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OSCILLATIONS OF CERTAIN EVEN ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract: In this paper, a class of even order half-linear differential equations with distributed deviating arguments are considered. Some new oscillatory criteria are obtained.

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1. Introduction

In recent years there has been an increasing interest in studying the oscillation of differential equations with distributed deviating arguments, we can see the papers of Yu and Fu [9], Liu and Fu [4], Bainov and Petrov [2], Wang and Li [7], Wang and Shi [8] and their references cited therein. In this paper, We consider the following even order half-linear differential equations with distributed deviating arguments of the form

$$[a(t)\phi(x(t))|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + \int_a^b F(t,\xi,x[g(t,\xi)])\,d\sigma(\xi) = 0, \qquad (E)$$

where n is an even, $\alpha > 0$ is a constant. For the case of half linear problem,

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Manojlović [5] considered the oscillation of solutions of the second-order half-linear differential equation. Agarwal, Grace and Oregan [1] investigated the even order half-linear differential equation

$$[|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + F[t, x[g(t)]] = 0.$$

We notice that the most of oscillatory criteria for the special case of high-order equation (E) have been obtained by using the relation of $F[t, x(g(t))]$ and x, or the methods by introducing the parameter functions $H(t, s)$ and $h(t, s)$. In this paper, we obtain new oscillatory criteria for equation (E) by choosing different conditions about the function $f(x)$ which is satisfying $F(t, \xi, x)\operatorname{sgn} x \geq p(t, \xi)f(x)\operatorname{sgn} x$.

We assume throughout this paper that the following conditions hold.

(H_1) $a(t) \in C^1([t_0, \infty), (0, \infty))$, $\phi(x(t)) \in C(R, (0, \infty))$, $[a(t)\phi(x(t))]' > 0$, there exists a nonnegative function $\varphi(t)$, such that $0 < \phi(x(t)) \leq \varphi(t)$, and $\int_{t_0}^{t} \frac{1}{a(s)\varphi(s)} ds = \infty$;

(H_2) $g(t, \xi) \in C([t_0, \infty) \times [a, b], R)$, $g(t, \xi) \leq t$, $\xi \in [a, b]$, $g(t, \xi)$ is nondecreasing with respect to t and ξ, respectively, and $\lim_{t\to\infty} \inf_{\xi\in[a,b]} g(t, \xi) = \infty$;

(H_3) $F(t, \xi, x) \in C([t_0, \infty) \times [a, b] \times R, R)$, there exist function $p(t, \xi) \in C([t_0, \infty) \times [a, b], R^+$ and $f(x) \in C(R, R)$, $xf(x) > 0$ $(x \neq 0)$ such that $F(t, \xi, x)\operatorname{sgn} x \geq p(t, \xi)f(x)\operatorname{sgn} x$; $-f(-x)$ $f(x)$, $x > 0$;

(H_4) $\sigma(\xi) \in ([a, b], R)$ is nondecreasing, integral of equation (E) is a Stieltjes one.

In order to establish oscillatory criteria, we first need the following lemmas.

Lemma 1.1. (see [6]) *Let $x(t) \in C^n([t_0, \infty), R^+)$. If $x^n(t)$ is eventually of one sign for all large t, say, $t_1 \geq t_0$, then there exists a $t_x \geq t_0$, and an integer l, $0 \leq l \leq n$ with $n + l$ even for $x^{(n)}(t) \geq 0$, or $n + l$ odd for $x^{(n)}(t) \leq 0$ such that $l > 0$ implies that $x^{(k)}(t) > 0$, for $t \geq t_x$, $k = 0, 1, \cdots, l - 1$; $l \leq n - 1$ implies that $(-1)^{k+l}x^{(k)}(t) > 0$, for $t \geq t_x$, $k = l, l + 1, \cdots, n - 1$.*

Lemma 1.2. (see [6]) *If the function $x(t)$ is as in Lemma 1.1 and $x^{(n-1)}(t)x^{(n)}(t) \leq 0$, for $t \geq t_x$, then there exists a constant θ, $0 < \theta < 1$, such that*

$$x'(\frac{t}{2}) \geq \frac{\theta}{(n-2)!} t^{n-2} x^{(n-1)}(t), \quad \textit{for all large } t.$$

Lemma 1.3. (see [3]) *If X and Y are nonnegative numbers, then*

$$X^\lambda - \lambda X Y^{\lambda-1} + (\lambda - 1)Y^\lambda \geq 0, \quad \lambda > 1.$$

In the above inequalities the equality holds if and only if $X = Y$.

