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OPTIMALITY CONDITIONS AND DUALITY IN NONLINEAR PROGRAMMING INVOLVING SEMI-LOCALLY B-SEMI-PREINVEX AND RELATED FUNCTIONS

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Abstract: A nonlinear programming problem is considered where the functions involved are semi-local differentiable. Necessary optimality conditions and sufficient optimality conditions are given. Mixed type duality results are formulated in terms of semi-local differentiable. These results are given by using the concepts of semi-locally B-semi-preinvex and related functions. Our results generalize the ones obtained by [4], [3], [5], [6].

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1. Introduction

Convexity and generalized convexity play a key role in many aspects of optimization. So the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. During the past

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several decades, various significant generalizations of convexity have been presented. Yang and Chen [8] introduced the semi-preinvex function, which includes the classes of preinvex functions and arc-connected convex functions, Stancu-Minasian [4] brought forward the semi-local B-preinvex functions by extending semi-local convex function and B-preinvex function.

In this paper, we introduce a wider class of nonconvex function – semi-local B-semi-preinvex – which includes the semipreinvex and semi-local B-preinvex functions and study some optimality conditions and duality theorems for the nonlinear programming under the semi-local B-semipreinvex and related functions. Our results generalize the ones obtained by [4], [3], [5], [6].

2. Preliminaries and Definitions

Definition 2.1. A set K is said to satisfy the "semi-local-connected" property, where $K \subseteq R^n$, if for any $x, y \in K$, there exist a maximum positive number $a(x,y) \le 1$ and a vector function $\eta(y,x,\lambda) : K \times K \times [0,1] \to R^n$ such that $x + \lambda\eta(y,x,\lambda) \in K, \forall \lambda \in (0, a(x,y))$.

Definition 2.2. Let K be a set in R^n having the "semi-local-connected" property with $\eta(y,x,\lambda) : K \times K \times [0,1] \to R^n$ and $f(x)$ be a real function on K. Then f is called as semi-local B-semi-preinvex with respect to $\eta(y,x,\lambda)$ if there exist a positive number $d(x,y) \le a(x,y)$ and a function $b : K \times K \times [0,1] \to R_+$ such that $b(x,y,\lambda)$ is continuous at $\lambda = 0$ and

$$f(y + \lambda\eta(y,x,\lambda)) \le \lambda b(x,y,\lambda) f(y) + (1 - \lambda b(x,y,\lambda)) f(x), \forall 0 < \lambda < d(x,y),$$

$$\lambda b(y,x,\lambda) \le 1, \lim_{\lambda \to 0^+} \lambda\eta(y,x,\lambda) = 0.$$

Definition 2.3. A real function f is said to be semi-local quasi B-semi-preinvex on K, where $K \subseteq R^n$ is a set which satisfies "semi-local-connected " property with $\eta(y,x,\lambda) : K \times K \times [0,1] \to R^n$, if for any $x, y \in K$, there exists a positive number $d(x,y) \le a(x,y)$ and a function $b : K \times K \times [0,1] \to R_+$such that $b(x,y,\lambda)$ is continuous at $\lambda = 0$ and

$$f(x) \le f(y)\,,$$

$$0 < \lambda < d(x,y), \quad \lambda b(y,x,\lambda) \le 1, \quad \lim_{\lambda \to 0^+} \lambda\eta(y,x,\lambda) = 0$$

imply that

$$b(x,y,\lambda) f(y + \lambda\eta(y,x,\lambda)) \le b(x,y,\lambda) f(y).$$

Definition 2.4. Let $f : K \to R$ be a function, where $K \subseteq R^n$, we say that f is semi-local differentiable at $x \in K$ if

$$(df)^+(x,h) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}[f(x + w(\lambda)) - f(x)]$$

exists for each function $w(\lambda) : [0,1] \to R^n$ such that $w(0) = 0, w'(0) = h$.

