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POLAR FLUID FLOW BETWEEN TWO ECCENTRIC ROTATING CYLINDERS: INERTIAL EFFECTS

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Abstract: The steady flow of an incompressible polar fluid between two rotating eccentric cylinders is studied. Expressions for velocity, stream function and spin are obtained. The torque acting on the inner cylinder is calculated and the effects of the coupling number, the eccentricity and the relative rotational speed of the cylinders on the flow behaviour are discussed.

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1. Introduction

The interest in developing theories to model complex fluids has been steadily increasing. Cowin [5] has introduced the theory of polar fluids to deal with a class of fluids which respond to certain microscopic effects arising from the presence of microstructure and are influenced by spin inertia. The theory appeared under a variety of different names, including micropolar fluids [7] and asymmetric hydrodynamics [1]. It has been applied to blood flow [12], suspensions

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[11], and lubrication [2]. In the polar fluids theory, there are two independent kinematical vector fields: the usual velocity field $\mathbf{V}$ and an axial vector field $\mathbf{G}$ that models the fluid's spin particle. The continuity, linear momentum and angular momentum equations of an incompressible polar fluid are given by:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$(\mu+\tau)\,\nabla^2\,\mathbf{V} + 2\,\tau\,(\nabla \times \mathbf{G}) + \rho\mathbf{b} - \nabla\,p = \rho\,\dot{\mathbf{V}}, \tag{2}$$

$$\begin{aligned}(\alpha+2\beta)\,\nabla\nabla \cdot \mathbf{G} - (\beta+\gamma)\nabla \times (\nabla \times \mathbf{G}) - 4\tau\,\mathbf{G}\\ +2\,\tau(\nabla \times \mathbf{V}) + \rho\,\mathbf{C} = \rho\,k^2\;\dot{\mathbf{G}},\end{aligned} \tag{3}$$

where ρ the density, p the pressure, $\mathbf{b}$ the body force, $\mathbf{C}$ the body couple, k the radius of gyration of the particle. The quantities α, β, γ, μ and τ are viscosity coefficients and are assumed to be constants.

In order to develop a characterization of the polar fluids theory, Cowin [6] introduced two dimensionless parameters. The coupling number $N = \left(\frac{\tau}{\mu+\tau}\right)^{1/2}$, $0 \leq N \leq 1$ measures the ratio of the relational viscosity τ to the linear viscosity μ. It also indicates the coupling between the linear and angular momentum equations. The second is called the length ratio L defined by $L = \frac{L_0}{\ell}$ where L_0 is a geometric length and ℓ is the material characteristic length and is given by $\ell = \left(\frac{\beta+\gamma}{\mu}\right)^{1/2}$, $0 < \ell < \infty$.

In the present work, the flow of a polar fluid between two infinitely long eccentric rotating cylinders is studied. This flow is of importance in rheometry and lubrication ([14], [10], [4]). Several authors have considered this flow using various methods of approximation for Newtonian fluids. Urban [13] has examined the various approximations used and the conditions under which they are valid. Ballal and Rivilin [3] have made the most extensive investigation of this flow.

Kamel [8] has solved the present eccentric flow for a polar fluid in which inertia terms were neglected. The eccentricity ϵ was taken as a perturbation parameter in the perturbation series and terms of $O(\epsilon^2)$ were not included. It was found that up to that order of approximation, the torque acting on the inner cylinder is the same as for the concentric case. In this work Kamel's analysis is extended by including inertia terms and terms of $O(\epsilon^2)$, in order to predict the effect of eccentricity on the torque acting on the cylinders. The problem of a micropolar fluid flow between two eccentric coaxially rotating spheres was investigated by Kamel and Chan Man Fong [9].

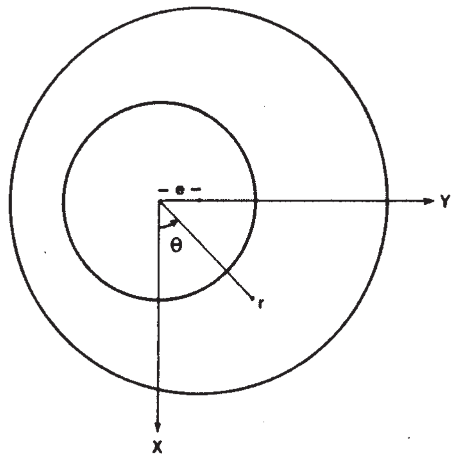

Figure 1: Geometry of eccentric cylinders

2. Basic Equations

As in [8], cylindrical polar coordinate system (r, θ, z) is employed with the z axis coinciding with the axis of the inner cylinder (Figure 1). Let e be the distance between the axes of the two cylinders, R_1, Ω_1 and R_2, Ω_2 be the radii and angular velocities of the inner and outer cylinders about their axes, respectively. Consider R_2 as a typical length and $U(= R_2\Omega_2)$ as the typical velocity.

