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CONDITIONS FOR ALMOST SURE CONVERGENCE

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Abstract: We study the almost sure convergence of normalized random vectors when the limit of the cumulant generating functions does not exist or one is unable to find it. As such, familiar tools like differentiability, convexity, exponential tightness, etc. are no longer at hand. We give examples and applications in both finite and infinite dimension.

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1. Introduction

We consider a σ-algebra $\mathcal{E}$ of subsets of a Hausdorff topological vector space E and, for each $n \geq 1$, an E-valued $\mathcal{E}$-measurable random vector X_n defined on a probability space $(\Omega, \mathcal{A}, P)$. Given $\xi \in E^*$, the topological dual space of E, and a strictly positive sequence $\{a_n\}_{n\geq 1}$ with $\lim_{n\to\infty} a_n = +\infty$, let us consider the cumulant generating functions

$$\phi_n(\xi) := \frac{1}{a_n} \log E(\exp < \xi, X_n >),$$

where $< \cdot, \cdot >$ represents the duality between E and E^*. If the convex function $\lim_{n\to\infty} \phi_n(\xi)$ exists, is finite and differentiable at any $\xi \in E = \mathbb{R}^d$ $(d \geq 1)$, then $\{X_n\}_{n\geq 1}$ satisfies the large deviation principle (see Ellis [6] and Gärtner [9]). Essentially the same holds in infinite dimension if in addition $\{X_n\}_{n\geq 1}$ is expo-

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nentially tight (see Dembo and Zeitouni [4]). The large deviation principle is closely related to the almost sure (a.s.) convergence properties of $\{X_n/a_n\}_{n\geq 1}$. For instance, suppose that $\lim_{n\to\infty} \phi_n(\xi)$ exists and is finite for all $\xi \in E = \mathbb{R}^d$ ($d \geq 1$), and the following technical condition holds:

$$\sum_{n=1}^{\infty} \exp(-a_n M) < +\infty \text{ for all } M > 0; \tag{1}$$

then $\{X_n/a_n\}_{n\geq 1}$ converges a.s. if and only if $\lim_{n\to\infty} \phi_n(\xi)$ is differentiable at $\xi = 0$ (see Ellis [6], Theorem IV.1). Also, in infinite dimension, the existence and differentiability of $\lim_{n\to\infty} \phi_n(\xi)$ at $\xi = 0$ are intimately related to $\{X_n\}_{n\geq 1}$ being exponentially tight (see Cox and Griffeath [3] and Sun [12]).

It is the purpose of this paper to analyze the a.s. behavior of $\{X_n/a_n\}_{n\geq 1}$ under more pessimistic scenarios, namely $\lim_{n\to\infty} \phi_n(\xi)$ *does not exist or one is unable to find it*; what type of hypotheses will still ensure the a.s. convergence of $\{X_n/a_n\}_{n\geq 1}$? We present two such results, with examples and practical applications; they both complement the large and moderate deviation problems treated in the literature in both finite and infinite dimension.

2. A Convex Dominating Function

In the first result, one needs to find a finite convex function ϕ dominating $\limsup_{n\to\infty} \phi_n(\xi)$, and with no supplementary properties (differentiability, etc).

Theorem 1. *Assume (1) and the existence of a convex function ϕ on E^* with $\phi(0) = 0$ and*

$$-\infty < \liminf_{n\to\infty} \phi_n(\xi) \leq \limsup_{n\to\infty} \phi_n(\xi) \leq \phi(\xi) < +\infty \text{ for } \xi \in E^*. \tag{2}$$

If $D^{\pm}\phi_\xi(0)$ denote the right-hand and left-hand derivatives at $t = 0$ of the function $\phi_\xi(t) := \phi(t\xi), t \in \mathbb{R}$, we then have:

$$D^-\phi_\xi(0) \leq \liminf_{n\to\infty} \frac{<\xi, X_n>}{a_n} \leq \limsup_{n\to\infty} \frac{<\xi, X_n>}{a_n} \leq D^+\phi_\xi(0)$$
$$\text{a.s. for } \xi \in E^*.$$

