p^\alpha$ we have $q|s+1$ and hence $q \le s+1$. Therefore $s=q$ or $s=q-1$. But easy calculation shows that in each case $p^\alpha - 1 \ne s.D(q)$, which is a contradiction. Therefore $p^\alpha > q^{23}$ and hence $p^\alpha = q^{23}p^m$ for some $m>0$. Thus we have

$$p^\alpha - 1 = q^{23}p^m - 1 = p^m(q^{23}+1) - p^m - 1.$$

Therefore $D(q)|p^m+1$ which implies that $p^m = q^{23}$, by the above statement and hence $p^\alpha = q^{46}$.

If $p^\alpha > q^{46}$ and $p^\alpha | q^{253}$, then by a similar method we conclude that $p^\alpha = q^{92}$, q^{138}, q^{184} or q^{230}.

(c) Similar to part (b) we conclude that p^α must be equal to q^{23}, q^{69}, q^{115}, q^{161}, q^{207} or q^{253} and the proof is completed. □

Remark. In the sequel of this paper and specially in the proof of Main Theorem, for convenience we suppose that $X = \{q^{46}, q^{92}, q^{138}, q^{184}, q^{230}\}$ and $Y = \{q^{23}, q^{69}, q^{115}, q^{161}, q^{207}, q^{253}\}$. Therefore if $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha - 1 \equiv 0 \pmod{D(q)}$ if and only if $p^\alpha \in X$, and $p^\alpha + 1 \equiv 0 \pmod{D(q)}$ if and only if $p^\alpha \in Y$.

Lemma 2.11. *Let G be a finite group and $M = PSU(23,q)$ and $OC(G) = OC(M)$. Then G is neither a Frobenius group nor a $2-$Frobenius group.*

Proof. We will use some results about Frobenius groups. For example we know that if G is a Frobenius group, by Lemma 2.4, $OC(G) = \{|H|, |K|\}$ where K and H are the Frobenius kernel and the Frobenius complement of G, respectively. Also we know that $|H| \mid (|K|-1)$, and hence $|H| < |K|$. So $|H| = \frac{q^{23}+1}{(q+1)(23,q+1)}$, $|K| = |G|/|H|$. There exists a prime p such that $p^\alpha | 11(q-1)^{11}$.

If P is a $p-$Sylow subgroup of K, then since K is nilpotent, $P \lhd G$ and hence $D(q) \mid |P|-1$ by Lemma 2.6, which implies that $p^{\alpha} \in Y$ by Lemma 2.1o(b). But obviously $11(q-1)^{11} < q^{46}$ which is a contradiction. Therefore, G is not a Frobenius group.

Let G be a 2-Frobenius group. By Lemma 2.5, there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that $|K/H| = \frac{q^{23}+1}{(q+1)(23,q+1)} < 2^{19}(q+1)^{22}$ and $|G/K| < |K/H|$. Thus there exists a prime p such that $p \mid 2^{19}(q+1)^{22}$ and $p||H|$. If P is a $p-$Sylow subgroup of H, since H is nilpotent, P must be a normal subgroup of K with $P \subseteq H$ and $|K| = \frac{q^{23}+1}{k(q+1)}|H|$. Therefore, $\frac{q^{23}+1}{k(q+1)} \mid (|P|-1)$, by Lemma 2.6 and hence $q^{46}| \; |P|$, which is impossible since $|P| \leq 2^{19}(q+1)^{22}$. Therefore, G is not a 2-Frobenius group. □

Lemma 2.12. *Let G be a finite group. If the order components of G are the same as those of $M = PSU(23,q)$, then G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd order component of M is equal to some of those of K/H, and in particular, $t(K/H) \geq 2$.*

Proof. The first part of the Lemma follows from the above lemmas since the prime graph of M has two components. For primes p and q, if K/H has an element of order pq, then G has one. Hence, by the definition of prime graph component, the odd order component of G must be an odd order component of K/H. □

3. Proof of Main Theorem

By Lemma 2.12, G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H and G/K are π_1-groups, K/H is a non-Abelian simple group, $t(K/H) \geq 2$ and the odd order component of M is an odd order component of K/H.

We now proceed the proof in the following steps.

Step 1. If $K/H \cong A_n$, where $n = p,\ p+1,\ p+2$ and $p \geq 5$ is a prime number, then we have two cases.

Case 1. $k = 1$. In this case, p or $p-2$ are equal to $\frac{q^{19}+1}{q+1}$. If $p = \frac{q^{19}+1}{q+1}$, then $p-1 = q(q-1)(q^2-q+1)(q^2+q+1)(q^6-q^3+1)(q^6+q^3+1)$ and

$$p-2 = q^{18}-q^{17}+q^{16}-q^{15}+q^{14}-q^{13}+q^{12}-q^{11}+q^{10}-q^{9}+q^{8}-q^{7} + q^{6}-q^{5}+q^{4}-q^{3}+q^{2}-q-1, \quad (1)$$

But easy calculation shows that $(p-2,|G|) \mid 3^{31}\times 5^{6}\times 7^{2}5\times 13\times 19^{2}\times 37\times 53\times 113\times 239$. But for $q \geq 223$, $D(q) > 3^{31}\times 5^{6}\times 7^{2}5\times 13\times 19^{2}\times 37\times 53\times 113\times 239$.