Then under the conditions stated in [8] retaining the inertia terms but not the microinertia, the dimensionless equations governing the flow are:

$$\mathrm{Re}\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+2N^2\frac{1}{r}\frac{\partial G_z}{\partial\,\theta}, \tag{4}$$

$$\mathrm{Re}\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)-2N^2\frac{\partial G_z}{\partial\,r}, \tag{5}$$

$$\nabla^2 G_z - 4\lambda^2 G_z + 2\lambda^2\left(\frac{\partial\, v_\theta}{\partial\, r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial\, v_r}{\partial \theta}\right) = 0, \tag{6}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial\, r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\, \theta^2}$, $\lambda^2 = \frac{N^2 L^2}{1-N^2}$, Re $= \frac{\rho U R_2}{\mu+\tau}$ is the Reynolds number, N is the coupling number, L is the length ratio, v_r, v_θ and G_z are the radial, transversal, and axial components of the velocity and microstructure spin respectively.

The no-slip and no-spin boundary conditions may be written as:

at the inner cylinder:

$$v_r = 0\ ,\ v_\theta = \mu_1 \eta\ ,\ G_z = \mu_1, \tag{7a}$$

at the outer cylinder:

$$v_r = \epsilon\ \cos\theta\ ,\ v_\theta = r_2 - \epsilon\ \sin\theta\ ,\ G_z = 1\,, \tag{7b}$$

where $\eta = \frac{R_1}{R_2}$, $\mu_1 = \frac{\Omega_1}{\Omega_2}$, $\epsilon = \frac{e}{R_2}$ and $r_2 = \epsilon\ \sin\ \theta + (1 - \epsilon^2\ \cos^2\ \theta)^{1/2}$ is the equation of the outer cylinder.

3. Solutions

In order to obtain the solution of (4) − (6) for small ϵ and Re, it is assumed that v_r, v_θ, G_z and p can be expanded in power series of the form:

$$\left.\begin{aligned} v_\theta &= v_0(r) + \epsilon\, v_1(r,\theta) + \epsilon\text{Re}\ v_2(r,\theta) + \epsilon^2\, v_3(r,\theta) + \dots\,, \\ v_r &= \epsilon\, u_1(r,\theta) + \epsilon\text{Re}\ u_2(r,\theta) + \epsilon^2 u_3(r,\theta) + \dots\,, \\ G_z &= G_0(r) + \epsilon\, G_1(r,\theta) + \epsilon\text{Re}\ G_2(r,\theta) + \epsilon^2\, G_3(r,\theta) + \dots\,, \\ p &= p_0(r) + \epsilon\, p_1(r,\theta) + \epsilon\text{Re}\ p_2(r,\theta) + \epsilon^2\, p_3(r,\theta) + \dots\,. \end{aligned}\right\} \tag{8}$$

Substituting (8) into (4) − (6) and comparing powers of ϵ, the resulting equations are:

$$O(\epsilon^0):\ \frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} - \frac{v_0}{r^2} - 2N^2\frac{dG_0}{dr} = 0, \tag{9a}$$

$$\frac{d^2 G_0}{dr^2} + \frac{1}{r}\frac{dG_0}{dr} - 4\lambda^2 G_0 + 2\lambda^2\left(\frac{dv_0}{dr} + \frac{v_0}{r}\right) = 0. \tag{9b}$$

$$O(\epsilon):\ -\frac{\partial\, p_1}{\partial r} + \left(\nabla^2 u_1 - \frac{u_1}{r^2} - \frac{2}{r^2}\frac{\partial\, v_1}{\partial\, \theta}\right) + 2N^2\frac{1}{r}\frac{\partial G_1}{\partial\, \theta} = 0, \tag{10a}$$

$$-\frac{1}{r}\frac{\partial p_1}{\partial \theta}+\left(\nabla^2 v_1-\frac{v_1}{r^2}+\frac{2}{r^2}\frac{\partial u_1}{\partial \theta}\right)-2N^2\frac{\partial G_1}{\partial r}=0, \tag{10b}$$

$$\nabla^2 G_1-4\lambda^2 G_1+2\lambda^2\left(\frac{\partial v_1}{\partial r}+\frac{v_1}{r}-\frac{1}{r}\frac{\partial u_1}{\partial \theta}\right)=0. \tag{10c}$$