Remarks. As the function ϕ_ξ in Theorem 1 is convex on $\mathbb{R}$, notice that both $D^{\pm}\phi_\xi(0)$ exist, but as extended real numbers. If

$$\underline{\phi}(\xi) \leq \liminf_{n\to\infty} \phi_n(\xi) \leq \limsup_{n\to\infty} \phi_n(\xi) \leq \phi(\xi) \leq \overline{\phi}(\xi) \text{ for } \xi \in E^*,$$

for some convex functions $\underline{\phi}$ and $\overline{\phi}$, and under additional assumptions (including differentiability of $\overline{\phi}$), large deviation principles for $\{X_n\}_{n\geq 1}$ are proved in Cox and Griffeath [3], Lemma 7, Lee and Rémillard [10], Lemma 9, and Wu [13], Theorem 1.2; therefore they guarantee the logarithmic order of decay of $P\{X_n/a_n\}$. Under milder assumptions, our Theorem 1 complements their results in describing the a.s. asymptotic behavior of $\{X_n/a_n\}_{n\geq 1}$.

Proof. First we have the following:

a) if $\alpha \geq D^-\phi_\xi(0)$, there exists $t > 0$ such that $\alpha t > \phi_\xi(t)$.

b) if $\alpha \leq D^+\phi_\xi(0)$, there exists $t > 0$ such that $-\alpha t > \phi_\xi(-t)$.

Indeed, by Lemma VII 3.1 in Ellis [7], for α as in a) and b) above, the Legendre-Fenchel transform $\phi^*_\xi(\alpha)$ of $\phi_\xi(\alpha)$ is strictly positive. As, by definition, we have

$$\phi^*_\xi(\alpha) = \sup_{t\in\mathbb{R}} \{\alpha t - \phi_\xi(t)\} = \sup_{t\in\mathbb{R}} \{-\alpha t - \phi_\xi(-t)\},$$

and using again the above quoted result, we obtain that each sup in the above formulas is attained at some strictly positive t, hence a) and b) are proved.

For fixed $\xi \in E^*$, let $\alpha > D^+\phi_\xi(0)$, and $t > 0$ given by b). We have

$$\frac{1}{a_n}\log P\left[\frac{<\xi, X_n>}{a_n} \geq \alpha\right]$$
$$\leq \frac{1}{a_n}\log\left[\exp(-a_n\alpha t)\int_{\{<\xi,X_n>\geq\alpha a_n\}} \exp(t<\xi, X_n>)dP\right] \leq -\alpha t + \phi_n(t\xi).$$

Pass to lim sup in the above inequalities and use b) to deduce that

$$\limsup_{n\to\infty} \frac{1}{a_n}\log P\left[\frac{<\xi, X_n>}{a_n} \geq \alpha\right] \leq -\alpha t + \phi(t\xi) = -\alpha t + \phi_\xi(t) < 0. \qquad (3)$$

For $m \geq 1$, consider the following events in $\mathcal{A}$:

$$T^\xi_{mn} = \left\{\frac{<\xi, X_n>}{a_n} \geq D^+\phi_\xi(0) + \frac{1}{m}\right\}.$$

By formula (3) there exist $M > 0$ and $N \geq 1$ such that

$$\frac{1}{a_n}\log P(T^\xi_{mn}) \leq -M$$

for $n \geq N$, therefore

$$\sum_{n\geq 1} P(T^{\xi}_{mn}) = \sum_{n<N} P(T^{\xi}_{mn}) + \sum_{n\geq N} P(T^{\xi}_{mn}) < \infty$$

by using hypothesis (1). Borel-Cantelli's Lemma gives $P(\limsup_{n\to\infty} T^{\xi}_{mn}) = 0$ for $m \geq 1$, and hence

$$P\left[\limsup_{n\to\infty} \frac{< \xi, X_n >}{a_n} \leq D^{+}\phi_{\xi}(0)\right] = 1.$$

