A NOTE ON OPERATORS CONJUGATE TO
d-HOMOMORPHISMS

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Abstract: Let $X, Y$ be two Banach $f$-modules over the same Banach $f$-algebra $A$ and suppose that $T$ is an $A$-orthomorphism continuous linear operator from $X$ into $Y$. It is shown that $T' \in dh(Y', X')$, where $T'$ is the continuous adjoint of $T$.

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1. Introduction

Let $X$ be a Banach space and $A$ be a Banach $f$-algebra with unit. By $L(X)$ we denote the set of all continuous linear operators from $X$ into $X$. We say that $X$ is a Banach $f$-module if there exists a bilinear mapping $p : A \times X \to X, (a, x) \to a.x$ satisfying the following conditions:

(i) $1.x = x$ for all $x \in X, 1 \in A$;
(ii) $(ab).x = a.(b.x)$ for all $a, b \in A, x \in X$;
(iii) $\|a.x\| \leq \|a\|\|x\|$ for all $a \in A, x \in X$.

Bilinear mapping $p$ induces $m : A \to L(X), (a, x) \to m(a)x = a.x$ is a unital norm $\|\| \text{ to } SOT$ (the strong operator topology) continuous algebra homomorphism. We let $X'$ denote the dual of a Banach $X$. We can establish
the following bilinear mappings:

\[(A) \quad X' \times X \to A', (f, x) \to (f.x)(a) = f(a.x), \ f \in X', x \in X;\]

\[(B) \quad A'' \times X' \to X', (a, f) \to (a.f)(x) = a(f.x), \ a \in A'', f \in X', x \in X;\]

\[(C) \quad A'' \times X'' \to X'', (a, z) \to (a.z)(f) = z(a.f), \ f \in X', a \in A'', z \in X''.\]

When \(X\) is taken as \(A\), then \((C)\) becomes the Arens product on \(A''\). The bilinear mapping \((B)\) defines a Banach \(A''\)-module structure on \(X'\) that gives a homomorphism \(m^\star: A'' \to L(X')\) defined by \(m^\star(a) = a.f\). The bilinear mapping \((C)\) defines a Banach \(A''\)-module structure on \(X''\). These are called the Arens extensions of the module multiplication on \(X\), Arens [3]. Note that for each \(a \in A, m^\star(a)\) is the adjoint in \(L(X')\) of the operator \(m(a)\) in \(L(X)\), and \(m^\star\) is (weak*, weak* -operator) continuous, and also for each \(f \in X'\) the linear mapping from \(A''\) into \(X'\) that sends \(a\) to \(a.f\) is \(\sigma(A'', A'), \sigma(X', X)\) continuous, Hadwin et al [4]. For terminology and unproved properties of Banach modules, Banach lattices, and f-algebra we refer to the reader to Abramovich et al [1, Aliprantis et al [2], Meyer [5]. We need the following result from Abramovich et al [1, Theorem 3.3].

**Theorem 1.** Let \(X\) and \(Y\) be vector lattices, and \(T : X \to Y\) a linear operator. The following conditions are equivalent:

1. \(T\) is an order bounded disjoint homomorphism.
2. \(|x_1| \leq |x_2|\) implies that \(|Tx_1| \leq |Tx_2|\).

**Definition.** If \(X\) is a Banach f-module over a Banach f-algebra \(A\) with unit and \(x \in X\), then

\[\Delta(x) = \text{Cl}_X \{a.x : a \in A, \|a\| \leq 1\},\]

where \(\text{Cl}_X\) denotes the norm closure in \(X\). A linear subspace \(Y\) of a Banach f-module \(X\) over a Banach f-algebra \(A\) is said to be an order ideal if for each \(x \in Y\) the whole interval \(\Delta(x)\) belongs to \(Y\). Two elements \(x, y\) of a Banach f-module \(X\) over a Banach f-algebra \(A\) are called disjoint \((x \perp y)\) if \(\Delta(x + y) = \Delta(x) + \Delta(y)\) and \(\Delta(x) \cap \Delta(y) = \{0\}\). Let \(X\) be a Banach f-module over a Banach f-algebra \(A\) and let \(f \in X'\), then

\[\Delta(f) = \text{Cl}_{X'} = \{a.f : a \in A'', \|a\| \leq 1\},\]

where \(\text{Cl}_{X'}\) denotes the closure in \(X'\).
Lemma 2. Let $X$ be a Banach $f$-module over a Banach $f$-algebra $A$ and $x \in X$. Let 
\[ X(x)_+ = \text{Cl}_X \{ a.x : 0 \leq a \in A \}. \]
Then, the space $X(x)$ ordered by the cone $X(x)_+$ is a Banach lattice.