$$O(\epsilon\mathrm{Re}):\ \frac{v_0}{r}\frac{\partial u_1}{\partial \theta}-\frac{2v_0v_1}{r}=-\frac{\partial p_2}{\partial r}+\left(\nabla^2 u_2-\frac{u_2}{r^2}-\frac{2}{r^2}\frac{\partial v_2}{\partial \theta}\right)$$

$$+2N^2\frac{1}{r}\frac{\partial G_2}{\partial \theta}, \tag{11a}$$

$$u_1\frac{\partial v_0}{\partial r}+\frac{v_0}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1v_0}{r}=-\frac{1}{r}\frac{\partial p_2}{\partial \theta}+\left(\nabla^2 v_2-\frac{v_2}{r^2}+\frac{2}{r^2}\frac{\partial u_2}{\partial \theta}\right),$$

$$-2N^2\frac{\partial G_2}{\partial r}, \tag{11b}$$

$$\nabla^2 G_2-4\lambda^2 G_2+2\lambda^2\left(\frac{\partial v_2}{\partial r}+\frac{v_2}{r}-\frac{1}{r}\frac{\partial u_2}{\partial \theta}\right)=0. \tag{11c}$$

$$O(\epsilon^2):-\frac{\partial p_3}{\partial r}+\left(\nabla^2 u_3-\frac{u_3}{r^2}-\frac{2}{r^2}\frac{\partial v_3}{\partial \theta}\right)+2N^2\frac{1}{r}\frac{\partial G_3}{\partial \theta}=0, \tag{12a}$$

$$-\frac{1}{r}\frac{\partial p_3}{\partial \theta}+\left(\nabla^2 v_3-\frac{v_3}{r^2}+\frac{2}{r^2}\frac{\partial u_3}{\partial \theta}\right)-2N^2\frac{\partial G_3}{\partial r}=0, \tag{12b}$$

$$\nabla^2 G_3-4\lambda^2 G_3+2\lambda^2\left(\frac{\partial v_3}{\partial r}+\frac{v_3}{r}-\frac{1}{r}\frac{\partial u_3}{\partial \theta}\right)=0. \tag{12c}$$

The boundary conditions (7a) on the inner cylinder can be written down directly and they are:

$$\left.\begin{array}{l} u_1(\eta,\theta)=u_2(\eta,\theta)=u_3(\eta,\theta)=0\,,\\ v_0(\eta)=\mu_1\eta\,,\ v_1(\eta,\theta)=v_2(\eta,\theta)=v_3(\eta,\theta)=0\,,\\ G_1(\eta,\theta)=G_2(\eta,\theta)=G_3(\eta,\theta)=0\,. \end{array}\right\} \tag{13}$$

Since the small parameter ϵ appears in r_2, the equation of the outer cylinder, the boundary conditions on the outer cylinder, can be obtained by expanding v_r, v_θ and G_z in Taylor series about $r=\eta$, the concentric case, then by comparing powers of ϵ the following are obtained:

$$\left.\begin{array}{l} v_0(1)=1\,,\quad v_1(1,\theta)=-\sin\theta\, v_0'(1),\quad v_2(1,\theta)=0,\\ v_3(1,\theta)=-\frac{1}{2}\cos^2\theta+\frac{1}{2}\cos^2\theta\, v_0'(1)-\frac{1}{2}\sin^2\theta\, v_0''(1)\\ -\sin\theta\, v_1'(1,\theta),\\ u_1(1,\theta)=\cos\theta,\quad u_2(1,\theta)=0,\quad u_3(1,\theta)=-\sin\theta\, u_1'(1,\theta),\\ G_0(1)=1,\quad G_1(1,\theta)=-\sin\theta\, G_0'(1),\quad G_2(1,\theta)=0,\\ G_3(1,\theta)=\frac{1}{2}\cos^2\theta\, G_0'(1)-\frac{1}{2}\sin^2\theta\, G_0''(1)-\sin\theta\, G_1'(1)\,. \end{array}\right\} \tag{14}$$

Equations (9) and (10) have been solved in [8] and the results are quoted:

$$v_0 = A_0(\frac{N}{L})\, I_1(2NLr) - B_0(\frac{N}{L})K_1(2NLr) + \frac{C_0}{2(1-N^2)}r + D_0\,\frac{1}{r}\,, \tag{15a}$$

$$G_0 = A_0 I_0(2NLr) + B_0 K_0(2NLr) + \frac{C_0}{2(1-N^2)}\,, \tag{15b}$$

$$u_1 = F(r)\cos\theta\ ,\ v_1 = -\frac{d}{dr}[r\,F(r)]\sin\theta\ ,\ G_1 = -g_1(r)\sin\theta\,, \tag{15c}$$

where $F(r) = \frac{A_1}{8N^2L^4}\,\frac{I_1(2NLr)}{r} + \frac{B_1}{8N^2L^4}\,\frac{K_1(2NLr)}{r} + C_1\,\frac{r^2}{4},$

$$+ D_1(\ln\, r - \frac{1}{2}) + E_1 + \frac{F_1}{r^2},$$

$$g_1(r) = \frac{A_1}{4N^2L^2}\, I_1(2NLr) + \frac{B_1}{4N^2L^2}\, K_1(2NLr) + C_1 r + D_1\,\frac{1}{r},$$

