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ON THE CONSTRUCTION OF OPERATOR RANGE TOPOLOGY AND TOPOLOGICAL UNIFORM DESCENT OF KATO OPERATORS IN BANACH SPACES

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Abstract: We consider a Kato operator A on a Banach space X (i.e., a closed operator A satisfying the following condition: the null space $N(A)$ of A has a topological complement L invariant by A) such that for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$, where $R(A^n)$ denotes the range of A^n. We first prove that given a Kato operator A, we can always give an operator range topology on $R(A^n)$ under which it becomes a Banach space continuously embedded in X for every positive integer n. So, in our result, to construct the range topology of A^n we only require that A is closed, but not A^n which can be not closed.

Using this result, we study the structure of the class of Kato operators. Our study focuses on the sequences of ranges $\{R(A^n)\}$.

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1. Construction of the Range Topology of Kato Operators

Throughout this section we consider a Banach space $(X, \|.\|_X)$ and a linear

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closed operator $A : \mathcal{D}(A) \subset X \longrightarrow X$.

We denote the null space and range of A by $N(A)$ and $R(A)$. The following definition describes the classes of operators we will study.

Definition 1. If the null space $N(A)$ has a topological complement L in X, L is invariant by A (i.e., $A(\mathcal{D}(A) \bigcap L) \subset L$), we say that A is Kato operator.

Definition 2. Assume that for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Then A induces a linear transformation from the vector space $R(A^n)/R(A^{n+1})$ to the space $R(A^{n+1})/R(A^{n+2})$. We will let $k_n(A)$ be the dimension of the null space of the induced map and let $k(A) = \sum\limits_{0}^{\infty} k_n(A)$.

If there is a nonnegative integer d for which $k_n(A) = 0$ for $n \geq d$ (i.e., if the induced maps are isomorphisms for $n \geq d$), we say that A has eventual uniform descent; and, more precisely, that A has uniform descent for $n \geq d$. If $k(A)$ is finite, we say that A has almost uniform descent.

It is clear that if A is Kato operator, then A has uniform descent for $n \geq 1$.

Proposition 3. *Assume that the null space $N(A)$ has a topological complement L in X. Then, there exists a norm $\|.\|_{R(A)}$ on $R(A)$ under which it becomes a Banach space continuously embedded in X. The topology generated by $\|.\|_{R(A)}$ is called the range topology of A.*

If in addition, L is invariant by A ($A(\mathcal{D}(A) \bigcap L) \subset L$) and $A(\mathcal{D}(A^2)) = R(A)$, then there exists a norm $\|.\|_{R(A^2)}$ on $R(A^2)$ under which $R(A^2)$ becomes a Banach space continuously embedded in $(R(A), \|.\|_{R(A)})$. Moreover, the operator inducted by A from $(R(A), \|.\|_{R(A)})$ into $(R(A^2), \|.\|_{R(A^2)})$ is bounded and bijective.

Proof. Let $A_1 : \mathcal{D}(A) \bigcap L \to R(A)$ the map defined from $\mathcal{D}(A) \bigcap L$ to $R(A)$ by $A_1 x = Ax$. Then A_1 is one-to-one and $R(A_1) = R(A)$. Put $A_2 = A_1^{-1}$. Let

$$\widehat{A} : \mathcal{D}(\widehat{A}) \equiv \mathcal{D}(A) \bigcap L \subset X \longrightarrow X$$

be a linear operator defined by $\widehat{A}x = A_1 x = Ax$, and let

$$\breve{A} : \mathcal{D}(\breve{A}) \equiv R(A) \subset X \longrightarrow X$$

be a linear operator defined by $\breve{A}x = A_2 x = {A_1}^{-1} x$.

Since $\widehat{A}$ is closed, then $\breve{A}$ is also closed.

Let $x \in R(A)$. Put $\|x\|_{R(A)} = \|x\|_X + \|\breve{A}x\|_X$. It is clear that $\|.\|_{R(A)}$ defines a norm on $R(A)$ and $(R(A), \|.\|_{R(A)})$ is continuously embedded in X. Since $\breve{A}$ is closed, we deduce that under the norm $\|.\|_{R(A)}$, $R(A)$ becomes a Banach space.