2. Main Results

Theorem 2.1. *Suppose that equation* (E) *satisfies the condition:*

(A_1) $f'(x) \geq u > 0$, $x \neq 0$, $t \geq t_0$, u *is a constant.*

If there exists a function $\rho(t) \in C^1([t_0, \infty), R^+)$, $\rho'(t) > 0$, *and a constant* θ, $0 < \theta < 1$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\rho(s) \int_a^b p(s,\xi)\, d\sigma(\xi) - \frac{a(s)\varphi(s)[\rho'(s)]^2}{4M\rho(s)[g(s,a)]^{n-2}g'(s,a)}\right] ds = \infty, \quad (2.1)$$

where $M = \frac{u\theta}{2(n-2)!|x^{(n-1)}(T)|^{|\alpha-1|}} > 0$ *is a constant, then equation* (E) *is oscillatory.*

Proof. Suppose to the contrary that equation (E) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution for $t \geq t_0$. From equation (E) and (H_3), we have

$$[a(t)\phi(x(t))|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' \leq -\int_a^b p(t,\xi)f(x[g(t,\xi)])\, d\sigma(\xi) \leq 0\,. \quad (2.2)$$

It follows that $a(t)\phi(x(t))|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If $x^{(n-1)}(t) \leq 0$ eventually for $t \geq t_1$, there exists a $t_2 \geq t_1$ such that $x^{(n-2)}(t_2) \leq 0$. Integrating both sides of the (2.2) from t_2 to t, we have

$$a(t)\phi(x(t))|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t) \leq a(t_2)\phi(x(t_2))|x^{(n-1)}(t_2)|^{\alpha-1}x^{(n-1)}(t_2) = L \leq 0.$$

From (H_1), we have

$$x^{(n-1)}(t) \leq -\left(\frac{-L}{a(t)\phi(x(t))}\right)^{\frac{1}{\alpha}} \leq -\left(\frac{-L}{a(t)\varphi(t)}\right)^{\frac{1}{\alpha}} \leq 0.$$

Integrating both sides of the above inequality from t_2 to t, we get

$$x^{(n-2)}(t) \leq x^{(n-2)}(t_2) - \int_{t_2}^t \left(\frac{-L}{a(s)\varphi(s)}\right)^{\frac{1}{\alpha}} ds.$$

Letting $t \to \infty$, by(H_1), we obtain $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$. Then $\lim_{t\to\infty} x(t) = -\infty$, we get a contradiction with $x(t) > 0$. So we find that $x^{(n-1)}(t) > 0$ eventually. From (2.2), we have

$$\begin{aligned} 0 \geq [a(t)\phi(x(t))|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' &= [a(t)\phi(x(t))(x^{(n-1)}(t))^{\alpha}]' \\ &= [a(t)\phi(x(t))]'[x^{(n-1)}(t))]^{\alpha} + \alpha a(t)\phi(x(t))[x^{(n-1)}(t))]^{\alpha-1}x^{(n)}(t). \end{aligned}$$

Therefore

$$x^{(n)}(t) \leq -\frac{[a(t)\phi(x(t))]'x^{(n-1)}(t)}{\alpha a(t)\phi(x(t))} \leq 0.$$

It implies that $x^{(n)}(t) \leq 0$ eventually. Then by Lemma1.1, we have

$$x^{(n-1)}(t) > 0,\ x^{(n)}(t) \leq 0,\ x'(t) > 0,\ t \geq t_3 \geq t_2. \tag{2.3}$$

Defining

$$w(t) = \frac{a(t)\phi(x(t))\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)}{f(x[g(t,a)/2])}, \quad t \geq t_4 \geq t_3.$$