Similarly one can prove the second inequality in Theorem 1, by considering the events $\left\{\frac{<\xi, X_n>}{a_n} \leq D^{-}\phi_{\xi}(0) - \frac{1}{m}\right\}$ and using a) in showing that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P\left[\frac{< \xi, X_n >}{a_n} \leq \alpha\right] < 0$$

for $\alpha < D^{-}\phi_{\xi}(0)$. □

Examples. (i) Assume $E = \mathbb{R}$ and take $\xi = 1$ in Theorem 1; one obtains

$$D^{-}\phi(0) \leq \liminf_{n\to\infty} \frac{X_n}{a_n} \leq \limsup_{n\to\infty} \frac{X_n}{a_n} \leq D^{+}\phi(0) \text{ a.s.},$$

where $D^{\pm}\phi(0)$ are the right-hand and left-hand derivatives at 0 of ϕ.

For instance, if $X_n = (-1)^n n$, we have $\phi_{2n}(\xi) = \xi$ and $\phi_{2n+1}(\xi) = -\xi$. A dominating function satisfying hypothesis (2) in Theorem 1 is $\phi(\xi) = |\xi|$, and hence $D^{\pm}\phi_{\xi}(0) = \pm 1$, precisely the lim sup and lim inf values of $\{X_n/n\}_{n\geq 1}$.

Consider a sequence $\{X_n\}_{n\geq 1}$ such that $P[X_n = b_n] = p_n$ and $P[X_n = c_n] = 1 - p_n$. Here $0 < p_n \leq 1$ for $n \geq 1$, and $\{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}$ are strictly positive numbers with $\lim_{n\to\infty} \max(b_n, c_n) = +\infty$. We have

$$-\xi^{-} + \frac{\log[2\min(p_n, 1-p_n)]}{\max(b_n, c_n)} \leq \phi_n(\xi) \leq \xi^{+} + \frac{\log[2\max(p_n, 1-p_n)]}{\max(b_n, c_n)},$$

where $\xi^{\pm}$ denote the positive and negative part of ξ, hence we deduce that a dominating convex function satisfying hypothesis (2) is $\phi(\xi) = \xi^{+}$. In this case we have $D^{-}\phi_{\xi}(0) = 0$ and $D^{+}\phi_{\xi}(0) = 1$, precisely the lim sup and lim inf values of $\{X_n/\max(b_n, c_n)\}_{n\geq 1}$.

(ii) In infinite dimension, let us consider $E = L^2[0,1]$, $a_n = n$, and $X_n = (-1)^n n$ on the set $[0, 1 - 1/n]$ and $X_n = 0$ otherwise. By Riesz' Theorem, each element in E^* is represented by a function $\xi \in L^2[0,1]$. As such, we can identify in Theorem 1: $\phi(\xi) = ||\xi||_{L^1[0,1]}$ and $D^{\pm}\phi_\xi(0) = \pm||\xi||_{L^1[0,1]}$, precisely the lim inf and lim sup values of $\{< \xi, X_n > /n\}_{n\geq 1}$.

3. An Exponential Submartingale Condition

An alternative to Theorem 1 is to check the following non-convex hypotheses for a.s. convergence.

Theorem 2. *Assume (1) and let $F \subset E^*$ be a linear subspace such that all $\xi \in F$ are $\mathcal{E}$-measurable. Consider $A_n, A : F \times E \to \mathbb{R}$ such that $A_n(\xi, \cdot)$ are $\mathcal{E}$-measurable, $A(\xi, \cdot)$ is continuous,*

$$\lim_{n\to+\infty} \sup_{x\in E} \left| \frac{1}{a_n} A_n(\xi, x) - A(\xi, x) \right| = 0 \text{ for } \xi \in F, \tag{4}$$

and

$$E[\exp(< \xi, X_n > - A_n(\xi, X_n))] \leq 1 \text{ for } \xi \in F \text{ and } n \geq 1. \tag{5}$$