Assume now that L is invariant by A $(A(\mathcal{D}(A) \bigcap L) \subset L)$ and $A(\mathcal{D}(A^2)) = R(A)$.

Let us remark first that if $x \in R(A^2)$, then $\breve{A}x \in R(A)$. Indeed, let $y \in \mathcal{D}(A^2)$ such that $A^2 y = x$. It follows immediately from the fact that L is invariant by A the following $Ay \in \mathcal{D}(A) \bigcap L$. Hence,

$$\breve{A}x = \breve{A}A^2 y = A_2(A_1(Ay)) = Ay \in R(A).$$

For every $x \in R(A^2)$, put $\|x\|_{R(A^2)} = \|x\|_X + \|\breve{A}x\|_{R(A)}$.

Let us prove that under the norm $\|.\|_{R(A^2)}$, $R(A^2)$ becomes a Banach space. Let (x_n) a Cauchy sequence in $(R(A^2), \|.\|_{R(A^2)})$. Then, there exists $x \in X$ and $y \in R(A)$ such that $x_n \to x$ in $(X, \|.\|_X)$ and $\breve{A}x_n \to y$ in $(R(A), \|.\|_{R(A)})$. Consequently, $\breve{A}^2 x_n \to \breve{A}y$ in $(X, \|.\|_X)$. On the other hand, $\breve{A}$ is closed with respect to the norm $\|.\|_X$ and $(R(A), \|.\|_{R(A)})$ is continuously embedded in $(X, \|.\|_X)$. Therefore, $x \in R(A)$ and $y = \breve{A}x$. Consequently, since $y \in R(A)$, we deduce that there exists $y_1 \in \mathcal{D}(A^2)$ such that $y = Ay_1 = \breve{A}x$. This implies that $x = A^2 y_1$. Hence, $x \in R(A^2)$. Thus, $R(A^2)$ under the norm $\|.\|_{R(A^2)}$ is a Banach space.

Let $\tilde{A} : R(A^2) \longrightarrow R(A)$ the map from $R(A^2)$ into $R(A)$ defined by: $\tilde{A}x = \breve{A}x$. It follows immediately from the above established results that $\tilde{A}$ is closed as a map from $(R(A^2), \|.\|_{R(A^2)})$ into $(R(A), \|.\|_{R(A)})$.

Therefore, since $\tilde{A}$ is one-to-one and onto, it follows by the Closed Graph Theorem that $\tilde{A}^{-1}$ which is equal to A on $R(A)$ is bounded. Thus, we achieve the proof. □

Corollary 4. *Assume that A is a Kato operator and for every positive integer n, $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Then, there exists a family of norms $\{\|.\|_{R(A^n)}\}_n$ such that for all positive integer n the following conditions hold:*

(a) $R(A^n)$ under the norm $\|.\|_{R(A^n)}$ becomes a Banach space continuously embedded in $(X, \|.\|_X)$;

(b) $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n), \|.\|_{R(A^n)})$;

(c) The operator $\dot{A}_n$ inducted by A from $R(A^n)$ into $R(A^{n+1})$ is bounded and bijective.

Proof. Since A is a Kato operator, then there is a topological complement L of $N(A)$ invariant by A. Let $A_1 : \mathcal{D}(A) \bigcap L \to R(A)$ the map defined from $\mathcal{D}(A) \bigcap L$ to $R(A)$ by $A_1 x = Ax$, $\breve{A} : \mathcal{D}(\breve{A}) \equiv R(A) \subset X \longrightarrow X$ the linear operator defined by $\breve{A}x = A_2 x = {A_1}^{-1}x$. Since $N(A)$ has a topological complement L in X, $(A(\mathcal{D}(A) \bigcap L) \subset L)$ and $A(\mathcal{D}(A^2)) = R(A)$, then by Proposition 3, it follows that there exists a norm $\|.\|_{R(A)}$ on $R(A)$ and a norm

$\|.\|_{R(A^2)}$ on $R(A^2)$ such that the following conditions hold: $\|x\|_{R(A)} = \|x\|_X + \|\breve{A}x\|_X$; $\|x\|_{R(A^2)} = \|x\|_X + \|\breve{A}x\|_{R(A)}$; $(R(A), \|.\|_{R(A)})$ and $(R(A^2), \|.\|_{R(A^2)})$ are Banach spaces continuously embedded in $(X, \|.\|_X)$; $(R(A^2), \|.\|_{R(A^2)})$ is continuously embedded in $(R(A), \|.\|_{R(A)})$ and the operator inducted by A from $R(A)$ into $R(A^2)$ is bounded and bijective.

