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GLOBAL CLASSICAL SOLUTIONS ON A SEMI-BOUNDED INITIAL AXIS FOR QUASILINEAR HYPERBOLIC SYSTEMS

Ping Yan
College of Mathematics and System Science
Xinjiang University
Urumqi, 830046, P.R. CHINA
e-mail: yanp2002cn@163.com

Abstract: In this paper, by re-constructing the continuous Glimm functional, we consider global classical solutions to the Cauchy problem with initial data given on a semi-bounded axis for quasilinear hyperbolic systems without assuming the systems to be weakly linearly degenerate. An application is given for the system of traffic flow.

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1. Introduction

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By the definition of strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u). \tag{1.2}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) =$

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$(r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \text{ (resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det |l_{ij}(u)| \neq 0 \qquad (\text{resp. } \det |r_{ij}(u)| \neq 0). \tag{1.4}$$

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.5}$$

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

Then all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

For the Cauchy problem of system (1.1) with the initial data

$$t = 0 : u = \varphi(x) \quad (-\infty < x < +\infty), \tag{1.7}$$

where $\varphi(x)$ is a C^1 vector function with bounded C^1 norm. If $\varphi(x)$ has small total variation, it was proved in [3] and [4] that, for a strictly hyperbolic system or a hyperbolic system with eigenvalues with constant multiplicity, Cauchy problem (1.1) and (1.7) admits a unique global C^1 solution for all $t \in \mathbb{R}$, provided that system (1.1) is weakly linearly degenerate (WLD) (cf. [7]), i.e. all the eigenvalues are WLD. The result on the global existence of the C^1 solution may be incorrect if system (1.1) is not WLD.

In this paper we will present the global existence of the C^1 solution to the Cauchy problem only assuming that the leftmost or rightmost eigenvalue is WLD.

For the Cauchy problem of system (1.1) with the following initial data

$$t = 0 : u = \varphi(x) \quad (0 \leq x < +\infty), \tag{1.8}$$

if the initial data with small C^1 norm are delaying, Li Ta-Tsien et al have proved the global existence of the C^1 solution to cauchy problem (1.1) and (1.8) (cf. [8]). We assume that the initial data belong to another kind of function whose C^1 norm is not small any more, the main result in this paper is the following

Theorem 1.1. *Suppose that in a neighborhood of $u = 0$, $A(u) \in C^2$ and system (1.1) is strictly hyperbolic. For $i = 1, \cdots, n-1$, we suppose that there exists a constant $b \geq 0$ such that*

$$\nabla\lambda_i(u) \cdot r_i(u) = O(|u|^{b+1}). \tag{1.9}$$

Suppose furthermore that $\lambda_n(u)$ is WLD. Suppose finally that the initial data satisfy the following properties:

(i) $\varphi(x) \in C^1$ has compact support: supp $\varphi \subseteq [0, \alpha_0]$, where $\alpha_0 > 0$;

(ii) The initial total variation is small enough, namely,

$$\theta \stackrel{\triangle}{=} \int_0^{+\infty} |\varphi'(x)| dx \ll 1. \tag{1.10}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (1.1) and (1.8) admits a unique global C^1 solution $u = u(t,x)$ on the domain $D = \{(t,x) | t \geq 0,\ x \geq x_n(t)\}$, where $x = x_n(t)$ is the n-th characteristic passing through the origin O(0,0):

$$\begin{cases} \dfrac{dx_n(t)}{dt} = \lambda_n(u(t, x_n(t))), \\ x_n(0) = 0. \end{cases} \tag{1.11}$$

On the other hand, under the assumption that in a neighborhood of $u = 0$, $A(u) \in C^1$ and system (1.1) is strictly hyperbolic, if for any given $\varphi(x)(x \geq 0)$ satisfies the condition (i)-(ii), Cauchy problem (1.1) and (1.8) always admits a unique global C^1 solution $u = u(t,x)$ on the domain D, Then $\lambda_n(u)$ must be WLD.

Remark 1.1. Suppose that in a neighborhood of $u = 0$,

$$\lambda_1(u) < \cdots < \lambda_p(u) < \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u), \tag{1.12}$$

where $\lambda(u) \stackrel{\triangle}{=} \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u)$ is an eigenvalue with constant multiplicity $n - p$. Suppose furthermore that $\lambda_{p+1}(u), \cdots, \lambda_n(u)$ are WLD and (1.9) holds for $i = 1, \cdots, p$. Then the conclusions of Theorem 1.1 are still valid.

