International Journal of Pure and Applied Mathematics

Volume 22 No. 4 2005, 519-527

INFLUENCE ANALYSIS OF BLUE IN RESTRICTED LINEAR REGRESSION MODELS

Shangli Zhang

School of Science
Beijing Jiaotong University
Beijing, 100044, P.R. CHINA
e-mail: shlzhang@center.njtu.edu.cn

Abstract: This paper is devoted to discussing the influence problem of Best Linear Unbiased Estimate (BLUE) in restricted linear regression models. The influence on BLUE with covariance matrix disturbing or data missing is studied. The relationships between $\hat{\beta}_L(G)$ and $\hat{\beta}_L$ are established, where $\hat{\beta}_L(G)$ is the BLUE of β in mode (1.4) and $\hat{\beta}_L$ is the BLUE of β in model (1.5). We define the generalized Cook distance D_G to assess the disturbance influence and obtain a formula for D_G.

AMS Subject Classification: 62J20
Key Words: linear regression model, best linear unbiased estimate, influence analysis

1. Introduction

Consider Gauss-Markov model:

$$\begin{cases} Y = X\beta + e\,, \\ E(e) = 0, \quad \mathrm{Cov}\,(e) = \sigma^2 I_n \end{cases} \tag{1.1}$$

where X is a known $n \times p$ matrix with $\mathrm{rank}\,(X) = p$, Y is a $n \times 1$ observed vector, e is $n \times 1$ random error vector, σ^2 and $p \times 1$ vector β are the unknown

Received: May 28, 2005

parameters, I_n is the $n \times n$ identity matrix. R.D. Cook [2]-[4] studied the influence on the estimate of parameters and fitting value with deleting one set or several sets of data. If the G-M condition is not satisfied, A.C. Aitken [1], C.R. Rao [9] discussed the LS estimate of the following linear regression model:

$$\begin{cases} Y = X\beta + e, \\ E(e) = 0,\ \mathrm{Cov}\,(e) = \sigma^2 G^{-1}\,, \end{cases} \tag{1.2}$$

where $G > 0$. Pregibon Dargl [5], Xiujuan Zhu, Ning Li [11], John Haslett, Kevin Hayes [8], John Haslett [7] studied the influence problem for (1.2) with covariance disturbing and data deletion. If some additional information about the parameters are known and can be described with restriction conditions. N.R. Draper, J.A. John [6], C.R. Rao [9] and Yilang Yu, Shiyang Wu [10] studied the following restricted G-M model:

$$\begin{cases} Y = X\beta + \epsilon, \quad E(\epsilon) = 0, \quad \mathrm{Cov}\,(\epsilon) = \sigma^2 I_n\,, \\ L\beta = 0\,, \end{cases} \tag{1.3}$$

where L is a $q \times p$ matrix with $\mathrm{rank}\,(L) = q$.

In this paper, we consider the following restricted linear regression model:

$$\begin{cases} Y = X\beta + e, \quad E(e) = 0, \quad \mathrm{Cov}\,(e) = \sigma^2 G^{-1}\,, \\ L\beta = 0\,, \end{cases} \tag{1.4}$$

where $G > 0$. The data deleting model discussed in this paper is as follows:

$$\begin{cases} Y(J) = X(J)\beta + e(J), \quad E(e(J)) = 0, \quad \mathrm{Cov}\,(e(J)) = \sigma^2 I_{n-m}\,, \\ L\beta = 0\,, \end{cases} \tag{1.5}$$

where $J = \{i_1, i_2, \dots, i_m\}$, $1 \le i_1 < i_2 < \cdots < i_m \le n$, $Y(J), X(J), e(J)$ are just Y, X and e in (1.3) with the rows in J deleted respectively.

2. Some Lemmas

Let A^{-1} and A^τ denote the inverse, transpose of matrix A respectively. $A > 0$ denotes that A is positive definite.