Let n be a positive integer. Remark first that if $x \in R(A^{n+1})$ then $\breve{A}x \in R(A^n)$. Indeed, let $y \in \mathcal{D}(A^{n+1})$ such that $A^{n+1}y = x$. It follows immediately from the fact that L is invariant by A the following $A^n y \in \mathcal{D}(A) \bigcap L$. Hence, $\breve{A}x = \breve{A}A^{n+1}y = A_2(A_1(A^n y)) = A^n y \in R(A^n)$.

Let x in $R(A^{n+1})$. Put $\|x\|_{R(A^{n+1})} = \|x\|_X + \|\breve{A}x\|_{R(A^n)}$.

Let us prove by recurrence that $\forall n \geq 1$ $R(A^n)$ under the norm $\|.\|_{R(A^n)}$ is a Banach space continuously embedded in $(X, \|.\|_X)$; $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n), \|.\|_{R(A^n)})$ and the operator $\dot{A}_n$ inducted by A from $R(A^n)$ into $R(A^{n+1})$ is bounded and bijective.

The assumption is true for $n = 1$. Assume now that the above assumption is true for n. Let (x_s) a Cauchy sequence in $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$. Then, there exists $x \in X$ and $y \in R(A^n)$ such that $x_s \to x$ in $(X, \|.\|_X)$ and $\breve{A}x_s \to y$ in $(R(A^n), \|.\|_{R(A^n)})$. We have $\breve{A}$ is closed with respect to the norm $\|.\|_X$ and $(R(A^n), \|.\|_{R(A^n)})$ is continuously embedded in $(X, \|.\|_X)$. Therefore, $y = \breve{A}x$. Consequently, since $y \in R(A^n)$, we deduce that there exists $y_1 \in \mathcal{D}(A^n)$ such that $A^n y_1 = y$. On the other hand, $A^n y_1 \in \mathcal{D}(A) \bigcap L$. Therefore, $x = A_1 y = A_1(A^n y_1) = A^{n+1}y_1$. Hence, $x \in R(A^{n+1})$. So $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is a Banach space continuously embedded in $(X, \|.\|_X)$.

Let us prove now that $(R(A^{n+2}), \|.\|_{R(A^{n+2})})$ is continuously embedded in $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$. Let $x \in R(A^{n+2})$. Then $\breve{A}x \in R(A^{n+1})$. On the other hand, $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n), \|.\|_{R(A^n)})$. Therefore, $\|\breve{A}x\|_{R(A^n)} \leq \|\breve{A}x\|_{R(A^{n+1})}$. This implies that $\|x\|_{R(A^{n+1})} \leq \|x\|_{R(A^{n+2})}$. Hence, $(R(A^{n+2}), \|.\|_{R(A^{n+2})})$ is continuously embedded in $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$.

Let $\tilde{A}_n : R(A^{n+2}) \longrightarrow R(A^{n+1})$ the map from $R(A^{n+2})$ into $R(A^{n+1})$ defined by: $\tilde{A}_n x = \breve{A}x$. It follows immediately that $\tilde{A}_n$ is closed as a map from $(R(A^{n+2}), \|.\|_{R(A^{n+2})})$ into $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$. Therefore, since $\tilde{A}_n$ is one-to-one and onto, it follows by the Closed Graph Theorem that $\tilde{A}_n^{-1}$ which is equal to $\dot{A}_n$ is bounded and bijective. Thus we achieve the proof. □

Definition 5. Assume that A is a Kato operator such that for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. If $R(A^n)$ is closed in the operator range topology of $R(A)$ for every $n \geq 1$, then we say that A has topological uniform descent.

Let us give now a useful lemma which we will need repeatedly in the sequel.