Remark 1.2. Suppose that in a neighborhood of $u = 0$, $\lambda_1(u)$ is WLD and (1.9) holds for $i = 2, \cdots, n$, or

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{1.13}$$

where $\lambda_1(u), \cdots, \lambda_p(u)$ are WLD and (1.9) holds for $i = p+1, \cdots, n$. For the initial data

$$t = 0 : u = \varphi(x) \quad (x \leq 0) \tag{1.14}$$

satisfies the following:

(i) $\varphi(x) \in C^1$ has compact support: *supp* $\varphi \subseteq [-\alpha_0, 0]$, where $\alpha_0 > 0$.

(ii) The initial total variation is small enough, namely,

$$\theta \triangleq \int_{-\infty}^{0} |\varphi'(x)| dx << 1. \tag{1.15}$$

Then similar results hold as in Theorem 1.1 and Remark 1.1.

In Section 2 we give some preliminaries. Then, the main result is proved in Section 3. Finally, an application of Theorem 1.1 is given in Section 4.

2. Preliminaries

For general quasilinear hyperbolic systems, by the proof of Lemma 2.5 in [9], for any given complete system of right eigenvectors $r_1(u), \cdots, r_n(u)$ of $A(u)$, there exists a suitably smooth invertible transformation $u = u(\widetilde{u})$ $(u(0) = 0)$ such that in the $\widetilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,

$$\widetilde{r}_i(\widetilde{u}_i e_i) // e_i, \quad \forall\ |\widetilde{u}_i| \quad (i = 1, \cdots, n), \tag{2.1}$$

where $\widetilde{r}_i(\widetilde{u})$ denotes the i-th right eigenvector corresponding to $r_i(u)$ and

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T.$$

This transformation is called a *generalized normalized transformation*, and the unknown variables $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)^T$ are called *generalized normalized variables* or *generalized normalized coordinates*. Without loss of generality, we suppose that

$$\widetilde{r}_i^T(\widetilde{u}) \widetilde{r}_i(\widetilde{u}) \equiv 1. \tag{2.2}$$

Then, (2.1) can be written as

$$\widetilde{r}_i(\widetilde{u}_i e_i) = e_i, \quad \forall\ |\widetilde{u}_i| \quad (i = 1, \cdots, n). \tag{2.3}$$

Let

$$v_i = l_i(u) u \quad (i = 1, \cdots, n), \tag{2.4}$$

$$w_i = l_i(u) u_x \quad (i = 1, \cdots, n). \tag{2.5}$$

By (1.5), we have

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.6}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.7}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.8}$$

denotes the directional derivative with respect to t along the i-th characteristic. We have (cf. [9] or [2])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \cdots, n), \tag{2.9}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.10}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall\ i, j. \tag{2.11}$$

Moreover, in the corresponding generalized normalized coordinates we have

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall\ |u_j|, \quad \forall i, j. \tag{2.12}$$

Noting (2.7), by (2.9) we have

$$d[v_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k dt \wedge dx, \tag{2.13}$$

where

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla \lambda_i(u) r_k(u) \delta_{ij}. \tag{2.14}$$

By (2.11), it is easy to see that

$$B_{iji}(u) \equiv 0, \quad \forall\ j \neq i \tag{2.15}$$

and

$$B_{iii}(u) = \nabla \lambda_i(u) r_i(u), \quad \forall\ i. \tag{2.16}$$

Moreover, by (2.12), in the corresponding generalized normalized coordinates we have

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall\ |u_j|, \quad \forall\ j \neq i. \tag{2.17}$$

On the other hand, we have (cf. [9] or [2])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \cdots, n), \tag{2.18}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik} + (j|k)\}, \tag{2.19}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\ j \neq i \tag{2.20}$$

and

$$\gamma_{iii}(u) = -\nabla \lambda_i(u) r_i(u) \quad (i = 1, \cdots, n). \tag{2.21}$$

Noting (2.7), by (2.18) we have

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dt \wedge dx, \tag{2.22}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u) [\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.23}$$

Hence

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall\ i, j. \tag{2.24}$$

Using Stokes formula, by (2.13) and (2.22) it is easy to prove the following lemma.