Then

$$\limsup_{n\to\infty} \frac{< \xi, X_n >}{a_n} \leq \sup_{x\in E} A(\xi, x) \text{ a.s. for } \xi \in F.$$

If, in addition,

$$\limsup_{n\to\infty} A_n(\cdot,\cdot) \leq 0, \text{ or } \limsup_{n\to\infty} [A_n(\xi,\cdot) + A_n(-\xi,\cdot)] \leq 0 \text{ for } \xi \in F, \tag{6}$$

then

$$\inf_{x\in E} A(\xi, x) \leq \liminf_{n\to\infty} \frac{< \xi, X_n >}{a_n} \text{ a.s. for } \xi \in F.$$

Remarks. Although the function $A(\xi, \cdot)$ in Theorem 2 is assumed continuous, the topological vector space E is not assumed compact, hence both $\inf A(\xi, \cdot)$ and $\sup A(\xi, \cdot)$ exist, but as extended real numbers. Formula (5) might be thought of as an exponential submartingale condition; it supplies the lack of information on the existence of $\lim_{n\to\infty} \phi_n(\xi)$ by a smooth behavior upon the correction terms $A_n(\xi, \cdot)$. For a general submartingale, $A_n(\xi, \cdot)$ are typically negative, yielding the first condition in (6). The second hypothesis in (6) is natural (and obviously satisfied) when E is a space of functions and $A_n(\xi, x)$ are (Daniell or Bochner) integrals of x against the measure ξ.

Proof. Fix $\xi \in F$ and denote $b = \sup_{x\in E} A(\xi, x)$; for $\alpha > b$ we want to prove that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P\left[\frac{<\xi, X_n>}{a_n} \geq \alpha\right] < 0.$$

By hypothesis (4), there exist $N \geq 1$ such that $A_n(\xi, x) \leq b a_n$ for $x \in E$ and $n \geq N$. Hence

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P\left[\frac{<\xi, X_n>}{a_n} \geq \alpha\right] \leq \limsup_{n\to\infty} \frac{1}{a_n}$$
$$\times \log\left[\exp[a_n(-\alpha+b)] \int_{\{<\xi, X_n> \geq \alpha a_n\}} \exp(<\xi, X_n> - A_n(\xi, X_n)) dP\right]$$
$$\leq (-\alpha+b) + \limsup_{n\to\infty} \frac{1}{a_n} \log E[\exp(<\xi, X_n> - A_n(\xi, X_n))] \leq -\alpha + b < 0,$$

by using hypothesis (5). Similarly one can prove that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P\left[\frac{<\xi, X_n>}{a_n} \leq \alpha\right] < 0$$

for $\alpha < a$, where $a = \inf_{x\in E} A(\xi, x)$. Indeed, by (4), there exist $N \geq 1$ such that $A_n(\xi, x) \geq a a_n$ for $x \in E$ and $n \geq N$; hence

$$\frac{1}{a_n} \log P\left[\frac{<\xi, X_n>}{a_n} \leq \alpha\right] \leq \frac{1}{a_n}$$
$$\times \log\left[\exp[a_n(\alpha - a)] \int_{\{<\xi, X_n> \leq \alpha a_n\}} \exp(-<\xi, X_n> + A_n(\pm\xi, X_n)) dP\right].$$

By using hypothesis (5) and (6), we deduce that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P\left[\frac{<\xi, X_n>}{a_n} \leq \alpha\right]$$
$$\leq (\alpha - a) + \limsup_{n\to\infty} \frac{1}{a_n} \log E[\exp(<-\xi, X_n> - A_n(-\xi, X_n))] \leq \alpha - a < 0.$$