But for $q < 223$, first we compute $D(q)$. It must be a prime number. If $D(q)$ is a prime number, then $D(q) - 2$ must be a divisor of $3^{31} \times 5^6 \times 7^2 5 \times 13 \times 19^2 \times 37 \times 53 \times 113 \times 239$. By checking the cases $q \leq 223$ we can see that equation (1) is not satisfied in each case, which is a contradiction.

If $p - 2 = \frac{q^{23}+1}{q+1}$, then we proceed similarly for $p - 4$ since $p > 5$.

Case 2. $k = 23$. Then p or $p - 2$ is equal to $\frac{q^{23}+1}{23(q+1)}$ and $p - 2$ or $p - 4$ must be equal to $\frac{q^{22}-q^{21}+q^{20}-q^{19}+q^{18}-q^{17}+q^{16}-q^{15}+\cdots-q-45}{23}$, respectively. Now we proceed similarly to the last case and get a contradiction.

Step 2. If K/H is a sporadic simple group, then $D(q)$ must be equal to 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71, which has no solution, since $D(2) = 2796203$.

Therefore K/H is a simple group of Lie type.

Step 3. If $K/H \cong E_6(q')$ or ${}^2E_6(q')$, then we get a contradiction.

Step 4. If $K/H \cong A_r(q')$, then we distinguish the following 6 cases:

4.1. $K/H \cong A_{p'-1}(q')$, where $(p', q') \neq (3, 2), (3, 4)$.

In fact we have $D(q) = (q'^{p'} - 1)/((q' - 1)(p', q' - 1))$ and hence $q'^{p'} - 1 \equiv 0 \pmod{D(q)}$. Now by using Lemma 2.10(b), we have $q'^{p'} \in X$, which implies that $q'^{p'} = q^{46}$, q^{92}, q^{138}, q^{184}, or q^{230}. If $p' > 13$, then $q'^{\frac{p'(p'-1)}{2}} > q^{253}$, which is impossible by Lemma 2.11(a). Therefore we must check cases $p' = 3, 5, 7, 11$ and 13. If $p' = 3$ and $q'^3 = q^{46}$, then

$$(q^{23} - 1)(q + 1)(23, q + 1) = (q' - 1)(3, q' - 1), \qquad q'^3 = q^{46}.$$

But these equations have no common solution in $\mathbb{Z}$, and hence this case is also impossible. If $p' = 3$ and $q'^3 = q^{92}$, q^{138}, q^{184}, or q^{230}; or if $p' = 5, 7, 11$ and 13, then we get a contradiction similarly.

4.2. $K/H \cong A_{p'}(q')$, where $(q' - 1)|(p' + 1)$. Then $q'^{p'} \in X$. But for $p' > 11$, we have $q'^{\frac{p'(p'+1)}{2}} > q^{253}$, which is impossible. If $p' = 3, 5, 7$ or 11, then $q' - 1 < 12$, since $q' - 1 \mid p' + 1$. Now easily we can get a contradiction.

4.3. $K/H \cong A_1(q')$, where $4|(q' + 1)$.

The odd order components of K/H are q' and $(q' - 1)/2$.

If $D(q) = \frac{q'-1}{2}$, then $q' \in X$, which implies that $q' = q^{46}$, q^{92}, q^{138}, q^{184} or q^{230}. If $q' = q^{46}$, then we have $(q23 + 1)/k(q + 1) = \frac{q^{46}-1}{2}$. Therefore, $2 = k(q + 1)(q^{23} - 1)$ and it is impossible.

If $D(q) = q'$, then by similar method in Step 1, we get a contradiction.

4.4. $K/H \cong A_1(q')$, where $4|(q' - 1)$.

The odd order components of K/H are q' and $(q' + 1)/2$.

If $D(q) = \frac{q'+1}{2}$, then $q' \in Y$, which implies that $q' = q^{23}$, q^{69}, q^{115}, q^{161}, q^{207}, or q^{253}. By similar method in (4.3), we get a contradiction

If $D(q) = q'$, then we proceed similarly to above case and get a contradiction.

4.5. $K/H \cong A_1(q')$ where $4 \mid q'$.

The odd order components of K/H are $q'+1$ and $q'-1$. By similar method in last case, we get a contradiction.

4.6. $K/H \cong A_2(2)$ or $K/H \cong A_2(4)$. Then $D(q)$ must be equal to 3, 5, 7, 9 which is impossible.

Step 5. If $K/H \cong B_r(q')$, or $C_r(q')$, or $D_r(q')$, or $F_4(q')$, or ${}^3D_4(q')$, or $E_8(q')$, or ${}^2G_2(q')$, by a similar method we get contradictions.

Step 6. If $K/H \cong E_7(2)$, $E_7(3)$, ${}^2E_6(2)$ or ${}^2F_4(2)'$, then $D(q)$ must be equal to 13, 17, 19, 73, 127, 757, 1093 which is impossible.