We need the following lemma and we refer to Abramovich et al [1, Lemma 7.11] for a Banach $C(K)$-module case.

Lemma 3. Let $X$ be a Banach $f$-module and let $f, g \in X'$. The following statements are true.

(i) An element $x \in X$ (as a functional on $X'$) is orthogonal to $X'(f)$ if and only if $f.x = 0$.

(ii) An element $x \in X$ (as a functional on $X'$) is positive on $X'(f)_+$ if and only if $0 \leq f.x$.

(iii) $f \in \Delta(g)$ if and only if $\|f.x\| \leq g.x$ for all $x \in X$ if and only if $|f.x| \leq |g.x|$ for all $x \in X$.

(iv) $f \perp g$ if and only if the elements $f, g$ belong to the lattice $X'(f + g)$ and are disjoint in it if and only if $(f.x) \perp (g.x)$ for all $x \in X$.

Definition. Let $X, Y$ be two Banach $f$-modules over a Banach $f$-algebra $A$. A linear operator $T : X \to Y$ is called an $A$-orthomorphism if $T(a.x) = a.Tx$ for all $a \in A, x \in X$.

Definition. Let $T : X \to Y$ be a linear operator from a Banach $f$-module $X$ into a Banach $f$-module $Y$ over the Banach $f$-algebra $A$. We call operator $T$ a (disjoint) $d$-homomorphism if $x \perp z$ in $X$ implies $Tx \perp Tz$ in $Y$. By $dh(X, Y)$ we denote $d$-homomorphisms from $X$ into $Y$.

Theorem 4. Let $T$ be a continuous linear operator from a Banach $f$-module $X$ over a Banach $f$-algebra $A$ into a Banach $f$-module $Y$ over the same Banach $f$-algebra $A$. The following are equivalent:

(i) $T' \in dh(Y', X')$;

(ii) $f \in \Delta(g) \Rightarrow T'f \in \Delta(T'g)$ for $f, g \in Y'$.

Proof. (i)$\Rightarrow$(ii). Let $f \in \Delta(g)$. Fix $x \in X$ and consider an operator $S : Y'(g) \to A'$ defined by $Sh = T'h.x$. By Lemma 3 (iv) the operator $S$ is a continuous $d$-homomorphism from $Y'(g)$ into $A'$. But each continuous $d$-homomorphism between Banach lattices is regular. Since $f \in \Delta(g)$, Theorem 1 implies that $Sf \in \Delta(g)$ or $|T'f.x| \leq |T'g.x|$. By Lemma 3(iii), we conclude that $T'f \in \Delta(T'g)$.

(ii)$\Rightarrow$(i). Let $f, g \in Y'$ and $f \perp g$. Then, by Lemma 3 (iv) $f \perp g$ in the vector lattice $Y'(f + g)$, and (ii) implies that the restriction of $T'$ to $Y'(f + g)$ takes values in the vector lattice $X'(T'(f + g))$ and satisfies the condition (2)
of Theorem 1. By this theorem, $T'$ is a d-homomorphism between the vector lattices $Y'(f + g)$ and $X'(T'(f + g))$, and hence $T'f \perp T'g$. }

**Theorem 5.** Let $T$ be an $A$-orthomorphism continuous linear operator from a Banach $f$-module $X$ into a Banach $f$-module $Y$ over the same Banach $f$-algebra $A$. Then, $T' \in dh(Y', X')$, where $T'$ is the continuous adjoint of $T$.

**Proof.** It suffices to show that if $f \in \Delta(g)$ then $T'f \in \Delta(T'g)$. Suppose that $f \in \Delta(g)$. Then, there exists a net $(a_\alpha)$ in $A''$, $\|a_\alpha\| \leq 1$ such that $a_\alpha.g \to f$ in $X'$. Since $T'$ is continuous, it implies that $T'(a_\alpha.g) \to T'(f)$. $T$ is an $A$-orthomorphism operator, i.e., for each $a \in A$, $Ta = aT$. Therefore, $T'(a.f) = a.T'f$ for all $a \in A''$. In fact, for an arbitrary $x \in X$ and $f \in X'$ and $a \in A$, $T'(a.f)x = T'(m^*(a)f)x = T'((m(a))^*f)x = f(m(a))x = f(Tm(a))x = (m^*(a)T'f)x = (a.T'f)x$, i.e., $T'(a.f) = a.T'f$. It is well known that $A$ is $\sigma(A'', A')$ dense in $A''$. We obtain that for each $a \in A''$, $a.T'f = T'(a.f)$. Since $T'(a_\alpha.g) = a_\alpha.T'g$ and $\Delta(T'g)$ is closed, it follows that $T'f \in \Delta(T'g)$. Hence, $T' \in dh(Y', X')$. \qed

**References**