By (H_2), we have $g(t,\xi) \geq g(t,a) \geq \frac{g(t,a)}{2} > 0, g'(t,a) > 0$. Then from (H_3) and (A_1) and using $x(t) > 0$ and $x'(t) > 0$, there exists a $t_5 \geq t_4$, such that

$$f(x[g(t,\xi)]) \geq f(x[g(t,a)]) \geq f(x[g(t,a)/2]) > 0,$$

for all $t \geq t_5$. Using (H_1) and (A_1) and (2.2), it follows that

$$\begin{aligned} w'(t) &\leq -\frac{\rho(t)\int_a^b p(t,\xi)f(x[g(t,\xi)])\,d\sigma(\xi)}{f(x[g(t,a)/2])} + \frac{\rho'(t)w(t)}{\rho(t)} \\ &\quad - \frac{f'(x[g(t,a)/2])x'[g(t,a)/2]g'(t,a)a(t)\phi(x(t))\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)}{2f^2(x[g(t,a)/2])} \\ &\leq -\frac{\rho(t)f(x[g(t,a)])\int_a^b p(t,\xi)\,d\sigma(\xi)}{f(x[g(t,a)/2])} + \frac{\rho'(t)w(t)}{\rho(t)} \\ &\quad - \frac{f'(x[g(t,a)/2])x'[g(t,a)/2]g'(t,a)w^2(t)}{2a(t)\phi(x(t))\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)} \\ &\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{ux'[g(t,a)/2]g'(t,a)w^2(t)}{2a(t)\phi(x(t))\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)} \\ &\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{ux'[g(t,a)/2]g'(t,a)w^2(t)}{2a(t)\varphi(t)\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)}. \end{aligned} \tag{2.4}$$

For $t \geq t_6 \geq t_5$, we have $g(t,a) \leq g(t,\xi) \leq t$. In view of (H_2) and (2.3), we have

$$x^{(n-1)}(t) \leq x^{(n-1)}[g(t,a)].$$

Furthermore by Lemma 1.2, there exists a $t_7 \geq t_6$, $0 < \theta < 1$ such that

$$x'[g(t,a)/2] \geq \frac{\theta [g(t,a)]^{n-2} x^{(n-1)}(t)}{(n-2)!}$$

for all $t \geq t_7$. From (2.4), we conclude that

$$w'(t) \leq -\rho(t) \int_a^b p(t,\xi)\, d\sigma(\xi) + \frac{\rho'(t) w(t)}{\rho(t)} - \frac{u\theta [g(t,a)]^{n-2} g'(t,a) w^2(t)}{2(n-2)! a(t)\varphi(t)\rho(t) |x^{(n-1)}(t)|^{\alpha-1}}. \quad (2.5)$$

Noting that $x^{(n)}(t) \leq 0$, it implies that

$$|x^{(n-1)}(t)|^{\alpha-1} \leq |x^{(n-1)}(T)|^{\alpha-1}, \quad \alpha \geq 1,$$
$$|x^{(n-1)}(t)|^{\alpha-1} \leq |x^{(n-1)}(t)|^{1-\alpha} \leq |x^{(n-1)}(T)|^{1-\alpha}, \quad 0 < \alpha < 1.$$

Hence

$$|x^{(n-1)}(t)|^{\alpha-1} \leq |x^{(n-1)}(T)|^{|\alpha-1|},$$

in which T satisfies $T \geq t_7, t \geq T$. Then from (2.5), we have

$$w'(t) \leq -\rho(t) \int_a^b p(t,\xi)\, d\sigma(\xi) + \frac{\rho'(t) w(t)}{\rho(t)} - \frac{u\theta [g(t,a)]^{n-2} g'(t,a) w^2(t)}{2(n-2)! a(t)\varphi(t)\rho(t) |x^{(n-1)}(T)|^{|\alpha-1|}}.$$

Let $M = \frac{u\theta}{2(n-2)! |x^{(n-1)}(T)|^{|\alpha-1|}} > 0$, then

$$\begin{aligned} w'(t) &\leq -\rho(t) \int_a^b p(t,\xi)\, d\sigma(\xi) + \frac{\rho'(t) w(t)}{\rho(t)} - \frac{M [g(t,a)]^{n-2} g'(t,a) w^2(t)}{a(t)\varphi(t)\rho(t)} \\ &= -\rho(t) \int_a^b p(t,\xi)\, d\sigma(\xi) + \frac{a(t)\varphi(t)[\rho'(t)]^2}{4M\rho(t)[g(t,a)]^{n-2} g'(t,a)} \\ &\quad - \left[\sqrt{\frac{M[g(t,a)]^{n-2} g'(t,a)}{a(t)\varphi(t)\rho(t)}}\, w(t) - \frac{\rho'(t)}{2\rho(t)} \sqrt{\frac{a(t)\varphi(t)\rho(t)}{M[g(t,a)]^{n-2} g'(t,a)}} \right]^2 \end{aligned}$$

$$\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{a(t)\varphi(t)[\rho'(t)]^2}{4M\rho(t)[g(t,a)]^{n-2}g'(t,a)}. \quad (2.6)$$