To finish the proof, apply to the events

$$\left\{\frac{<X_n, \xi>}{a_n} \geq b + \frac{1}{m}\right\} \text{ and } \left\{\frac{<\xi, X_n>}{a_n} \leq a - \frac{1}{m}\right\}$$

a similar argument as we did to T^{ξ}_{mn} in Theorem 1. □

Examples. (i) Consider a seuence $\{X_n\}_{n\geq 1}$ with $P[X_n = 1] = P[X_n = -1] = 1/2$; Theorem 2 applies with $a_n = n$, $A_n(\xi, x) = \xi \operatorname{sgn}(x)$ and $A(x, \xi) = 0$. Consider a seuence $\{X_n\}_{n\geq 1}$ with $P[X_n = n] = P[X_n = -n] = 1/2$; Theorem 1 applies with $a_n = n$ and $\phi(\xi) = |\xi|$. However, in the latter case, hypotheses (5)-(6) in Theorem 2 are satisfied with $A_n(\xi, x) = |x|\xi$, but hypothesis (4) is not satisfied. Therefore, Theorem 1 is preferred to apply in finite dimension.

(ii) To check condition (2) and to compute $D^{\pm}\phi_\xi(0)$ in Theorem 1 may become rather cumbersome in infinite dimension, hence Theorem 2 may be a better solution. For instance, consider the case of Itô stochastic equations driven by Brownian motion W_t and a Poisson random measure $\mu(dt)$, with $E = D[0, 1]$, Skorohod's space, and $F =$ the space of Borel measures on $[0, 1]$:

$$X_t^n = X_t^0 + \int_0^t b(s, X_s^n)ds + n^{-1/2}\int_0^t \sigma(s, X_s^n)dW_s + n^{-1}\int_0^t f(s, X_s^n)\mu(ds),$$

for $t \in (0, 1]$; if the above coefficients are non-degenerate, bounded and uniformly Lipschitz, there is a unique a.s. strong solution X_n. As proved in de Acosta [1], Proposition 4.4, assumptions (4)-(6) are satisfied for some function A, hence our Theorem 2 says that the $\liminf$ and $\limsup$ values of $\frac{1}{n}\int X_n d\xi$, for $\xi \in F$, both exist and are finite a.s. It is interesting to remark that, under additional assumptions (including convexity and Gâteaux differentiability of A), non-convex large deviation principles for $\{X_n\}_{n\geq 1}$ are proved in Dupuis and Ellis [5], de Acosta [1], Feng [8].

4. Applications

In statistical mechanics, X_n represent configuration-dependent quantities in a sequence of physical systems which are proportional to a_n. As the cumulant generating functions associated to $\{X_n\}_{n\geq 1}$ may fail to converge (or the limit is difficult to compute), the total spin in Ising and related models of ferromagnetism may fail to converge, too. Theorem 1 and Theorem 2 offer bounds for the limiting values of almost all configurations, that is, estimates of how bad the above lack of convergence might be. In physical terminology (see Ellis [7], Chapters 3-4), *our theorems estimate the ferromagnetic phase transition or the spontaneous magnetization.*

In mathematical economics, X_n refers to the total excess demand at a given price in an economy of size a_n. In spite of the fact that hypotheses (2) or (4)-(6) do not ensure validity of the law of large numbers for random economies (see Nummelin [11], Theorem 2), however our Theorem 1 and Theorem 2 measure *the distance between the limit values of random excess demands and their counterparts in the "expectation economy"*, that is, in which the excess demand equals its expected value.

Let $\{X_n\}_{n\geq 1}$ be a Galton-Watson branching process in varying environments. In the supercritical case, the suitably normalized population size converges a.s. to a non-degenerate limit; the latter is strictly positive on the survival set $\{X_n \to +\infty\}$ of the process. Theorem 1 measures how far $\{X_n/a_n\}_{n\geq 1}$ is from a.s. convergence in the critical and subcritical cases. In the *uniformly supercritical case*, Theorem 2 measures the degree of degeneracy of a.s.-$\lim_{n\to\infty} X_n/a_n$, where $\{a_n\}_{n\geq 1}$ is the *Seneta-Heyde norming* of $\{X_n\}_{n\geq 1}$ (see Cohn [2], Section 4).

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