Lemma 2.1. *Suppose that $u = u(t, x)$ is a C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and $\mathbb{D}$ is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+. Then*

$$\int_{\tau_1} |v_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |v_i(dx - \lambda_i(u)dt)| + \iint_{\mathbb{D}} |\sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k| dt dx\,, \tag{2.25}$$

and

$$\int_{\tau_1} |w_i(dx - \lambda_i(u)dt)| \leq \int_{\tau_2} |w_i(dx - \lambda_i(u)dt)| + \iint_{\mathbb{D}} |\sum_{\substack{j,k=1 \\ j\neq k}}^{n} \Gamma_{ijk}(u)w_j w_k|dtdx. \quad (2.26)$$

3. Proof of Theorem 1.1

Without loss of generality, we prove our result in generalized normalized coordinates.

We first prove the necessity – the second part of Theorem 1.1.

By the definition of the weak linear degeneracy, the n-th eigenvalue $\lambda_n(u)$ is weakly linearly degenerate iff $\lambda_n(u_n e_n) \equiv \lambda_n(0),\ \forall\ |u_n|$. By means of the method in [5], [6], we can immediately get the necessity in Theorem 1.1.

Next, we prove the sufficiency – the first part of Theorem 1.1.

We may assume that

$$0 < \lambda_1(0) < \cdots < \lambda_n(0). \quad (3.1)$$

Then there exist positive constants δ_0 and δ so small that

$$\lambda_{i+1}(u) - \lambda_i(u) \geq 4\delta_0 \quad \forall |u| \leq \delta\ (i = 1, \cdots, n-1) \quad (3.2)$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2},\ \forall |u|, |v| \leq \delta\ (i = 1, \cdots, n). \quad (3.3)$$

By (3.1), without loss of generality, we may also assume that

$$\lambda_i(0) \geq \delta_0 \quad (i = 1, \cdots, n). \quad (3.4)$$

By (3.3)-(3.4), it is easy to have

$$x_n(t) \geq (\lambda_n(0) - \frac{\delta_0}{2})t \geq \frac{\delta_0}{2}t, \quad (3.5)$$

where $x = x_n(t)$ is the n-th characteristic passing through the origin $O(0,0)$:

$$\begin{cases} \dfrac{x_n(t)}{dt} = \lambda_n(u(t, x_n(t))), \\ x_n(0) = 0. \end{cases}$$

For any given $T > 0$, let

$$D^T = \{(t,x)|0 \le t \le T,\ x \ge x_n(t)\}. \tag{3.6}$$

Suppose that $u = u(t,x)$ is the C^1 solution to Cauchy problem (1.1) and (1.8) on the domain D^T. Let

$$L(t) = \sum_{i=1}^{n} \int_{x_n(t)}^{+\infty} |w_i(t,x)|dx \quad (0 \le t \le T) \tag{3.7}$$

and

$$Q(t) = \sum_{i>j} \iint_{x_n(t)<x<y<+\infty} |w_i(t,x)||w_j(t,y)|dxdy \quad (0 \le t \le T). \tag{3.8}$$

Then it is easy to see that

$$\frac{dL(t)}{dt} \le -\sum_{i=1}^{n}(\lambda_n(u) - \lambda_i(u))|w_i(t,x_n(t))| + \sum_{i>j} \int_{x_n(t)}^{+\infty} (\lambda_i(u) - \lambda_j(u))|w_j(t,x)w_k(t,x)|dx\,, \tag{3.9}$$

and

$$\frac{dQ(t)}{dt} \le -(1 - n \cdot L(t)) \sum_{i>j} \int_{x_n(t)}^{+\infty} (\lambda_i(u) - \lambda_j(u))|w_i(t,x)||w_j(t,x)|dx. \tag{3.10}$$

By (3.9)-(3.10), there must exist a positive constant M so big that $F(t) = L(t) + MQ(t)$ is non-increasing with respect to t, namely

$$\frac{dF(t)}{dt} \le 0, \quad \forall\, t \in [0,T], \tag{3.11}$$

provided that $L(t)$ is small enough. Using (1.10) and (2.7), it is easy to get the following lemma.