Definition 2.5. Let $f : K \to R$ be a semi-local differentiable function and $\frac{d}{d\lambda}(\lambda\eta(y,x,\lambda)) \mid_{\lambda=0} = \widehat{\eta}(y,x)$. We say that f is semi-local pseudo B-semi-preinvex if $(df)^+(x,\eta(\widehat{y},x)) \geq 0$, $\lim_{\lambda \to 0^+} \lambda\eta(y,x,\lambda) = 0 \Longrightarrow \bar{b}(x,y)f(x) \geq \bar{b}(x,y)f(y)$, where $K \subseteq R^n$ have the "semi-local-connected" property with $\eta(y,x,\lambda) : K \times K \times [0,1] \to R^n$, and $b(x,y,\lambda)$ is continuous at $\lambda = 0$ and $\bar{b}(x,y) = \lim_{\lambda \to 0^+} b(x,y,\lambda)$.

Remark 2.1. These definitions above generalize the corresponding ones in [8] and [4].

Definition 2.6. (see [1]) A function $f : X^0 \to R^k$ is a convex-like function if for any $x, y \in X^0$ and $0 \leq \lambda \leq 1$, there is $z \in X^0$ such that $f(z) \leq \lambda f(x) + (1-\lambda)f(y)$.

Lemma 2.1. (see [2]) *Let S be a nonempty set in R^n and $\varphi : S \to R^k$ be a convex-like function, then $\varphi(x) < 0$ has a solution $x \in S$, or $\lambda^T\varphi(x) \geq 0$ for all $x \in S$ and some $\lambda \in R^k, \lambda \geq 0$ and $\lambda \neq 0$, but both alternatives are never true.*

3. Necessary Optimality Conditions

Consider the nonlinear programming problem

$$(\mathrm{NP}) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i = 1,2,\ldots,m, \\ & x \in X^0. \end{array}$$

where $X^0 \subseteq R^n$ is a nonempty set which satisfy the "semi-local-connected " property and $f : X^0 \to R$ is a semi-local differentiable function.

Let $X = \{x \in X^0 \mid g_i(x) \leq 0, i = 1,2,\ldots,m\}$ be the set of all feasible solutions to (NP).

For $\bar{x} \in X$, denote $I = \{i \mid g_i(\bar{x}) = 0, i = 1,2,\ldots,m\}$, $J = \{i \mid g_i(\bar{x}) < 0, i = 1,2,\ldots,m\}$.

Definition 3.1. We say g satisfies the *generalized Slater's constraint qualification* (GSCQ) at $\bar{x} \in X$, if g_I is semi-local pseudo B-semi-preinvex at $\bar{x}$ and there exists $\widehat{x} \in X$ such that $g_I(\widehat{x}) < 0$.

Lemma 3.1 *Let $\bar{x} \in X$ be a local minimum solution to (NP), we assume that g_i is continuous at $\bar{x}$ for any $i \in J$, and f, g_I are semi-local differentiable at $\bar{x}$ satisfying*

$$\frac{d(\lambda\eta(y,x,\lambda))}{d\lambda}\,|_{\lambda=0}= \widehat{\eta}(y,x)$$

for any $x, y \in X$. Then the system

$$\begin{cases} (df)^+(\bar{x}, \widehat{\eta}(x,\overline{x})) < 0, \\ (dg_I)^+(\bar{x}, \widehat{\eta}(x,\overline{x})) < 0 \end{cases}$$

has no solution $x \in X^0$.

Proof. By contradiction, suppose that there is $x^* \in X^0$ such that

$$\begin{cases} (df)^+(\bar{x}, \hat{\eta}(x^*,\overline{x})) < 0, \\ (dg_I)^+(\bar{x}, \hat{\eta}(x^*,\overline{x})) < 0. \end{cases}$$

Let $\varphi(\lambda) = f(\bar{x} + \lambda\eta(x^*, \bar{x}, \lambda)) - f(\bar{x})$, where $\lambda \in (0, a(\bar{x}, x^*))$.