Lemma 1. (see [9]) *The BLUE $\hat\beta_L$ of β in model (1.3) is*

$$\hat\beta_L = \hat\beta - T^{-1}M\hat\beta\,, \tag{2.1}$$

where $T = X^\tau X$, $M = L^\tau(LT^{-1}L^\tau)^{-1}L$, $\hat\beta = T^{-1}X^\tau Y$.

Lemma 2. (see [9]) *The BLUE $\hat\beta(G)$ of β in model (1.2) is*

$$\hat\beta(G) = (X^\tau GX)^{-1}X^\tau GY. \tag{2.2}$$

From Lemma 2 and by using a similar method as in the proof of Lemma 1, we can prove the following result.

Lemma 3. *The BLUE $\hat\beta(G)$ of β in model (1.4) is*

$$\hat\beta(G) = \hat\beta(G) - (X^\tau GX)^{-1}L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}L\hat\beta(G). \tag{2.3}$$

Lemma 4. *For model (1.2), if $(I - H\bar G)$ is invertible, then*

$$\hat\beta(G) = \hat\beta - T^{-1}X^\tau\bar G(I - H\bar G)^{-1}\cdot\delta\,, \tag{2.4}$$

where $\bar G = I - G, \quad H = X(X^\tau X)^{-1}X^\tau, \quad \delta = (I - H)Y.$

Proof. Noting $G = I - \bar G$, from $I + (I - H\bar G)^{-1}H\bar G = (I - H\bar G)^{-1}$ and inverse matrix formula, we have

$$\begin{aligned}\hat\beta(G) = (X^\tau GX)^{-1}X^\tau GY &= (X^\tau X - X^\tau\bar GX)^{-1}X^\tau(I - \bar G)Y \\ &= [T^{-1} - T^{-1}X^\tau\bar G(I - H\bar G)^{-1}XT^{-1}]\cdot[X^\tau Y - X^\tau\bar GY] \\ = \hat\beta - T^{-1}X^\tau\bar G(I - H\bar G)^{-1}\cdot(I - H)Y &= \hat\beta - T^{-1}X^\tau\bar G(I - H\bar G)^{-1}\cdot\delta. \quad \square\end{aligned}$$

3. Relations of BLUEs in Models (1.3), (1.4) and (1.5)

Theorem 1. *For model (1.4), assume $(I - H\bar G)$ and $I + N\bar G(I - H\bar G)^{-1}$ are both invertible. Then*

$$\begin{aligned}\hat\beta_L(G) &= \hat\beta_L \\ &\quad - (I - T^{-1}M)T^{-1}X^\tau\bar G(I - H\bar G)^{-1}\cdot[I + N\bar G(I - H\bar G)^{-1}]^{-1}\hat e,\end{aligned} \tag{3.1}$$

where $N = XT^{-1}MT^{-1}X^\tau$ and $\hat e = (I - H + N)Y$.

Proof. Since $G = I - \bar G$, it follows from (2.3) and (2.4) that

$$\begin{aligned}\hat\beta_L(G) &= \hat\beta(G) - (X^\tau GX)^{-1}L^\tau\cdot[L(X^\tau GX)^{-1}L^\tau]^{-1}L\hat\beta(G) \\ &= \hat\beta - (X^\tau GX)^{-1}L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}L\hat\beta - T^{-1}X^\tau\bar G(I - H\bar G)^{-1}\delta\end{aligned}$$

$$+ (X^\tau GX)^{-1}L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}LT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1}\delta. \quad (3.2)$$

Note that

$$I - [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1} \cdot N\bar{G}(I - H\bar{G})^{-1} = [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}, \quad (3.3)$$

$$I - N\bar{G}(I - H\bar{G})^{-1} \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1} = [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}. \quad (3.4)$$