$A_0, B_0, ..., C_1, D_1$, are arbitrary constants to be determined from the boundary conditions, I_n and K_n are modified Bessel functions of order n of the first and second kind, respectively.

Eliminating p_2 between (11a) and (11b) and introducing a stream function Ψ_2 such that:

$$u_2 = \frac{1}{r}\,\frac{\partial\,\Psi_2}{\partial\,\theta}\quad,\quad v_2 = -\frac{\partial\,\Psi_2}{\partial\,r}\,,$$

the resulting equations are:

$$u_1\left[\frac{d^2v_0}{dr^2} + \frac{1}{r}\,\frac{dv_0}{dr}\right] + v_0\left[\frac{1}{r}\,\frac{\partial^2 v_1}{\partial r\partial\theta} - \frac{1}{r^2}\,\frac{\partial^2 u_1}{\partial\theta^2} + \frac{1}{r^2}\,\frac{\partial v_1}{\partial\theta} - \frac{u_1}{r^2}\right]$$

$$= -\nabla^4\Psi_2 - 2N^2\,\nabla^2 G_2, \tag{16a}$$

$$(\nabla^2 - 4\lambda^2)G_z - 2\lambda^2\nabla^2\Psi_2 = 0. \tag{16b}$$

From equations (13),(14) and (15) it can be seen that Ψ_2 and G_2 are of the form

$$\Psi_2 = \psi_2(r)\,\cos\,\theta\,,\quad G_2 = g_2(r)\,\cos\,\theta \tag{17}$$

Substituting (17) into (16) and using (15) the following equations are obtained:

$$D^4\psi_2 + 2N^2D^2g_2 = P(r)\,, \tag{18a}$$

$$D^2g_2 - 4\lambda^2 g_2 - 2\lambda^2 D^2\psi_2 = 0, \tag{18b}$$

where $D^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ and

$$P(r) = -F\left[\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr}\right] + H\left[\frac{d^2F}{dr^2} + \frac{3}{r}\frac{dF}{dr} + \frac{F}{r^2}\right].$$

Eliminating ψ_2 between (18a) and (18b), a differential equation in g_2 is obtained whose solution may be written as

$$g_2 = r\,A(r) + \frac{B(r)}{r}, \tag{19}$$

where:
$A(r) = \int^r [D(\xi)I_1(2NL\xi) + E(\xi)K_1(2NL\xi)]\,d\xi$,
$B(r) = -\int^r \xi^2[D(\xi)I_1(2NL\xi) + E(\xi)K_1(2NL\xi)]\,d\xi$,
$D(r) = \lambda^2\int^r \xi\,K_1(2NL\xi)\,P(\xi)\,d\xi$,
$E(r) = -\lambda^2\int^r \xi\,I_1(2NL\xi)\,P(\xi)\,d\xi$.

From (18b), and after lengthy calculations, ψ_2 can be written as

$$\psi_2 = r\,A_\psi(r) + \frac{B_\psi(r)}{r}, \tag{20}$$

where:

$$A_\psi(r) = \frac{1}{2\lambda^2}A(r) - \int^r\left[\xi\,A(\xi) + \frac{B(\xi)}{\xi}\right]d\xi,$$

$$B_\psi(r) = \frac{r^2}{2\lambda^2}A(r) + \int^r\left[\left(\frac{\xi}{\lambda^2} + \xi^3\right)A(\xi) + \xi\,B(\xi)\right]d\xi.$$

From (13) and (14) the following boundary conditions are deduced:

$$\psi_2(\eta) = \psi_2'(\eta) = \psi_2(1) = \psi_2'(1) = g_2(\eta) = g_2(1) = 0. \tag{21}$$

In order to obtain u_3, v_3 and G_3, a similar procedure to the above is followed. Eliminating p_3 between (12a) and (12b) and introducing a stream function Ψ_3 defined by

$$u_3 = \frac{1}{r}\frac{\partial\Psi_3}{\partial\theta}, \quad v_3 = -\frac{\partial\Psi_3}{\partial r}. \tag{22}$$