We have $\varphi(0) = 0$ and $\lim\limits_{\lambda\to 0^+} \frac{\varphi(\lambda)-\varphi(0)}{\lambda} = \lim\limits_{\lambda\to 0^+} \frac{f(\bar{x}+\lambda\eta(x^*,\bar{x},\lambda))-f(\bar{x})}{\lambda} = (df)^+(\bar{x}, \hat{\eta}(x^*,\overline{x})) < 0$.

So there is some open interval $(0, \delta_1)$ such that $\varphi(\lambda) < 0$ for $\lambda \in (0, \delta_1)$, where $\delta_1 > 0$, i.e.,

$$f(\bar{x} + \lambda\eta(x^*, \bar{x}, \lambda)) < f(\overline{x}), \quad \forall\lambda \in (0, \delta_1).$$

Similarly, there exists a $\delta_2 > 0$ such that $g_I(\bar{x}+\lambda\eta(x^*, \bar{x}, \lambda)) < g_I(\bar{x}) = 0, \ \ \forall\lambda \in (0, \delta_2), \delta_2 > 0$.

Noting that g_J is continuous at $\bar{x}$, then there is $\delta_3 > 0$ such that

$$g_J(\bar{x} + \lambda\eta(x^*, \overline{x}, \lambda)) < 0, \quad \forall\lambda \in (0, \delta_3).$$

Since X_0 is "semi-local connected", there is $\delta_4 > 0$ such that

$$\bar{x} + \lambda\eta(x^*, \overline{x}, \lambda) \in X_0, \quad \forall\lambda \in (0, \delta_4).$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\} > 0$, then

$$f(\bar{x} + \lambda\eta(x^*, \overline{x}, \lambda)) < f(\bar{x}), \bar{x} + \lambda\eta(x^*, \overline{x}, \lambda) \in X, \quad \forall\lambda \in (0, \delta),$$

which contradicts the assumption that $\bar{x}$ is a local minimum solution to (NP). Thus the lemma is proved. □

Theorem 3.1. (Fritz John Type Necessary Optimality Conditions) *Let us suppose that f, g are semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y,x,\lambda)) \mid_{\lambda=0}= \hat{\eta}(y,x)$ and g_i is continuous at x for $i \in J$. Further assume that*

$$((df)^+(\bar{x}, \hat{\eta}(x,\bar{x})), \quad (dg_I)^+(\bar{x}, \hat{\eta}(x,\bar{x})))$$

is a convex-like function of x on X^0. If $\bar{x}$ is a local minimum solution to (NP), then there exist $\bar{u_0} \in R, \bar{u} \in R^m$ such that

$$\bar{u_0}(df)^+(\bar{x}, \hat{\eta}(x,\bar{x})) + \bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x,\bar{x})) \geq 0, \quad \forall x \in X^o,$$

$$\bar{u}^T g(\bar{x}) = 0, \quad (\bar{u_0}, \bar{u}) \neq 0, \quad (\bar{u_0}, \bar{u}) \geq 0.$$

Proof. Define vector function

$$\psi(x) = ((df)^+(\bar{x}, \hat{\eta}(x,\bar{x})), (dg_I)^+(\bar{x}, \hat{\eta}(x,\bar{x}))).$$

Then $\psi(x)$ is a convex-like function. By Lemma 3.1, the system $\psi(x) < 0$ has no solution in X^0. So, from Lemma 2.1, there is $\bar{u_0} \in R$, $\bar{u}_{I(\bar{x})} \in R^{|I(x^*)|}$ such that

$$\bar{u_0}(df)^+(\bar{x}, \hat{\eta}(x,\bar{x})) + \bar{u}_{I(\bar{x})}^T(dg)^+(\bar{x}, \hat{\eta}(x,\bar{x})) \geq 0, \quad \forall x \in X^o,$$

$$(\bar{u_0}, \bar{u}_{I(\bar{x})}) \neq 0, \quad (\bar{u_0}, \bar{u}_{I(\bar{x})}) \geq 0.$$