We have

$$\begin{aligned}
&(X^\tau GX)^{-1}L^\tau \cdot [L(X^\tau GX)^{-1}L^\tau]^{-1}L \\
&= (X^\tau X - X^\tau\bar{G}X)^{-1} \cdot L^\tau[L(X^\tau X - X^\tau\bar{G}X)^{-1}L^\tau]^{-1}L \\
&= [T^{-1} + T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1}XT^{-1}] \\
&\quad \cdot L^\tau[LT^{-1}L^\tau - LT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1}XT^{-1}L^\tau]^{-1}L \\
&= [T^{-1} + T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1}XT^{-1}] \\
&\quad \{M - MT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1} \cdot XT^{-1} \cdot M\} \\
&= T^{-1}M - T^{-1}MT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}XT^{-1}M \\
&\quad + T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}XT^{-1}M \\
&= T^{-1}M + (I - T^{-1}M)T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1}[I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}XT^{-1}M,
\end{aligned} \quad (3.5)$$

$$\begin{aligned}
&(X^\tau GX)^{-1}L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}LT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \\
&= T^{-1}MT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} + (I - T^{-1}M)T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \\
&\quad \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1} \cdot N\bar{G}(I - H\bar{G})^{-1} \\
&= T^{-1}MT^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} + (I - T^{-1}M)T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \\
&\quad \cdot \{I - [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}\} \\
&= T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} + (I - T^{-1}M)T^{-1}X^\tau\bar{G}(I - H\bar{G})^{-1} \\
&\quad \cdot [I + N\bar{G}(I - H\bar{G})^{-1}]^{-1}.
\end{aligned} \quad (3.6)$$

Substituting (3.5) and (3.7) into (3.2) immediately yields (3.1). □

Theorem 2. *Let $\hat{\beta}_L(J)$ be the BLUE of $\beta(J)$ in model (1.5). If $G = I - \sum_{j\in J}(1-\omega_j)d_j d_j^\tau$, $0 < \omega_j \le 1$, $j \in J$. Then*

$$\lim_{\omega_j \to 0^+, j\in J} \hat{\beta}_L(G) = \hat{\beta}_L(J)\,, \tag{3.7}$$

where d_j is the column vector whose j-th component is 1 and the others are 0.

Proof. Note that

$$\hat{\beta}_L(J) = \hat{\beta}_L(J) - (X^\tau(J)X(J))^{-1} \cdot L^\tau[L(X^\tau(J)X(J))^{-1}L^\tau]^{-1}L\hat{\beta}_L(J)\,, \tag{3.8}$$

$$\hat{\beta}_L(G) = \hat{\beta}_L(J) - (X^\tau GX)^{-1} \cdot L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}L\hat{\beta}_L(G)\,. \tag{3.9}$$

If $G = I - \sum_{j\in J}(1-\omega_j)d_j d_j^\tau$, then

$$\hat{\beta}_L(G) = X^\tau GX)^{-1}X^\tau GY = (X^\tau X - \sum_{j\in J}(1-\omega_j)x_j x_j^\tau)^{-1}(X^\tau Y - \sum_{j\in J}(1-\omega_j)x_j y_j)\,, \tag{3.10}$$

$$(X^\tau GX)^{-1} \cdot L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}L = [X^\tau X - \sum_{j\in J}(1-\omega_j)x_j x_j^\tau]^{-1} \cdot L^\tau\{L[X^\tau X - \sum_{j\in J}(1-\omega_j)x_j x_j^\tau]^{-1}L^\tau\}^{-1}L\,. \tag{3.11}$$

Thus

$$\lim_{\omega_j \to 0^+, j\in J} \hat{\beta}_L(G) = \hat{\beta}_L(J)\,, \tag{3.12}$$

$$\lim_{\omega_j \to 0^+, j\in J} (X^\tau GX)^{-1} \cdot L^\tau[L(X^\tau GX)^{-1}L^\tau]^{-1}L = (X^\tau(J)\cdot X(J))^{-1} \cdot L^\tau \cdot [L(X^\tau(J)\cdot X(J))^{-1}L^\tau]^{-1}L. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.8) yields (3.7). □

It can be seen from Theorem 2, that if $G = I - \sum_{j\in J}(1-\omega_j)d_j d_j^\tau$ and $\omega_j \to 0^+, j \in J$, i.e., $\mathrm{Var}\,(y_i) \to \infty, j \in J$. Then covariance disturbing is euqivalent to deleting the data in J as for BLUE.