Then Ψ_3 and G_3 satisfy the following equations:

$$\nabla^4\Psi_3 + 2N^2\,\nabla^2 G_3 = 0, \tag{23a}$$

$$(\nabla^2 - 4\lambda^2)G_3 - 2\lambda^2\nabla^2\Psi_3 = 0. \tag{23b}$$

From above and the boundary conditions, forms for Ψ_3 and G_3 are assumed to be:

$$\Psi_3(r,\theta) = \psi_3(r)\cos^2\theta + \phi_3(r)\,, \tag{24a}$$

$$G_3(r,\theta) = g_3(r)\cos^2\theta + g_4(r)\,. \tag{24b}$$

Substituting (24) and (23), equating terms involving $\cos^2\theta$ and terms independent of θ, the following are obtained:

$$g_3^{(iv)} + \frac{2}{r}\,g_3''' - \frac{9}{r^2}\,g_3'' + \frac{9}{r^3}\,g_3' - 4N^2L^2\left(g_3'' + \frac{1}{r}\,g_3' - \frac{4}{r^2}\,g_3\right) = 0, \tag{25a}$$

$$g_4^{(iv)} + \frac{2}{r}\,g_4''' - \frac{1}{r^2}\,g_4'' + \frac{1}{r^3}\,g_4' - 4N^2L^2\left(g_4'' + \frac{1}{r}\,g_4'\right) = \frac{8N^2L^2}{r^2}\,g_3 + \frac{4}{r^3}g_3' - \frac{4}{r^2}g_3'', \tag{25b}$$

$$\psi_3'' + \frac{1}{r}\,\psi_3' - \frac{4}{r^2}\,\psi_3 = \frac{1}{2\lambda^2}\left(g_3'' + \frac{1}{r}\,g_3' - \frac{4}{r^2}g_3\right) - 2g_3, \tag{25c}$$

$$\phi_3'' + \frac{1}{r}\,\phi_3' = \frac{1}{2\lambda^2}\left(g_4'' + \frac{1}{r}\,g_4' + \frac{2}{r^2}g_3\right) - 2g_4 - \frac{2}{r^2}\psi_3. \tag{25d}$$

The solution of (25a) can be seen to be

$$g_3 = A_3 I_2(2NLr) + B_3 K_2(2NLr) + C_3 r^2 + D_3\,\frac{1}{r^2}, \tag{26}$$

where A_3, B_3, C_3, and D_3 are arbitrary constants. Knowing g_3, then the right hand side of (25b) is known and is denoted by $f(r)$. The complementary functions of (25b) are: constant, $\ln r$, $I_0(2NLr)$ and $K_0(2NLr)$. A particular integral of (25b) may be written as

$$g_{4p} = P_1(r) + P_2(r)\,\ln r + P_3(r)\,I_0(2NLr) + P_4(r)\,K_0(2NLr). \tag{27}$$

Substituting (27) into (25b), the expressions for $P_1(r) - P_4(r)$ are found to be given by:

$$P_1(r) = \frac{1}{4N^2L^2}\int^r \xi\, f(\xi)\ln\xi\, d\xi\ ,\ P_2(r) = -\frac{1}{4N^2L^2}\int^r \xi\, f(\xi)\, d\xi,$$

$$P_3(r) = \frac{1}{4N^2L^2}\int^r \xi\, K_0(2NL\xi)\, f(\xi)\, d\xi,$$

$$P_4(r) = -\frac{1}{4N^2L^2}\int^r \xi\, I_0(2NL\xi) f(\xi)\, d\xi,$$

$$f(r) = 4\left\{ A_3 \left[-\frac{2N^2L^2I_2(2NLr)}{r^2} - \frac{8I_2(2NLr)}{r^4} + \frac{4NLI_1(2NLr)}{r^3} \right] \right.$$
$$-B_3 \left[\frac{2N^2L^2K_2(2NLr)}{r^2} + \frac{8K_2(2NLr)}{r^4} + \frac{4NLK_1(2NLr)}{r^3} \right]$$
$$\left. +2N^2L^2C_3 + 2D_3 \left(\frac{N^2L^2}{r^4} - \frac{4}{r^6} \right) \right\}.$$

Thus g_4 can be written as

$$g_4(r) = A_4 I_0(2NLr) + B_4 K_0(2NLr) + C_4 \ln r + D_4 + g_{4p}\,, \tag{28}$$

where $A_4 - D_4$ are arbitrary constants.