Thus, by letting $\bar{u} = (\bar{u}_{I(\bar{x})}, 0_{I \setminus I(\bar{x})})$, we get

$$\bar{u_0}(df)^+(\bar{x}, \hat{\eta}(x,\bar{x})) + \bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x,\bar{x})) \geq 0, \quad \forall x \in X^o,$$

$$\bar{u}^T g(x) = 0, \quad 0 \neq (\bar{u_0}, \bar{u}) \geq 0.$$

Theorem 3.2. (Kuhn-Tucker Type Necessary Optimality Conditions) *Let $\bar{x} \in X$ be a local minimum solution to problem (NP), g_i be continuous at $\bar{x}$ for $i \in J$ and f, g be semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y,x,\lambda)) \mid_{\lambda=0}= \hat{\eta}(y,x)$. Assume that $((df)^+(\bar{x}, \hat{\eta}(x,\bar{x})), (dg_I)^+(\bar{x}, \hat{\eta}(x,\bar{x})))$ be a convex-like function of x on X^0. If g satisfies GSCQ at $\bar{x}$ and g is a semi-local B-semi-preinvex function, then there exists $\bar{u} \in R^m$such that*

$$(df)^+(\bar{x}, \hat{\eta}(x,\bar{x})) + \bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x,\bar{x})) \geq 0, \quad \forall x \in X^o, \tag{3.1}$$

$$\bar{u}^T g(x) = 0, \quad g(\bar{x}) \leq 0, \quad \bar{u} \geq 0. \tag{3.2}$$

Proof. First, by Theorem 3.1, we know that there is $u_0 \in R, u \in R^m$ such that

$$\bar{u_0}(df)^+(\bar{x}, \hat{\eta}(x, \bar{x})) + \bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \geq 0, \quad \forall x \in X^o,$$
$$\bar{u}^T g(x) = 0, \quad 0 \neq (\bar{u_0}, \bar{u}) \geq 0.$$

The next problem is to show that $u_0 \neq 0$, if there is not true, we have from above

$$\bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \geq 0, \quad \forall x \in X^o,$$
$$\bar{u}^T g(x) = 0, \quad g(\bar{x}) \leq 0, \quad u \geq 0.$$

Since g is a a semi-local B-semi-preinvex, we have

$$g(\hat{x}) - g(\bar{x}) \geq (dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})),$$

i.e., $\bar{u}^T g(\hat{x}) \geq 0$. Thus $u_0 > 0$. The proof is completed. □

4. Sufficient Optimality Conditions

Theorem 4.1. *Let $\bar{x} \in X$, and assume that f is pseudo semi-local B_1-semi-preinvex and g_I is quasi semi-local B_2-semi-preinvex. Further assume that f, g are semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y, x, \lambda)) \mid_{\lambda=0} = \hat{\eta}(y, x)$. If there exists $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ satisfy conditions (3.1)-(3.2) of Theorem 3.2 and $\bar{b}_1(x, \bar{x}) = \lim_{\lambda \to 0^+} b_1(x, \bar{x}, \lambda) > 0$, $\bar{b}_2(x, \bar{x}) = \lim_{\lambda \to 0^+} b_2(x, \bar{x}, \lambda)$, then $\bar{x}$ is an optimal solution to problem (NP).*

Proof. Since $g_I(x) \leq g_I(\bar{x}) = 0$, $\forall x \in X$, noting that g_I is quasi semi-local B_2-semi-preinvex, we have

$$(dg)^+(\bar{x}, \hat{\eta}(x, \bar{x}) \leq 0, \quad \forall x \in X,$$

which together with $\bar{u}_I \geq 0$ implies

$$\bar{u}_I^T(dg_I)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \leq 0, \quad \forall x \in X.$$