Lemma 3.1. *Suppose that in a neighborhood of $u = 0$, system (1.1) is hyperbolic, $A(u) \in C^2$ has eigenvalues with constant multiplicity and (1.5)-(1.6) hold. Suppose furthermore that $u = u(t,x)$ is the C^1 solution to Cauchy problem (1.1) and (1.8) on the domain D^T. Suppose finally that (1.10) holds. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0,\theta_0]$, there exist two* *positive constants κ_1 and κ_2 independent of θ and T, such that the following uniform estimates hold:*

$$\|u(t,\cdot)\|_{C^0} = \sup_{x\in\mathbb{R}} |u(t,x)| \le \kappa_1\theta,\ \forall t\in[0,T] \tag{3.12}$$

and

$$L(t) \le \kappa_1\theta + \kappa_2\theta^2,\ \forall t\in[0,T]. \tag{3.13}$$

Remark 3.1. By (3.12), there exists a positive constant δ such that on any existence domain D^T of the C^1 solution $u = u(t,x)$, we have

$$|u(t,x)| \le \delta, \tag{3.14}$$

provided that $\theta_0 > 0$ is small enough. This is the uniform a priori estimate on the C^0 norm of the C^1 solution $u = u(t,x)$.

Noting (3.14), by (3.1), it is easy to see that when $\delta > 0$ is small enough on the domain D^T of the C^1 solution $u = u(t,x)$, we have

$$\delta_0 \le \lambda_1(u) < \cdots < \lambda_n(u). \tag{3.15}$$

For any given constant $\mu > 0$, on the existence domain D^T, let

$$V_\infty^c(T) = \max_{i=1,\cdots,n-1} \sup_{(t,x)\in D^T} \{(1+t)^{1+\mu}|v_i(t,x)|\}, \tag{3.16}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n-1} \sup_{(t,x)\in D^T} \{(1+t)^{1+\mu}|w_i(t,x)|\}, \tag{3.17}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n-1} \sup_{(t,x)\in D^T} \{(1+t)^{1+\mu}|u_i(t,x)|\}, \tag{3.18}$$

$$\widetilde{W}_1(T) = \max_{j=1,\cdots,n} \max_{i\neq j} \sup_{c_i} \int_{c_i} |w_j(t,x)|dt, \tag{3.19}$$

where c_i denotes any given i-th characteristic on D^T,

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\ge 0}} |u_i(t,x)|, \tag{3.20}$$

$$V_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\ge 0}} |v_i(t,x)|, \tag{3.21}$$

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\ge 0}} |w_i(t,x)|. \tag{3.22}$$

Remark 3.2. By (3.12) and noting (2.4), there exists a positive constant C_0 independent of θ and T such that

$$U_\infty(T), \quad V_\infty(T) \leq C_0\theta. \tag{3.23}$$

Lemma 3.2. *Suppose that in a neighborhood of $u = 0$, $A(u) \in C^2$ and system (1.1) is strictly hyperbolic. Suppose furthermore that $\varphi(x)$ satisfies the assumptions given in Theorem 1.1. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain D^T of the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.8), we have the following uniform a priori estimates:*

$$\widetilde{W}_1(T) \leq \kappa_3\theta, \tag{3.24}$$

$$U^c_\infty(T),\ V^c_\infty(T) \leq \kappa_4\theta, \tag{3.25}$$

$$W^c_\infty(T) \leq \kappa_5, \tag{3.26}$$

henceforth, κ_i $(i = 3, 4, 5, \cdots)$ are positive constants independent of θ and T.

Proof. First of all, on any existence domain D^T of the C^1 solution $u = u(t, x)$, suppose that there exists a positive constant κ such that

$$\|w(t, x)\|_{C_0} \leq \kappa, \quad \forall t \in [0, T]. \tag{3.27}$$

At the end of the proof of Lemma 3.3, we shall explain that this hypothesis is reasonable.

Noting that the initial data possess compact support, it is sufficient to estimate the integral on the right hand side of (3.19) in a finite time interval $0 \leq t \leq T_0$.