The complementary functions of (25c) are r^2 and $\frac{1}{r^2}$. A particular integral ψ_{3p} can be obtained as previously by the method of variation of parameters which may be written as

$$\psi_{3p} = r^2 \gamma_1(r) + \frac{\gamma_2(r)}{r^2}\,, \tag{29}$$

where:

$$\gamma_1(r) = \int^r \alpha_3(\xi) d\xi\ ,\ \gamma_2(r) = -\int^r \xi^4\, \alpha_3(\xi) d\xi,$$

$$\alpha_3(\xi) = \frac{1}{8\lambda^2} \left\{ A_3 \left[\frac{4N^2L^2I_2(2NL\xi)}{\xi} + \frac{5I_2(2NL\xi)}{\xi^3} - \frac{2I_2(2NL\xi)}{\xi} \right] \right.$$
$$\left. +B_3 \left[\frac{4N^2L^2K_2(2NL\xi)}{\xi} + \frac{5K_2(2NL\xi)}{\xi^3} - \frac{2K_2(2NL\xi)}{\xi} \right] - 2C_3\,\xi - 2D_3\, \frac{1}{\xi^3} \right\}.$$

Thus ψ_3 is given by

$$\psi_3(r) = C_5 r^2 + D_5\, \frac{1}{r^2} + \psi_{3p}, \tag{30}$$

where C_5 and D_6 are arbitrary constants.

Similarly, the solution for ϕ_3 can be written as

$$\phi_3(r) = C_6\, \ln r + D_6 + \phi_{3p}, \tag{31}$$

where C_6 and D_6 are arbitrary constants and ϕ_{3p} is given by

$$\phi_{3p} = \gamma_3(r)\, \ln r + \gamma_4(r),$$

and

$$\gamma_3(r) = \int^r \xi\,\alpha_5(\xi)d\xi\ ,\ \gamma_4(r) = -\int^r \xi\,\ln\xi\,\alpha_5(\xi)d\xi,$$

$$\alpha_5(r) = \frac{1}{2\lambda^2}\ \{\,4N^2L^2(A_4I_0(2NLr) + B_4K_0(2NLr))$$

$$+4N^2L^2(P_3(r)I_0(2NLr) + P_4(r)K_0(2LNr))$$

$$+2A_3\,\frac{I_2(2LNr)}{r^2} + 2B_3\,\frac{K_2(2LNr)}{r^2} + 2C_3 + 2D_3\,\frac{1}{r^4}\Big\}$$

$$-2\,\{A_4I_0(2NLr) + B_4K_0(2NLr) + C_4\,\ln r + D_4 + P_1(r)$$

$$+P_2(r)\ln\,r + P_3(r)I_0(2LNr) + P_4K_0(2NLr)$$

$$+C_6 + D_6\,\frac{1}{r^2} + \gamma_1(r) + \frac{\gamma_2(r)}{r^4}\Big\}.$$

The appropriate boundary conditions are:

$$\left.\begin{array}{l} \psi_3(\eta) = \psi_3'(\eta) = \phi_3'(\eta) = g_3(\eta) = g_4(\eta) = 0, \\ \psi_3(1) = \frac{1}{2}F'(1)\,,\ \psi_3' = \frac{1}{2} - F''(1) - 2F'(1) - \frac{1}{2}v_0'(1) - \frac{1}{2}v_0''(1), \\ \phi_3'(1) = F''(1) + 2F'(1) + \frac{1}{2}v_0''(1), \\ g_3(1) = \frac{1}{2}[G_0''(1) + G_0'(1)] - G(1)\ ,\ g_4(1) = G(1) - \frac{1}{2}\,G_0''(1). \end{array}\right\} \tag{32}$$

In all above integrals the lower limit of integration is taken to be η.

4. Torque

The torque M acting on the inner cylinder, in dimensional form is given by

$$M = \int_0^{2\pi} r[r\,T_{\theta r} + \wedge_{zr}]_{r=R_1} d\,,\theta \tag{33}$$

where the shear stress $T_{\theta\,r}$ and the couple stress $\wedge_{zr}$ are given by:

$$T_{\theta r} = 2\mu\,D_{\theta r} - 2\tau H_{\theta r}\ ,\quad \wedge_{zr} = (\beta + \alpha)\,\frac{\partial\,G_z}{\partial\,r}\,,$$

$$D_{\theta r} = \frac{1}{2}\left\{\frac{\partial\,v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\,\frac{\partial\,v_r}{\partial\,\theta}\right\}\,, \tag{34}$$

$$H_{\theta r} = G_{\theta r} - W_{\theta r} = 2G_z - \frac{1}{2}\left[\frac{\partial\,v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\,\frac{\partial\,v_r}{\partial\,\theta}\right]\,.$$