Furthermore, using (3.2), we get

$$\bar{u}^T(dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \leq 0,$$

for all $x \in X$, so we have from (3.1)

$$(df)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \geq 0,$$

for all $x \in X$. Taking into account that f is pseudo-semi-local-B_1-semi-preinvex, this implies

$$\bar{b_1}(x, \bar{x}) f(x) \geq \bar{b_1}(x, \bar{x}) f(\bar{x}),$$

for all $x \in X$, Also noting $\bar{b_1}(x, \bar{x}) > 0$, hence $\bar{x}$ is an optimal solution to problem (NP). □

Theorem 4.2. *Let $\bar{x} \in X$, and assume that f, g are semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y, x, \lambda)) \mid_{\lambda=0} = \hat{\eta}(y, x)$. Further assume that there is $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ satisfy conditions (3.1)-(3.2) of Theorem 3.2. If $f + \bar{u_I}^T g_I$ is pseudo-semi-local-B-semipreinvex at $\bar{x}$ and $\bar{b}(x, \bar{x}) = \lim_{\lambda \to 0^+} b(x, \bar{x}, \lambda) > 0$, then $\bar{x}$ is an optimal solution to problem (NP).*

Proof. Take into account (3.1)-(3.2), we get

$$[d(f + \bar{u_I}^T g_I)]^+(\bar{x}, \hat{\eta}(x, \bar{x})) = d(f)^+(\bar{x}, \hat{\eta}(x, \bar{x})) + \bar{u}^T (dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \geq 0, \quad \forall x \in X.$$

Noting that $f + \bar{u_I}^T g_I$ is pseudo-semi-local-B-semi-preinvex and the relationship above. One obtains

$$\bar{b}(x, \bar{x})(f + \bar{u_I}^T g_I)(x) \geq \bar{b}(x, \bar{x})(f + \bar{u_I}^T g_I)(\bar{x}),$$

for all $x \in X$.

The inequality above together with $\bar{u_I}^T g_I(x) = 0$ and $\bar{b}(x, \bar{x}) > 0$, yields

$$f(x) + \bar{u_I}^T g_I(x) \geq f(\bar{x}), \quad \forall x \in X.$$

In view of $\bar{u_I} \geq 0$ and $g_I(x) \leq 0$, we get

$$f(x) \geq f(\bar{x}), \quad \forall x \in X.$$

Hence $\bar{x}$ is an optimal solution to problem (NP). □

A direct corollary of the two theorems above is as follows.

Corollary 4.1. *Let $\bar{x} \in X$, and assume that f, g are semi-local differentiable and f is B_1-semi-preinvex and g_I is semi-local B_2-semi-preinvex. Further assume that $\frac{d}{d\lambda}(\lambda\eta(y, x, \lambda)) \mid_{\lambda=0} = \hat{\eta}(y, x)$. If there exists $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ satisfies the conditions*

$$(df)^+(\bar{x}, \hat{\eta}(x, \bar{x})) + \bar{u}^T (dg)^+(\bar{x}, \hat{\eta}(x, \bar{x})) \geq 0, \quad \forall x \in X,$$

$$\bar{u}^T g(\bar{x}) = 0, \quad g(\bar{x}) \leq 0, \quad \bar{u} \geq 0,$$

with $\bar{b}_1(x, \bar{x}) = \lim_{\lambda \to 0^+} b_1(x, \bar{x}, \lambda) > 0$ and $\bar{b}_2(x, \bar{x}) = \lim_{\lambda \to 0^+} b_2(x, \bar{x}, \lambda)$, then $\bar{x}$ is an optimal solution to problem (NP).

5. Mixed Type Duality

Relative to the problem (NP), we consider the following Mixed type dual problem [7].

$$\text{(MD)}\qquad \begin{array}{ll} \max & f(u) + y_{J_1}^T g_{J_1}(u) \\ \text{s.t.} & (df)^+(u, \hat{\eta}(x,u)) + y^T (dg)^+(u, \hat{\eta}(x,u)) \geq 0, \forall x \in X, \\ & y_{J_2}^T g_{J_2}(u) \geq 0, \\ & y \geq 0,\ u \in X^0,\ y \in R^m, \end{array}$$

where $J_1 \bigcup J_2 = \{1, 2, \ldots, m\}$.