Integrating both sides of the (2.6) from T to t, we obtain

$$w(t) \leq w(T) - \int_T^t \left[\rho(s)\int_a^b p(s,\xi)\,d\sigma(\xi) - \frac{a(s)\varphi(s)[\rho'(s)]^2}{4M\rho(s)[g(s,a)]^{n-2}g'(s,a)}\right] ds.$$

Therefore

$$\int_T^t \left[\rho(s)\int_a^b p(s,\xi)\,d\sigma(\xi) - \frac{a(s)\varphi(s)[\rho'(s)]^2}{4M\rho(s)[g(s,a)]^{n-2}g'(s,a)}\right] ds \leq w(T) - w(t) \leq w(T) < \infty,$$

which contradicts assumption (2.1).

If $x(t)$ is an eventually negative solution of equation (E), let $y(t) = -x(t)$, then $y(t)$ is an eventually positive solution of equation (E), and equation (E) will transfer the following equation

$$[a(t)\phi(y(t))|y^{(n-1)}(t)|^{\alpha-1}y^{(n-1)}(t)]' + \int_a^b F^*(t,\xi,y[g(t,\xi)])\,d\sigma(\xi) = 0, \quad (E')$$

in which $F^*(t,\xi,y[g(t,\xi)]) = -F(t,\xi,-y[g(t,\xi)])$. By (H_3), we can obtain

$$F^*(t,\xi,-y[g(t,\xi)]) \geq p(t,\xi)\{-f[-y(g(t,\xi))]\} \geq p(t,\xi)f(y[g(t,\xi)]).$$

Then equation (E') satisfies the conditions of Theorem 2.1, and using the above-mentioned method, we can also get a contradiction. This completes the proof of Theorem 2.1.

Theorem 2.2. *Suppose that equation (E) satisfies the condition:*

(A_2) $\frac{f'(x)}{(|f(x)|^{\alpha-1})^{\frac{1}{\alpha}}} \geq k > 0$, *$k$ is a positive constant.*

If there exists a function $\rho(t)$ and a constant θ which be defined as in Theorem 2.1, such that

$$\limsup_{t\to\infty}\int_T^t \left[\rho(s)\int_a^b p(s,\xi)\,d\sigma(\xi) - \frac{\beta a(s)\varphi(s)[\rho'(s)]^{\alpha+1}}{[\rho(s)[g(s,a)]^{n-2}g'(s,a)]^{\alpha}}\right] ds = \infty, \quad (2.7)$$

where $\beta = \frac{1}{\alpha M^\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$, $M = \frac{k\theta}{2(n-2)!} > 0$ is a constant, then equation (E) is oscillatory.

Proof. Suppose to the contrary that equation (E) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution for $t \geq t_0$. From (H_3) and (A_2), we have $f(x) > 0$, $f'(x) > 0$. As in the proof of Theorem 2.1, (2.2) and (2.3) are fulfilled. We define the function $w(t)$ as in the proof of theorem 2.1, then we have

$$\begin{aligned} w'(t) &\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} \\ &\quad - |w(t)|^{\frac{\alpha+1}{\alpha}} \frac{f'(x[g(t,a)/2])x'[g(t,a)/2]g'(t,a)}{2[a(t)\phi(x(t))\rho(t)|f(x[g(t,a)/2])|^{\alpha-1}|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]^{\frac{1}{\alpha}}} \\ &\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} \\ &\quad - |w(t)|^{\frac{\alpha+1}{\alpha}} \frac{k\theta[g(t,a)]^{n-2}g'(t,a)x^{(n-1)}(t)}{2(n-2)![a(t)\phi(x(t))\rho(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]^{\frac{1}{\alpha}}} \\ &\leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} \\ &\quad - |w(t)|^{\frac{\alpha+1}{\alpha}} \frac{k\theta[g(t,a)]^{n-2}g'(t,a)}{2(n-2)![a(t)\varphi(t)\rho(t)]^{\frac{1}{\alpha}}}. \end{aligned} \tag{2.8}$$