Lemma 6. *Assume that A is a Kato operator and for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Denote by $\sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^n)})')$ the weak topology on $R(A^{n+1})$ with respect to the norm $\|.\|_{R(A^n)}$. Then for every positive integer n, the following assumptions are equivalent:*
(a) $R(A^{n+1})$ is closed in the operator range topology on $R(A^n)$;
(b) $R(A^{n+2})$ is closed in the operator range topology on $R(A^{n+1})$;
(c) $R(A^{n+2})$ is closed in the operator range topology on $R(A^n)$;
(d) $R(A^{n+2})$ is closed in $(R(A^{n+1}),\sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^n)})'))$.

Proof. Let n be a positive integer. By Corollary 4, the operator $\dot{A_n}$ inducted by A from $(R(A^n),\|.\|_{R(A^n)})$ into $(R(A^{n+1}),\|.\|_{R(A^{n+1})})$ is continuous and bijective. Consequently, $R(A^{n+2})$ is closed in $(R(A^{n+1}),\|.\|_{R(A^{n+1})})$ if and only if $\dot{A_n}^{-1}(R(A^{n+2})) \equiv R(A^{n+1})$ is closed in $(R(A^n),\|.\|_{R(A^n)})$. Thus (a) is equivalent to (b).

On the other hand, $(R(A^{n+1}),\|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n),\|.\|_{R(A^n)})$, then

$$\sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^n)})') \subset \sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^{n+1})})') \subset \|.\|_{R(A^{n+1})}.$$

Therefore, the operator $\dot{A_n}$ inducted by A from $(R(A^n),\|.\|_{R(A^n)})$ into $(R(A^{n+1}),\sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^n)})'))$ is continuous. Hence (a) is equivalent to (d). It is clear that (c)$\Rightarrow$ (b). Let us prove now that (a)$\Rightarrow$ (c). Assume that $R(A^{n+1})$ is closed in the operator range topology on $R(A^n)$. Then $R(A^{n+2})$ is closed in $(R(A^{n+1}),\sigma(R(A^{n+1}),(R(A^{n+1}),\|.\|_{R(A^n)})'))$. On the other hand, using the fact that $R(A^{n+1})$ is a convex set, it follows that $R(A^{n+1})$ is closed in $(R(A^n),\sigma(R(A^n),(R(A^n),\|.\|_{R(A^n)})'))$. Hence, $R(A^{n+2})$ is closed in $(R(A^n),\sigma(R(A^n),(R(A^n),\|.\|_{R(A^n)})'))$ and therefore in $(R(A^n),\|.\|_{R(A^n)})$. Thus, we achieve the proof. □

2. Characterization and Structure Theorems

In this section we will always assume that $(X,\|.\|_X)$ is a Banach space and $A:\mathcal{D}(A)\subset X \longrightarrow X$ is a linear closed operator on X.

We can give now several characterization of eventual topological uniform descent and we can also describe the structure of maps which are induced by Kato operators with eventual topological uniform descent. Put $R(A^\infty) = \bigcap_n R(A^n)$. Then, we have the following result.

Proposition 7. *Assume that A is a Kato operator and for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Then the following assertions hold:*

(a) The maps induced by A from $R(A^n)/R(A^{n+1})$ to $R(A^{n+1})/R(A^{n+2})$ are isomorphisms for each $n \geq 1$;

(b) $R(A^n) \bigcap N(A) = R(A^\infty) \bigcap N(A) = \{0\}$ for $n \geq 1$;

(c) $N(A^n) = N(A)$ for each $n \geq 1$.

Proof. The assertion (a) follows from the fact that every Kato operator has eventual uniform descent for $n \geq 1$. Since A is a Kato operator, then there is a topological complement L of $N(A)$ invariant by A. Consequently, $R(A) \subset \mathcal{D}(A) \bigcap L$. Therefore, $R(A^n) \bigcap N(A) = R(A^\infty) \bigcap N(A) = \{0\}$ and $N(A^n) = N(A)$ for each $n \geq 1$. □

Let us give now a characterization of eventual topological uniform descent for Kato operators.