Let W denote the set of all feasible solutions to problem (MD).

Theorem 5.1. (Weak Duality) *Let $\bar{x} \in X$ and $(\bar{u}, \bar{y}) \in W$, and assume that f, g are semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y, x, \lambda)) \mid_{\lambda=0} = \hat{\eta}(y, x)$. If f, g are semi-local B-semi-preinvex on X, with $\bar{b}(\bar{x}, \bar{u}) = \lim_{\lambda \to 0^+} b(\bar{x}, \bar{u}, \lambda) > 0$, then $f(\bar{x}) \geq f(\bar{u}) + y_{J_1}^T g_{J_1}(u)$.*

Proof. Noting that f, g are semilocal B-semipreinvex and the given conditions, we get

$$\bar{b}(\bar{x}, \bar{u}) f(\bar{x}) \geq \bar{b}(\bar{x}, \bar{u}) f(\bar{u}) + (df)^+(\bar{u}, \hat{\eta}(\bar{u}, \bar{x})),$$
$$\bar{b}(\bar{x}, \bar{u}) g(\bar{x}) \geq \bar{b}(\bar{x}, \bar{u}) g(\bar{u}) + (dg)^+(\bar{u}, \widehat{\eta}(\bar{u}, \bar{x})).$$

Combining the first constraint of (MD) and the relationships above, one gets

$$\begin{aligned} \bar{b}(\bar{x}, \bar{u}) f(\bar{x}) &\geq \bar{b}(\bar{x}, \bar{u}) f(\bar{u}) - \bar{y}^T (dg)^+(\bar{u}, \widehat{\eta}(\bar{u}, \bar{x})) \\ &\geq \bar{b}(\bar{x}, \bar{u}) f(\bar{u}) - \bar{b}(\bar{x}, \bar{u}) \bar{y}^T (g(\bar{x}) - g(\bar{u})). \end{aligned}$$

In view of $\bar{b}(\bar{x}, \bar{u}) > 0$, we have

$$f(\bar{x}) \geq f(\bar{u}) - \bar{y}^T (g(\bar{x}) - g(\bar{u})).$$

Taking into account

$$\bar{y} \geq 0, \quad g(x) \leq 0, \quad \bar{y}_{J_2}^T g_{J_2}(\bar{u}) \geq 0, \quad J_1 \bigcup J_2 = \{1, 2, \cdots, m\}.$$

So, we have

$$f(\bar{x}) \geq f(\bar{u}) + \bar{y}_{J_1}^T g_{J_1}(\bar{u}). \qquad \square$$

Theorem 5.2. (Direct Duality) *Let $\bar{x}$ be an optimal solution to (NP), and f, g be semi-local differentiable at $\bar{x}$, and assume that:*

(i_1) $\frac{d}{d\lambda}(\lambda\eta(y,x,\lambda)) \mid_{\lambda=0}= \hat{\eta}(y,x)$.

(i_2) $((df)^+(\bar{x},\hat{\eta}(x,\bar{x})),(dg)^+(\bar{x},\hat{\eta}(x,\bar{x})))$ *is a convex-like function on* X.

(i_3) f *and* g *are semi-local B-semipreinvex,* $g_i(i \in J)$ *are continuous at* $\bar{x}$.

(i_4) g *satisfies GSCQ at* $\bar{x}, \bar{b}(\bar{x},\bar{u}) = \lim_{\lambda\to 0^+} b(\bar{x},\bar{u},\lambda) > 0$.

Then there exists $\bar{y} \in R^m$ such that $(\bar{x},\bar{y})$ is an optimal solution to (MD) and their optimal value is equal.