Let $M = \frac{k\theta}{2(n-2)!}$, from (2.8) we have

$$w'(t) \leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\rho'(t)w(t)}{\rho(t)} - |w(t)|^{\frac{\alpha+1}{\alpha}} \frac{M[g(t,a)]^{n-2}g'(t,a)}{[a(t)\varphi(t)\rho(t)]^{\frac{1}{\alpha}}}.$$

If we take $X = [M[g(t,a)]^{n-2}g'(t,a)]^{\frac{\alpha}{\alpha+1}} \frac{|w(t)|}{[a(t)\varphi(t)\rho(t)]^{\frac{1}{\alpha+1}}}$, $\lambda = \frac{\alpha+1}{\alpha}$,

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{[a(t)\varphi(t)\rho(t)]^{\frac{\alpha}{\alpha+1}}}{[M[g(t,a)]^{n-2}g'(t,a)]^{\frac{\alpha^2}{\alpha+1}}} \left[\frac{\rho'(t)}{\rho(t)}\right]^{\alpha},$$

according to Lemma 1.3, we obtain

$$\frac{\rho'(t)}{\rho(t)}w(t) - \frac{M[g(t,a)]^{n-2}g'(t,a)|w(t)|^{\frac{\alpha+1}{\alpha}}}{[a(t)\varphi(t)\rho(t)]^{\frac{1}{\alpha}}} \leq \frac{\beta a(t)\varphi(t)[\rho'(t)]^{\alpha+1}}{[\rho(t)[g(t,a)]^{n-2}g'(t,a)]^{\alpha}}.$$

Hence

$$w'(t) \leq -\rho(t)\int_a^b p(t,\xi)\,d\sigma(\xi) + \frac{\beta a(t)\varphi(t)[\rho'(t)]^{\alpha+1}}{[\rho(t)[g(t,a)]^{n-2}g'(t,a)]^{\alpha}}. \tag{2.9}$$

We choose a sufficiently large T, and integrating both sides of the (2.9) from T to $t, (t \geq T \geq t_5)$, we have

$$w(t) \leq w(T) - \int_T^t \left[\rho(s) \int_a^b p(s,\xi)\, d\sigma(\xi) - \frac{\beta a(s)\varphi(s)[\rho'(s)]^{\alpha+1}}{[\rho(s)[g(s,a)]^{n-2}g'(s,a)]^{\alpha}}\right] ds.$$

Thus

$$\int_T^t \left[\rho(s) \int_a^b p(s,\xi)\, d\sigma(\xi) - \frac{\beta a(s)\varphi(s)[\rho'(s)]^{\alpha+1}}{[\rho(s)[g(s,a)]^{n-2}g'(s,a)]^{\alpha}}\right] ds$$
$$\leq w(T) - w(t) < w(T) < \infty,$$

which contradicts assumption (2.7).

The case of $x(t)$ is an eventually negative solution of equation (E) can be proved by the same argument of Theorem 2.1. This completes the proof of Theorem 2.2. □

The following two examples will illustrate our theory.

Example 2.1. Consider the even order half-linear differential equation

$$[|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + \int_{\frac{\sqrt{3}}{3}}^{1} tx(t\xi)\sqrt{1+x^2(t\xi)}\arctan\xi\, d\xi = 0,$$
$$t \geq 1,\ n > 2,$$

where $a = \frac{\sqrt{3}}{3}$, $b = 1$, $a(t) = \phi(x(t)) = 1$, $g(t,\xi) = t\xi$, $\sigma(\xi) = \xi$, $F(t,\xi,x) = tx(t\xi)\sqrt{1+x^2(t\xi)}\arctan\xi$. Choosing

$$p(t,\xi) = \pi/6, \quad f(x) = x(t\xi)\sqrt{1+x^2(t\xi)}, \quad \rho(t) = t^2, \varphi(t) = 1.$$

Then

$$F(t,\xi,x)\operatorname{sgn} x = tx(t\xi)\sqrt{1+x^2(t\xi)}\arctan\xi\ \operatorname{sgn} x$$
$$\geq \frac{\pi}{6}x(t\xi)\sqrt{1+x^2(t\xi)}\operatorname{sgn} x = p(t,\xi)f(x)\operatorname{sgn} x,$$

$$f'(x) = \frac{1}{2}[x^2(t\xi) + x^4(t\xi)]^{-1/2}[2\xi x(t\xi) + 4\xi x^3(t\xi)] = \frac{\xi[1+2x^2(t\xi)]}{\sqrt{1+x^2(t\xi)}} > \xi.$$