Theorem 8. *Let A a Kato operator on a Banach space X such that for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Then the following conditions are equivalent:*

(a) A has a topological uniform descent;

(b) There is an $n \geq 1$ and a positive integer k for which $R(A^{n+k})$ is closed in the operator range topology on $R(A^n)$;

(c) For each $n \geq 1$ and each positive integer k , $R(A^{n+k})$ is closed in the operator range topology on $R(A^n)$;

(d) There is a positive integer k for which $R(A^k)$ is closed in $(\mathcal{D}(A) \bigcap L, \| \,.\, \|_1)$, where $\| \,.\, \|_1$ is the norm on $\mathcal{D}(A) \bigcap L$ defined by $\| x \|_1 = \| x \|_X + \| Ax \|_X$;

(e) For each positive integer k $R(A^k)$ is closed in $(\mathcal{D}(A) \bigcap L, \| \,.\, \|_1)$

Proof. The equivalence of (a), (b), and (c) is immediate from Lemma 6. A induces a bijective bounded operator A_1 from $(\mathcal{D}(A) \bigcap L, \| \,.\, \|_1)$ to $(R(A), \| \,.\, \|_{R(A)})$. Hence it follows that $R(A^{k+1})$ is closed in the operator range topology on $R(A)$ if and only if $A_1^{-1}(R(A^{k+1})) = R(A^k)$ is closed in $(\mathcal{D}(A) \bigcap L, \| \,.\, \|_1)$. This completes the proof. □

Theorem 9 below is our result on the structure of Kato operators with eventual topological uniform descent.

Theorem 9. *Let A a Kato operator on a Banach space X such that for every positive integer n $A^n(\mathcal{D}(A^{n+1})) = R(A^n)$. Assume that A has a* *topological uniform descent. Then the following assertions hold:*

(a) The restriction of A to $R(A^\infty)$ is onto;

(b) The map induced by A on $R(A)/R(A^\infty)$ is bounded below.

Proof. Since A has a topological uniform descent, then for all positive integer n $R(A^n)$ is closed in $(R(A), \| \, . \, \|_{R(A)})$. Let $\hat{A} : \mathcal{D}(\hat{A}) \subset R(A) \longrightarrow R(A)$ the map induced by A on $\mathcal{D}(\hat{A}) \equiv R(A)$. Then $\hat{A}$ is closed with respect to the norm $\| \, . \, \|_{R(A)}$. On the other hand, it follows from Proposition 7 that $N(\hat{A}) = N(A) \bigcap R(A) = \{0\}$. Therefore, $N(\hat{A})$ has a topological complement $\hat{L} = R(A)$ in $R(A)$. Consequently, since $\hat{A}(\mathcal{D}(\hat{A}) \bigcap \hat{L}) \subset \hat{L}$, it follows that $\hat{A}$ is a Kato operator satisfying $\hat{A}^n(\mathcal{D}(\hat{A}^{n+1})) = R(\hat{A}^n)$. Hence, using the fact that $R(\hat{A}) = R(A^2)$, $R(\hat{A}^2) = R(A^3)$ and $R(A^3)$ is closed in $(R(A^2), \| \, . \, \|_{R(A^2)})$, we deduce that $R(\hat{A}^2)$ is closed in $(R(\hat{A}), \| \, . \, \|_{R(\hat{A})})$. This proves that $\hat{A}$ has a topological uniform descent. It follows from Corollary 4 that for every positive integer n the maps $\dot{A}_n$ induced by A from $R(A^n)$ into $R(A^{n+1})$ and $\dot{\hat{A}}_n$ induced by $\hat{A}$ from $R(\hat{A}^n)$ into $R(\hat{A}^{n+1})$ are bounded with respect to their operator range topologies and are bijective. Therefore,

$$\dot{A}_1^{-1}(R(A^\infty)) = \hat{A}^{-1}(R(\hat{A}^\infty)) = \bigcap_n \dot{\hat{A}}_n^{-1}(R(\hat{A}^{n+1}))$$

$$= \bigcap R(\hat{A}^n) = R(\hat{A}^\infty) = R(A^\infty).$$

This implies that $R(A^\infty) = \dot{A}_1(R(A^\infty)) = A(R(A^\infty))$. Hence, the map induced by A from $R(A^\infty)$ to $R(A^\infty)$ is onto. Since

$$\hat{A}^{-1}(R(\hat{A}^\infty)) = R(A) \bigcap A^{-1}(R(A^\infty)),$$

the map induced by A on $R(A)/R(A^\infty)$ is one-to-one; and since $\hat{A}$ has closed range, this induced map has also closed range. Hence, we have (b). □

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