Theorem 3. *For model (1.4), if* $G = I - \sum_{j\in J}(1-\omega_j)d_j d_j^\tau$, $0 < \omega_j \leq 1, j \in J$. *Then*

$$\hat{\beta}_L(G) = \hat{\beta}_L - (I - T^{-1}M)T^{-1}X_J^\tau \Lambda_J (I_J - H_J\Lambda_J)^{-1} \cdot [I_J + N_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1}\hat{e}_J\,, \quad (3.14)$$

and

$$\lim_{\omega_j\to 0^+, j\in J} \hat{\beta}_L(G) = \hat{\beta}_L - (I - T^{-1}M)T^{-1}X_J^\tau (I_J - H_J)^{-1} \cdot [I_J + N_J(I_J - H_J)^{-1}]^{-1}\hat{e}_J\,, \quad (3.15)$$

where $X_J, \hat{e}_J$ *is consisted of the rows of* X *and* $\hat{e}$ *in* J *respectively,* I_J *is the* $m \times m$ *identity matrix and* $H_J = X_J(X^\tau X)^{-1}X_J^\tau$, $N_J = X_J T^{-1} M T^{-1} X_J^\tau$, $\Lambda =$*diag*$(1-\omega_{i_1}, 1-\omega_{i_2}, \ldots, 1-\omega_{i_m},)$.

Proof. Without loss of generality, we can assume $J = \{1, 2, \ldots, m\}$. Then

$$\bar{G} = \sum_{j=1}^m (1-\omega_j)d_j d_j^\tau = \begin{pmatrix} \Lambda_J & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} X_J \\ X(J) \end{pmatrix}$$

$$H = \begin{pmatrix} H_J & * \\ * & * \end{pmatrix}, \quad N = \begin{pmatrix} N_J & * \\ * & * \end{pmatrix}, \quad \hat{e} = \begin{pmatrix} \hat{e}_J \\ \hat{e}(J) \end{pmatrix}$$

Therefore,

$$(I - H\bar{G})^{-1} = \begin{pmatrix} I_J - H_J\Lambda_J & 0 \\ * & I(J) \end{pmatrix}^{-1} = \begin{pmatrix} (I_J - H_J\Lambda_J)^{-1} & 0 \\ * & I(J) \end{pmatrix}, \quad (3.16)$$

$$[I + N\bar{G}(I - H\bar{G})^{-1}]^{-1} = \begin{pmatrix} [I_J + N_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1} & 0 \\ * & I(J) \end{pmatrix}. \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.1) yields

$$\begin{aligned}\hat{\beta}_L(G) &= \hat{\beta}_L - (I - T^{-1}M)T^{-1}X^\tau \bar{G}(I - H\bar{G})^{-1} \cdot [I + H\bar{G}(I - H\bar{G})^{-1}]^{-1}\hat{e} \\ &= \hat{\beta}_L - (I - T^{-1}M)T^{-1}X_J^\tau \Lambda_J (I_J - H_J\Lambda_J)^{-1}\end{aligned}$$

$$\cdot [I_J + H_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1}\hat{e}_J\,.$$

Finally, (3.15) is obvious. □

Theorem 3 described the BLUE's variation with covariance disturbing of m sets of data or deleting m sets of data.

Corollary. *For model (1.4), assume that* $G = I - (1-\omega_i)d_i d_i^\tau$. *Then*

$$\hat{\beta}_L(G) = \hat{\beta}_L - \frac{(I - T^{-1}M)T^{-1}x_i(1-\omega_i)\hat{e}_i}{1 - (1-\omega_i)h_{ii} + (1-\omega_i)n_{ii}} \tag{3.18}$$

and

$$\lim_{\omega_i \to 0^+} \hat{\beta}_L(G) = \hat{\beta}_L - \frac{(I - T^{-1}M)T^{-1}x_i\hat{e}_i}{1 - h_{ii} + n_{ii}}\,, \tag{3.19}$$

where $\hat{e}_i$ *is the* i*-th component of* $\hat{e}_i$, h_{ii} *is the* i*-th diagonal element of* H *and* n_{ii} *is the* i*-th diagonal element of* N.