For $j = 1, \cdots, n$, let c_i $(i \neq j)$ be the i-th characteristic in D^T passing through point (t, x) which intersects x-axis and $x = x_n(t)$ (or $t = T_0$) at points $(0, x^0_i)$ and $(t_0, x_i(t_0))(t_0 \leq T_0)$, respectively. Noting (3.2) and (3.14), by Lemma 2.1, (3.13) and (3.27), we have

$$\begin{aligned}\int_{c_i} |w_j(t, x)|dt \leq C_1\{\int_{x^0_i}^{x_i(t_0)} |w_j(t, x)|dt \\ + \int_0^{T_0}\int_{-\infty}^{+\infty}\sum_{k\neq l} |\Gamma_{jkl}w_kw_l(t, x)|dxdt\} \leq C_2(1 + \kappa T_0)\theta \leq C_3\theta,\end{aligned} \tag{3.28}$$

henceforth $C_i(i = 1, 2, \cdots)$ denote positive constants independent of θ and T. Then we can choose $\kappa_3 \geq C_3$ to get (3.24).

Next we estimate $V^c_\infty(T)$ and $W^c_\infty(T)$.

For any given point $(t, x) \in D^T$, and for $i = 1, \cdots n-1$, we draw the i-th characteristic passing through point (t, x) which intersects the x-axis at point $(0, x_i)$. By (3.6), if $x_i \leq \alpha_0$, α_0 is defined in Theorem 1.1, it is easy to see that the i-th characteristic must intersects the curve $x_n(t)$ at some point $(\bar{t}, \bar{x})$ satisfying $\bar{t} \leq T_0$. Integrating (2.9) and (2.18) along this i-th characteristic, we get

$$v_i(t, x) = v_i(0, x_i) + \int_0^t \sum_{\substack{j,k=1 \\ k \neq i}}^n \beta_{ijk}(u) v_j w_k d\tau, \tag{3.29}$$

$$w_i(t, x) = w_i(0, x_i) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k d\tau. \tag{3.30}$$

Multiplying $(1+t)^{(1+\mu)}$ on both sides of (3.29), and noting $0 \leq t \leq T_0$ and (3.14), by (3.23)-(3.24), we have

$$(1+t)^{(1+\mu)} |v_i(t, x)| \leq C_4 \theta (1 + \kappa_3 \theta) \leq 2C_6 \theta. \tag{3.31}$$

At the same time, by (2.6), it is easy to get

$$U_\infty^c(T) \leq 2C_4 \theta. \tag{3.32}$$

Then we can choose $\kappa_4 \geq 2C_4$ to get (3.25).

Multiplying $(1+t)^{(1+\mu)}$ on both sides of (3.30), and noting $0 \leq t \leq T_0$, by (1.9), (2.20) and (3.23)-(3.24), we have

$$|\gamma_{iii}(u)| \leq C_5 U_\infty$$

and

$$\begin{aligned}(1+t)^{(1+\mu)} |w_i(t, x)| \leq & C_6 (1 + \widetilde{W}_1(T) W_\infty^c(T) + C_5 U_\infty \kappa W_\infty^c(T)) \\ \leq & C_7 (1 + (\theta + \theta\kappa) W_\infty^c(T)).\end{aligned} \tag{3.33}$$

If $\theta > 0$ is so small that $C_7(\theta + \theta\kappa) < \frac{1}{2}$, from (3.33) we get

$$W_\infty^c(T) \leq 2C_7. \tag{3.34}$$

Hence we can choose $\kappa_5 \geq 2C_7$ independent of κ to get (3.26), provided that $\theta > 0$ is small enough.

The proof of Lemma 3.2 is finished. □

Lemma 3.3. *Under the assumptions of the first part of Theorem 1.1, in the corresponding generalized normalized coordinates. Then there exists* $\theta_0 > 0$

so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain D^T of the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.8), we have the following uniform a priori estimates:

$$W_\infty(T) \le \kappa_6. \tag{3.35}$$

Proof. For any given point $(t, x) \in D^T$, we draw the n-th characteristic passing through point (t, x) which intersects the x-axis at point $(0, x_0)$. Integrating (2.8) and (2.17) along this n-th characteristic, we get

$$\begin{aligned} w_n(t,x) = w_n(0,x_i) &+ \int_0^t \sum_{j,k=1}^{n-1} \gamma_{njk}(u) w_j w_k(\tau, x_n(\tau)) d\tau \\ &+ 2\int_0^t \sum_{j=1}^{n-1} \gamma_{njn}(u) w_j w_n(\tau, x_n(\tau)) d\tau + \int_0^t \gamma_{nnn}(u) w_n^2(\tau, x_n(\tau)) d\tau. \end{aligned} \tag{3.36}$$