Substituting (34) into (33) and non-dimensionalizing results in

$$\bar{M} = \frac{M}{UR_2(\mu+\tau)} = \int_0^{2\pi} \left\{ r^2 \left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} - 4N^2 G_2 \right.\right. \\ \left.\left. + N^2 \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right) \right] + \frac{(1-N^2)}{L^2} r \frac{\partial G_z}{\partial r} \right\}_{r=\eta} d\theta. \quad (35)$$

From equations (15), (17) and (24), it can be seen that terms of $O(\epsilon)$ and $O(\epsilon \mathrm{Re})$ do not contribute to M. Thus (35) may be written as

$$\bar{M} = M_0 + \epsilon^2 M_3 \,,$$

where

$$M_\circ = 2\pi \left\{ -2N^2\eta^2 I_\circ A_\circ - 2N^2\eta^2 K_\circ B_\circ \; -2D_\circ(1-N^2) - C_\circ \frac{N^2\eta^2}{1-N^2} \right\} ,$$

and

$$M_3 = \pi \left[-\eta^2 \Psi_3'' - 2\eta^2 \phi_3'' + \frac{\eta}{L^2}(1-N^2)\, g_3' + \frac{2\eta}{L}(1-N^2) g_4' \right] .$$

5. Numerical Results

In order to illustrate the results obtained in previous sections, the appropriate boundary conditions are applied to determine the arbitrary constants for particular values of the parameters. Those parameters include the coupling number N, the length ratio L, the eccentricity ϵ and the relative rotational speed of the cylinders μ_1.

5.1. Stream Function

The linear systems associated with the zeroth and first order approximation were solved [8] by the Gaussian elimination method with iterative improvement. However, the determination of $\psi_2(r)$ and $g_2(r)$ involves the computation of triple integrals numerically, the application of boundary conditions (21), the solution of a linear system and then the evaluation of $A_\psi(r)$ and $B_\psi(r)$ for several values of r. The approximate value of the stream function is computed using the formula:

$$\Psi(r,\theta) \approx \psi_0(r) + \epsilon\, \psi_1(r) \sin\theta + \epsilon \,\mathrm{Re}\; \psi_2(r) \cos\theta \,, \quad (36)$$

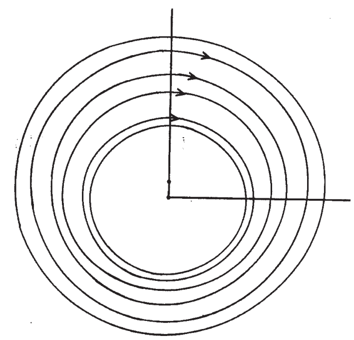

Figure 2: Streamlines $\Psi(r,\theta) = C$ for $\mu_1 = 0.5, \epsilon = 0.1$, $N = 0.5$, $L = 2$, $\eta = 0.5$, Re $= 4$

where

$$\psi_0(r) = -\int v_0(r)dr \quad \text{and} \quad \psi_1(r) = r\,F(r)\,.$$

The streamlines $\Psi(r,\theta) = C$ for $\epsilon = 0.1$, $\mu_1 = 0.5$, $N = 0.5$, $L = 2$, $\eta = 0.5$ and Re $= 4$ are plotted in Figure 2. In this case the outer cylinder is rotating faster than the inner cylinder.

The computer output shows that $\Psi(r,\theta)$ is a strictly monotonic function in r (Table 1) and therefore up to the order of approximation considered and the values of the parameters used, no eddies are observed. Since the solution is obtained using perturbation ϵ and Re are restricted to be small. Increasing the value of $\mu_1(\mu_1 > 0)$ where other parameters remain unchanged, has resulted in streamlines having the same direction as those observed in the case of $\mu_1 = 0.5$. When the cylinders rotate in opposite directions ($\mu_1 < 0$), it is noticed that $\Psi(r,\theta)$ is also strictly monotonic in r but the streamlines have the reverse direction of those when ($\mu_1 > 0$).

The variation of the coupling number N has resulted essentially in plots of streamlines similar to those illustrated in Figure 2. However, the numerical values show that the magnitude of $\Psi(r,\theta)$ is a strictly increasing function of N for all r and θ. This observation is based on the results for various values of μ_1 and for $0.1 \leq N \leq 0.9$ (Table 1). Also, it is observed that when the length ratio L increases the magnitude of $\Psi(r,\theta)$ increases.