Proof. From the given conditions and Theorem 3.2, we can conclude that there exists $\bar{y} \in R^m$ such that $(\bar{x},\bar{y}) \in W$.

Suppose that (u,y) is a feasible solution of (MD). In view of f,g are semi-local B-semipreinvex and the given conditions, one gets

$$\bar{b}(\bar{x},u)(f(\bar{x})-f(u)) \geq (df)^+(u,\hat{\eta}(\bar{x},u))\,,$$

$$\bar{b}(\bar{x},u)(g(\bar{x})-g(u)) \geq (dg)^+(u,\hat{\eta}(\bar{x},u))\,.$$

Taking into account the first constraint condition of (MD) and the two inequalities above, we obtain

$$\bar{b}(\bar{x},u)(f(\bar{x})-f(u)) \geq -\bar{y}^T(dg)^+(u,\widehat{\eta}(\bar{x},u)) \geq \bar{b}(\bar{x},u)\bar{y}^T(g(\bar{x})-g(u)).$$

Noting that $\overline{y} \geq 0, g(\bar{x}) \leq 0, \bar{y}^T_{J_2}g_{J_2}(\bar{u}) \geq 0$ and $\overline{b}(\overline{x},\overline{u}) > 0$,

One knows $f(\bar{x}) \geq f(\bar{u}) + \bar{y}^T_{J_1}g_{J_1}(\bar{u})$. Also noting that $\bar{y}^Tg(\bar{x}) = 0, \bar{y} \geq 0, g(\bar{x}) \leq 0$.

We obtain $\bar{y}^T_{J_1}g_{J_1}(\bar{x}) = 0$. So

$$f(\bar{x}) + \bar{y}^T_{J_1}g_{J_1}(\bar{x}) \geq f(\bar{u}) + \bar{y}^T_{J_1}g_{J_1}(\bar{u}),$$

i.e. $(\bar{x},\bar{y})$ is an optimal solution to (MD). □

Theorem 5.3. (Converse Duality) *Let $\bar{x} \in X, (\bar{u},\bar{y}) \in w$, and $f(\bar{x}) = f(\bar{u}) + \bar{y}^T_{J_1}g_{J_1}(\bar{u})$. Further assume that f,g are semi-local differentiable and $\frac{d}{d\lambda}(\lambda\eta(y,x,\lambda)) \mid_{\lambda=0}= \hat{\eta}(y,x)$. If f,g are semi-local B-semi-preinvex on X with $\bar{b}(x,u) = \lim_{\lambda\to 0^+} b(x,u,\lambda) > 0$, then $\bar{x}$ is an optimal solution to (NP).*

Proof. Taking into account f and g are semi-local B-semi-preinvex and the given conditions, one gets

$$\bar{b}(x,\bar{u})(f(x)-f(\bar{u})) \geq (df)^+(\bar{u},\hat{\eta}(x,\bar{u})), \quad \forall x \in X,$$

$$\bar{b}(x,\bar{u})(g(x)-g(\bar{u})) \geq (dg)^{+}(\bar{u},\hat{\eta}(x,\bar{u})), \quad \forall x \in X.$$

In view of $(\bar{u},\bar{y}) \in w$, we have from the first constraint condition of (MD) and the two inequalities

$$\bar{b}(x,\bar{u})(f(x)-f(\bar{u})) \geq -\bar{y}^{T}(dg)^{+}(\bar{u},\hat{\eta}(x,\bar{u})) \geq \bar{b}(x,\bar{u})\bar{y}^{T}(g(\bar{u})-g(x)).$$

Noting that $f(\bar{x}) = f(\bar{u}) + \bar{y}_{J_1}^{T} g_{J_1}(\bar{u}), \bar{y} \geq 0, g(x) \leq 0$ and $\bar{b}(x,\bar{u}) > 0$.

We obtain $f(x) \geq f(\bar{x}), \forall x \in X$. So $\bar{x}$ is an optimal solution to (NP). □

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