4. Generalized Cook Distance

In order to assess the influence on BLUE with covariance disturbing under restriction conditions. Similar to Cook distance (see [3], [4]), we define the Cook distance under restriction conditions as follows.

$$D_G(K, c) = \frac{(\hat{\beta}_L(G) - \hat{\beta}_L)^\tau K(\hat{\beta}_L(G) - \hat{\beta}_L)}{c}\,, \tag{4.1}$$

where K is a given positive definite matrix and c is a given constant.

We first give some notations. Let λ_i, μ_i, s_i $(i = 1, 2, \ldots, m)$ denote the eigenvalues of $\bar{G}_J$, H_J and N_J respectively. Γ is the orthogonal matrix consisted of the eigenvectors of H_J, a_i is the i-th component of $\Gamma\hat{e}_J$, $\hat{\sigma}_L^2 = (Y - X\hat{\beta}_L)^\tau \cdot (Y - X\hat{\beta}_L)/(n-t)$, $t = \operatorname{rank}\binom{L}{X} - \operatorname{rank}(L)$.

Theorem 4. *In model (1.4), let* $G = I - \sum_{j\in J}(1-\omega_j)d_j d_j^\tau$, $0 < \omega_j \le 1$, $J = \{i_1, i_2, \ldots, i_m\}$, $1 \le m \le n$. *Take* $K = X^\tau X$, $c = (p-q)\cdot\hat{\sigma}_L^2$. *If* H_J, $\bar{G}_J$, N_J *are pairwise exchangeable. Then*

$$D_G(X^\tau X, (p-q)\hat{\sigma}_L^2) = \frac{1}{(p-q)\hat{\sigma}_L^2} \cdot \sum_{i=1}^{m} \frac{(\mu_i - s_i)a_i^2\lambda_i^2}{(1 - \lambda_i\mu_i + s_i\lambda_i)^2}\,. \tag{4.2}$$

Proof. It follows from (3.14) that

$$\hat{\beta}_L(G) - \hat{\beta} = -(I - T^{-1}M)T^{-1}X_J^{\tau}\Lambda_J(I_J - H_J\Lambda_J)^{-1} \cdot [I_J + H_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1}\hat{e}_J\,. \quad (4.3)$$

Since $MT^{-1}M = M$, we have

$$\begin{aligned} &X_JT^{-1}(I - MT^{-1}) \cdot X^{\tau}X(I - T^{-1}M)T^{-1}X_J^{\tau} \\ &= X_JT^{-1}X_J^{\tau} - 2X_JT^{-1}MT^{-1}X_J^{\tau} + X_JT^{-1}MT^{-1}MT^{-1}X_J^{\tau} \\ &= X_JT^{-1}X_J^{\tau} - X_JT^{-1}MT^{-1}X_J^{\tau} = H_J - N_J. \quad (4.4) \end{aligned}$$

On the other hand, since the exchangeability of H_J, $\bar{G}_J$, N_J, there exists an orthogonal matrix Γ such that

$$H_J = \Gamma^{\tau}\mu\Gamma, \quad \bar{G}_J = \Gamma^{\tau}\Lambda\Gamma, \quad N_J = \Gamma^{\tau}S\Gamma, \quad (4.5)$$

where μ =diag$(\mu_1, \ldots, \mu_m)$, Λ =diag$(\lambda_1, \ldots, \lambda_m)$, S =diag$(s_1, \ldots, s_m)$. Combining (4.3), (4.4) and (4.5) yields:

$$\begin{aligned} D_G(X^{\tau}X, (p-q)\hat{\sigma}_L^2) &= \frac{(\hat{\beta}_L(G) - \hat{\beta}_L)^{\tau}X^{\tau}X(\hat{\beta}_L(G) - \hat{\beta}_L)}{(p-q)\hat{\sigma}_L^2} \\ &= \frac{\hat{e}_J^{\tau} \cdot [I_J + N_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1} \cdot (I_J - H_J\Lambda_J)^{-1} \cdot \Lambda_J(H_J - N_J)}{(p-q)\hat{\sigma}_L^2} \\ &\quad \times \Lambda_J(I_J - H_J\Lambda_J)^{-1} \cdot [I_J + N_J\Lambda_J(I_J - H_J\Lambda_J)^{-1}]^{-1} \cdot \hat{e}_J \\ &= \frac{1}{(p-q)\hat{\sigma}_L^2} \cdot \sum_{i=1}^{m} \frac{(\mu_i - s_i)a_i^2\lambda_i^2}{(1 - \lambda_i\mu_i + s_i\lambda_i)^2}\,. \quad \square \end{aligned}$$

Theorem 5. *In Theorem 4, if $K = X^{\tau}X$ is replaced with $K = X_J^{\tau}X_J$. Then*

$$D_G(X_J^{\tau}X_J, (p-q)\hat{\sigma}_L^2) = \frac{1}{(p-q)\hat{\sigma}_L^2} \cdot \sum_{i=1}^{m} [\frac{(\mu_i - s_i)a_i\lambda_i}{1 - \lambda_i\mu_i + s_i\lambda_i}]^2\,.$$

Proof. Nothing

$$\begin{aligned} &X_JT^{-1}(I - MT^{-1}) \cdot X_J^{\tau}X_J(I - T^{-1}M)T^{-1}X_J^{\tau} \\ &= X_JT^{-1}X_J^{\tau}X_JT^{-1}X_J^{\tau} - X_JT^{-1}X_J^{\tau}X_JT^{-1}MT^{-1}X_J^{\tau} \\ &- X_JT^{-1}MT^{-1}X_J^{\tau}X_JT^{-1}X_J^{\tau} + X_JT^{-1}MT^{-1}X_J^{\tau}X_JT^{-1}MT^{-1}X_J^{\tau} \\ &= (X_JT^{-1}X_J^{\tau} - X_JT^{-1}MT^{-1}X_J^{\tau})^2 = (H_J - N_J)^2. \end{aligned}$$

Applying a similar method as in the proof of Theorem 4 can yield the required conclusion. $\square$

References

[1] A.C. Aitken, On least Squares and linear combination of observations, *Proc. Roy. Soc. Edinburgh Sect*, **55** (1935), 42-48.

[2] R.D. Cook, Detection of influential observations in linear regression, *Technometrice*, **19** (1977), 15-18

[3] R.D. Cook, S. Weisberg, Characterization of an empirical influence function for detection influential cases in regression, *Technometrice*, **22** (1980), 495-508

[4] R.D. Cook, S. Weisberg, *Residuals and Influence in Regression*, New York, Chapman and Hall (1982).

[5] Rregibon Dargl, Logistic regression diagnostics, *The Annals of Atatistics*, **9** (1981), 705-724.

[6] N.R. Draper, J.A. John, Influential observations and outliers in regression, *Technometrice*, **23** (1981), 21-26.

[7] John Haslett, A simple derivation of deletion diagnostic results for the general linear model with correlated errors, *Journal of the Royal Statistical Society, Series B*, **3** (1999), 603-609.

[8] John Haslett, Kevin Hayes, Residuals for the linear model with general covariance structure, *Journal of the Royal Statistical Society, Series B*, **1** (1998), 201-205.

[9] C.R. Rao, *Linear Statistical Inference and its Applications. Second Edition*, New York, Wiley (1973).

[10] Yilang Yu, Shiyong Wu, Restricted Welsch-kuh statistic and restricted cook distance, *Chinese Journal of Applied Probability and Statistics*, **2** (1991), 136-142.

[11] Xiujuan Zhu, Ning Li, Influence analysis in regression model with covariance matrix disturbance, *Chinese Journal of Applied Probability and Statistics*, **4** (1993), 365-370.