Since $\lambda_n(u)$ is WLD, by (2.21), we have

$$\gamma_{nnn}(u_n e_n) \equiv 0. \tag{3.37}$$

For Hadarmard's formula, we have

$$\begin{aligned} &\gamma_{nnn}(u) - \gamma_{nnn}(u_n e_n) \\ &\qquad = \int_0^1 \sum_{j \ne n} \frac{\partial \gamma_{nnn}}{\partial u_j}(su_1, \cdots, su_{j-1}, u_j, su_{j+1}, \cdots, su_n) u_j ds. \end{aligned} \tag{3.38}$$

Noting $\varphi(x)$ is a C^1 function with compact support, using (3.25)-(3.26), we get

$$|w_n(t,x)| \le C_8(1 + (\widetilde{W}_1(T))^2 + \widetilde{W}_1(T) W_\infty(T) + W_\infty^2(T) U_\infty^c(T)). \tag{3.39}$$

Hence, we have

$$W_\infty(T) \le C_8 + C_9 \theta (W_\infty(T) + W_\infty^2(T)), \tag{3.40}$$

provided $C_8 \ge \kappa_5$.

Using the continuous induct, we can get

$$W_\infty(T) \le 2C_8, \tag{3.41}$$

provided that $\theta > 0$ is small enough.

Hence we can choose $\kappa_6 \ge 2C_8$ to get (3.35).

The proof of Lemma 3.3 is finished. □

Remark 3.4. By the choice of κ_6, we can choose κ satisfies $\kappa \geq 2\kappa_6$. Then by (3.35), we have $\|w(t,x)\|_{C_0} \leq \kappa_6 \leq \frac{1}{2}\kappa, \quad \forall t \in [0,T]$. So the hypothesis (3.27) is reasonable.

Proof of Theorem 1.1. By Lemma 3.1 and 3.3, it is easy to see that on any given existence domain D^T of the C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.8), we have the following uniform a priori estimate:

$$\|u(t,\cdot)\|_{C^1} \overset{\triangle}{=} \|u(t,\cdot)\|_{C^0} + \|u_x(t,\cdot)\|_{C^0} \leq C. \tag{3.43}$$

Hence, Cauchy problem (1.1) and (1.8) admits a unique C^1 solution $u = u(t,x)$ with bounded C^1 norm on the domain D^T .

This proves Theorem 1.1. □

4. Application

Consider the following Cauchy problem for the system of traffic flow (cf. [1])

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t (v + p(\rho)) + v\partial_x(v + p(\rho)) = 0, \end{cases} \tag{4.1}$$

with the initial data

$$t = 0 : (\rho, v) = (\tilde{\rho}_0 + \rho_0(x), \tilde{v}_0 + v_0(x)) \quad (x \geq 0), \tag{4.2}$$

where ρ and v are the density and velocity of cars at point x and time t respectively, $p(\cdot)$ is a suitably smooth and strictly increasing function and satisfies the following

$$p'(\rho) = O(\rho^{1+\alpha}), \tag{4.3}$$

where $\alpha \geq 0$ is a constant. $\tilde{\rho}_0$ and $\tilde{v}_0$ are constants, $(\rho_0(x), v_0(x)) \in C^1$ satisfies the assumptions in Theorem 1.1.

Let $U = (\rho, v)^T$. It is easy to see that when $\rho > 0$, system (4.1) is always a strictly hyperbolic system with the following real eigenvalues: $\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v$. The right eigenvectors corresponding the eigenvalues are the following: $r_1(U) = (1, -p'(\rho))^T, \quad r_2(U) = (1, 0)^T$. It is easy to see that $\lambda_2(U)$ is linearly degenerate, then weakly linearly degenerate. We also get from (4.3) that $\nabla\lambda_1(U) \cdot r_1(U) = O(\rho^{1+\alpha})$.

By Theorem 1.1 we have the following result.

Theorem 4.1. *Suppose that $p(\cdot)$ is a suitably smooth and strictly increasing function and satisfies (4.3). Suppose furthermore that $(\rho_0(x), v_0(x)) \in C^1$ satisfies the assumptions in Theorem 1.1. Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, Cauchy problem (4.1)-(4.2) admits a unique global C^1 solution $U = U(t,x)$ on the domain $D = \{(t,x)|t \geq 0, x \geq x_2(t)\}$, where $x = x_2(t)$ is the second characteristic passing through the origin $O(0,0)$.*

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