Step 7. If $K/H \cong G_2(q')$, then we consider 3 cases:

7.1. $K/H \cong G_2(q')$, where $2 < q' \equiv 1 (\mathrm{mod}\ 3)$. Then $D(q) = {q'}^2 - q' + 1$ and hence ${q'}^3 \in Y$, which implies that ${q'}^3 = q^{23}$, q^{69}, q^{115}, q^{161}, q^{207}, or q^{253}. If ${q'}^3 = q^{23}$, then

$$\frac{{q'}^3+1}{q'+1} = \frac{q^{23}+1}{k(q+1)}.$$

Obviously $k = 1$ implies that $q = q'$ which is impossible. If $k = 23$, then $q^{23} = (23q+22)^3$, which has no solution in $\mathbb{Z}$.

7.2. $K/H \cong G_2(q')$, where $2 < q' \equiv -1 (\mathrm{mod}\ 3)$. Then $D(q) = {q'}^2 + q' + 1$ and hence ${q'}^3 \in X$. Now we can proceed similar to 7.1 and get contradiction.

7.3. $K/H \cong G_2(q')$, where $3|q'$. Then ${q'}^2 \pm q' + 1 = D(q)$. This is similar to cases 7.1 and 7.2.

Step 8. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2r+1} > 2$, or ${}^2B_2(q')$, where $q' = 2^{2t+1} > 2$ we can get a contradiction by a similar method in above step.

Step 9. If $K/H \cong {}^2D_r(q')$, then we consider 6 cases:

9.1. $K/H \cong {}^2D_r(q')$, where $r = 2^t \geq 4$.

9.2. $K/H \cong {}^2D_r(2)$, where $r = 2^t + 1 \geq 5$.

9.3. $K/H \cong {}^2D_p(3)$, where $5 \leq p \neq 2^r + 1$.

9.4. $K/H \cong {}^2D_r(3)$, where $r = 2^t + 1 \neq p$, $t \geq 2$.

9.5. $K/H \cong {}^2D_p(3)$, where $p = 2^t + 1$, $t \geq 2$.

9.6. $K/H \cong {}^2D_{p+1}(2)$, where $p = 2^r - 1, r \geq 2$.

In all of above cases, we get a contradiction.

Step 10. If $K/H \cong {}^2A_r(q')$, then we consider 3 cases:

10.1. $K/H \cong {}^2A_3(2)$, ${}^2A_3(3)$ or ${}^2A_5(2)$. Then $D(q)$ must be equal to 5, 7, 11 which is impossible.

10.2. $K/H \cong {}^2A_{p'}(q')$, where $(q'+1)|(p'+1)$ and $(p',q') \neq (3,3),(5,2)$. Then $D(q) = q'^{p'}+1/q'+1$ and hence $q'^p \in Y$ which implies that $q'^{p'} = q^{23}$, q^{69}, q^{115}, q^{161}, q^{207}, or q^{253}. If $\frac{p+1}{2} > 11$, then $q'^{p(p+1)/2} > q^{253}$ which is a contradiction, by Lemma 2.10(a). Therefore $p' = 3$, 5, 7, 11, 13, 17 19. If $p' = 3$, then $q' = 3$, since $q'+1|p'+1$. But it is a contradiction, since $(p',q') \neq (3,3)$.

If $p' = 5$, then $q'+1|6$. Since $(p',q') \neq (5,2)$, we have $q' = 5$. But $q^{23} = 5^5$, which is a contradiction.

Similarly we get a contradiction in other cases.

10.3. $K/H \cong {}^2A_{p'-1}(q')$. Then $q'^{p'} = q^{23}$, q^{69}, q^{115}, q^{161}, q^{207}, or q^{253}. If $p' > 23$, then $q'^{\frac{p'(p'-1)}{2}} > q^{253}$, which is impossible. Otherwise, if $p' = 3$, 5, 7, 11, 13, 17 or 19, then

$$(q'+1)(p',q'+1) = (q+1)(23,q+1), \qquad q'^{p'} = q^{23}.$$

But these equations have no common solution in $\mathbb{Z}$. If $p' = 23$, then $q = q'$. Thus $|G| = |PSU(23,q)| = |K/H| = |K|/|H|$ which implies that $|H| = 1$ and $|K| = |G| = |PSU(23,q)|$. Therefore, $K = PSU(19,q)$ and hence $G = PSU(23,q)$.

The proof of Main Theorem is now completed.

Remark 3.1. It is a well known conjecture of J.G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-Abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture for the groups under discussion.

Corollary 3.2. *Let G be a finite group with $Z(G) = 1$, $M = PSU(23,q)$ and $N(G) = N(M)$, then $G \cong M$.*

Proof. By Lemma 2.8, if G and M are two finite groups satisfying the conditions of Corollary 3.2, then $OC(G) = OC(M)$. So Main Theorem implies this corollary. □

Acknowledgments

The author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) Tehran, Iran for the financial support (No. 83200016).

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