The conditions $(H_1) - (H_4)$ and (A_1) are satisfied. Furthermore

$$\limsup_{t\to\infty} \int_T^t \left[\rho(s) \int_a^b p(s,\xi)\, d\sigma(\xi) - \frac{a(s)\varphi(s)[\rho'(s)]^2}{4M\rho(s)[g(s,a)]^{n-2}g'(s,a)}\right] ds$$

$$= \limsup_{t\to\infty} \int_T^t \left[s^2 \int_{\sqrt{3}/3}^1 \frac{\pi}{6}\, d\xi - \frac{4s^2}{4Ms^2(\frac{\sqrt{3}}{3}s)^{n-2}\frac{\sqrt{3}}{3}} \right] ds$$

$$= \limsup_{t\to\infty} \left[\frac{\pi}{18}(1-\frac{\sqrt{3}}{3})(t^3-T^3) - \frac{1}{M(\frac{\sqrt{3}}{3})^{n-1}(3-n)}(t^{3-n}-T^{3-n}) \right]$$

$$= \infty.$$

The condition (2.1) is satisfied. Then all conditions of Theorem 2.1 are satisfied, so the equation is oscillatory.

Example 2.2. Consider the even order half-linear differential equation

$$[|x^{(n-1)}(t)|^2 x^{(n-1)}(t)]' + \int_{\frac{\sqrt{3}}{3}}^1 (t\xi)^\lambda x^3(t\xi) \arctan \xi \, d\xi = 0, \quad t \geq 1,$$

where $\alpha = 3$, $a(t) = \phi(x(t)) = 1$, $a = \frac{\sqrt{3}}{3}$, $b = 1, g(t,\xi) = t\xi$, $F(t,\xi,x) = (t\xi)^\lambda x^3(t\xi) \arctan \xi$, $\lambda \geq 1$. Choosing $p(t,\xi) = \pi/6$, $f(x) = x^3(t\xi)$, $\rho(t) = t^2, \varphi(t) = 1$. Then

$$F(t,\xi,x) \operatorname{sgn} x = (t\xi)^\lambda x^3(t\xi) \arctan \xi \operatorname{sgn} x > \frac{\pi}{6} x^3(t\xi) \operatorname{sgn} x$$
$$= p(t,\xi) f(x) \operatorname{sgn} x,$$

$$xf(x) = x^4(t\xi) > 0, \ \ (x \neq 0), \ \ \int_{t_0}^t \frac{1}{a(s)\varphi(s)} ds = t - t_0 = \infty \ \ (t \to \infty),$$

$$\frac{f'(x)}{(|f(x)|^{\alpha-1})^{\frac{1}{\alpha}}} = \frac{3\xi x^2(t\xi)}{|x(t\xi)|^2} = 3\xi \geq \sqrt{3}.$$

The conditions $(H_1) - (H_4)$ and (A_2) are satisfied. Furthermore

$$\limsup_{t\to\infty} \int_T^t \left[\rho(s) \int_a^b p(s,\xi)\, d\sigma(\xi) - \frac{\beta a(s)\varphi(s)[\rho'(s)]^{\alpha+1}}{[\rho(s)[g(s,a)]^{n-2} g'(s,a)]^\alpha} \right] ds$$

$$= \limsup_{t\to\infty} \int_T^t \left[s^2 \int_{\sqrt{3}/3}^1 \frac{\pi}{6}\, d\xi - \frac{16\beta s^4}{[s^2(\frac{\sqrt{3}}{3}s)^{n-2}\frac{\sqrt{3}}{3}]^3} \right] ds$$

$$= \limsup_{t\to\infty} \left[\frac{\pi}{18}(1-\frac{\sqrt{3}}{3})(t^3-T^3) - \frac{16\beta}{(\frac{\sqrt{3}}{3})^{3n-3}(5-3n)}(t^{5-3n}-T^{5-3n}) \right]$$

$$= \infty.$$

The condition (2.7) is satisfied. Then all conditions of Theorem 2.2 are satisfied, so the equation is oscillatory by Theorem 2.2.

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