N	0.2	0.5	0.8
0.50	-0.2468	-0.2858	-0.4037
0.55	-0.2596	-0.2985	-0.4161
0.60	-0.2736	-0.3121	-0.4291
0.65	-0.2896	-0.3276	-0.4436
0.70	-0.3081	-0.3456	-0.4605
0.75	-0.3296	-0.3666	-0.4805
0.80	-0.3547	-0.3912	-0.5040
0.85	-0.3837	-0.4197	-0.5318
0.90	-0.4169	-0.4524	-0.5642
0.95	-0.4547	-0.4899	-0.6014
1.00	-0.4973	-0.5323	-0.6438

Table 1: Values of the Stream Function for $L = 2$, $\eta = 0.5$, $\mu_1 = 0.5$, $\theta = \frac{\pi}{4}$, $\epsilon = 0.1$

5.2. Velocity and Spin

The dependence of the velocity and spin fields on the polar fluid parameters was discussed [8]. It was observed that when $\mu_1 = 2$, increasing N has resulted in increasing the velocity components and decreasing the spin. This result still holds when the inertia terms are included. However, in the case of $\mu_1 = 0.5$ it is noticed that the velocity is a decreasing function of N while the spin is an increasing function. Therefore, our results suggest that the dependence of **V** and **G** on N is also associated with the relative speed of the two cylinders μ_1. It seems that the effect of increasing N is to increase the velocity of the fluid and to decrease the spin of the polar fluid particles if the inner cylinder is rotating faster than the outer one and the effect is reversed if the inner cylinder is rotating slower than the outer one. Figure 3 (a,b,c) illustrates the dependence of v_r, v_θ and G_z on μ_1 for a fixed value of $N = 0.5$, and for $L = 2$, $\eta = 0.5$, $\epsilon = 0.05$ at $\theta = \frac{\pi}{3}$. It shows that when μ_1 increases,v_r, v_θ and G_z also increase.

5.3. Torque

The evaluation of the torque on the inner cylinders requires the calculation of ψ_3, ϕ_3, g_3 and g_4. As before, these values are evaluated numerically. Table 2 shows the effect of the coupling number N on the torque M for values of

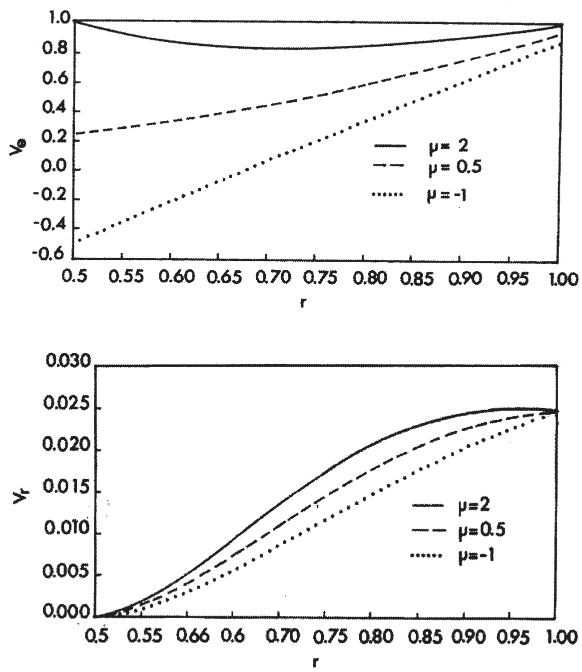

Figure 3: Effect of μ_1 on (a) v_θ; (b) v_r; (c) G_z with $N = 0.5$, $L = 2$, $\eta = 0.5$, $\epsilon = 0.05$ at $\theta = \pi/3$

$\mu_1 = 0.5$ and $\mu_1 = 2$. It is observed that when N increases, the magnitude of M and the individual components M_0 and M_3 decrease. Also, the effect of increasing L when N is fixed increases the magnitude of M. Thus the effect of increasing L is the opposite of increasing N on the torque.

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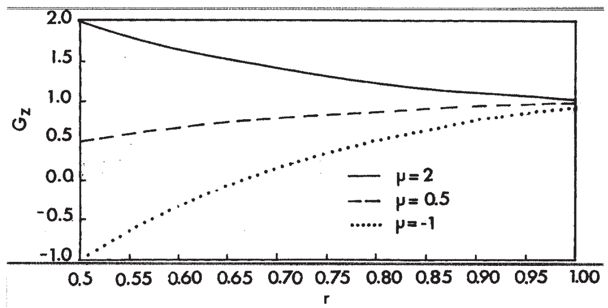

Figure 4: Continuation of Figure 3

N μ_1	0.5	2.0
0.2	31.16	-24.90
0.3	22.48	-22.07
0.4	17.40	-20.75
0.5	14.20	-19.02
0.6	12.40	-18.18
0.7	9.90	-17.05
0.8	7.70	-15.20
0.9	4.20	-13.30

Table 2: Torque on inner cylinder for $L = 2$, $\eta = 0.5$, $\in= 0.05$

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