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UNIFORMLY ERGODIC THEOREM AND FINITE CHAIN LENGTH FOR MULTIOPERATORS

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Abstract: The main purpose of this paper is to extend the technical assumption $(E-k)$ of [7] to multioperators. We give necessary and sufficient conditions for uniform ergodicity of a commuting multioperator satisfying the condition $(E-k)$. Those results are of interest in view of analogous results for valued operators established in [7] and in view of recent activity in the ergodic theory and its applications (see, for example [8]).

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Let X be a complex Banach space and let $L(X)$ be the algebra of linear continuous operators acting in X. Let $T \in L(X)$. If there is an integer n for which $T^{n+1}X = T^nX$, then we say that T has finite descent and the smallest integer $d(T)$ for which the equality holds is called the descent of T. If there is an integer m for which $\ker T^{m+1} = \ker T^m$, then we say that T has finite ascent and the smallest integer $a(T)$ for which this equality occurs is called the ascent of T.

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If both $a(T)$ and $d(T)$ are finite, then they are equal ([3], 38.3). In this case, we say that T is chain-finite and that its chain length is this common minimal value.

If T is chain-finite, then there is a decomposition of the vector space

$$X = T^{d(T)}X \bigoplus \ker T^{d(T)} \qquad \text{(see [3], 38.4).}$$

$$\text{Set } M_i(T) = i^{-1}(I + T + T^2 + \cdots + T^{i-1}), \quad i = 1, 2, 3, \ldots, \tag{1}$$

i.e. the averages associated with T, where $I = \mathrm{id}_X$ is the identity of X.

If now $T = (T_1, T_2, \ldots, T_n) \in L(X)^n$ is a commuting multioperator (briefly, a c.m.) we also set

$$M_v(T) = M_{v_1}(T_1)M_{v_2}(T_2), \ldots, M_{v_n}(T_n), \quad v \in Z_+^n, v \geq e, \tag{2}$$

where Z_+^n is the family of multi-indices of length n (i.e. n-tuples of nonnegative integers) and $e := (1, 1, \ldots, 1) \in Z_+^n$. In other words, (2) defines the averages associated with T.

Definition 1. A c.m. $T \in L(X)^n$ is said to be Cesàro quasi-bounded if the sequences

$$\Big(\prod_{i \neq j} M_{v_i}(T_i)\Big)_{v_1 \geq 1, \ldots, v_{j-1} \geq 1, v_{j+1} \geq 1, \ldots v_n \geq 1} \qquad (j = 1, \ldots, n) \tag{3}$$

are bounded in $L(X)$. If in addition the limit

$$\lim_{v \to \infty} M_v(T) \tag{4}$$

exists in the uniform topology of $L(X)$, then T is said to be uniformly ergodic.

Remark 2. (a) If $n = 1$, then the condition (3) is automatically satisfied, and therefore the above definition extends the usual concept of uniform ergodicity (see for example [5]).

(b) If $T = (I, \ldots, T_j, I, \ldots, I) \in L(X)^n$, then T is Cesàro quasi-bounded (resp. uniformly ergodic) if and only if the sequence $(M_{v_j}(T_j))_{v_j \geq 1}$ is bounded in $L(X)$ (resp. the $\lim\limits_{v_j} M_{v_j}(T_j)$ exists in the uniform topology of $L(X)$).

Definition 3. Let $k = (k_1, \ldots, k_n) \in Z_+^n$ and let $T \in L(X)^n$ be a c.m. We say that T satisfies the condition $(E - k)$ if for each $j \in \{1, \ldots, n\}$

$$\lim_v (I - T_j)^{k_j} M_v(T) = 0$$

in the uniform topology of $L(X)$.

Definition 4. Let $k = (k_1, \dots, k_n) \in Z_+^n$ and let $T \in L(X)^n$ be a c.m. We say that T satisfies the condition s-$(E - k)$ if for each $j \in \{1, \dots, n\}$ and for each finite sequence of nonnegative integers $s_1, \dots, s_n$ satisfying the equality $s_1 + \dots + s_n = k_j$

$$\lim_v \left(\prod_{i=1}^n (I - T_i)^{s_i} \right) M_v(T) = 0$$

in the uniform topology of $L(X)$.

It is clear that the condition $(E - k)$ implies the condition $(E - n)$ and the condition s-$(E - k)$ implies the condition s-$(E - n)$ for any $n \geq k$.

Let us remark also that the condition s-$(E-k)$ implies the condition $(E-k)$.

For each $j \in \{1, \dots, n\}$ put

$$R_{v,k,j}(T) = v_j^{-1} T_j^{v_j} (I - T_j)^{k_j - 1} \prod_{i \neq j} M_{v_i}(T_i).$$

Then we have the following result.

Lemma 5. *If a c.m $T \in L(X)^n$ is Cesàro quasi-bounded, then T satisfies the condition $(E - k)$ if and only if $\lim_v R_{v,k,j}(T) = 0$ for each $j \in \{1, \dots, n\}$.*

Proof. Let $j \in \{1, \dots, n\}$. Then

$$(I - T_j)^{k_j} M_{v_j}(T_j) = {v_j}^{-1} (I - T_j^{v_j})(I - T_j)^{k_j - 1}.$$

Therefore,

$$\begin{aligned} M_v(T)(I - T_j)^{k_j} &= {v_j}^{-1} (I - T_j^{v_j})(I - T_j)^{k_j - 1} \prod_{i \neq j} M_{v_i}(T_i) \\ &= {v_j}^{-1} (I - T_j)^{k_j - 1} \prod_{i \neq j} M_{v_i}(T_i) - R_{v,k,j}(T). \end{aligned}$$

Thus, using the fact that T is Cesàro quasi-bounded, we deduce the result. ☐

Remark 6. By Definition 3, it is easy to check that if $n = 1$, then $T \equiv T_1$ satisfies the condition $(E - k)$ if and only if $\lim_{v_1} (I - T)^k M_{v_1}(T) = 0$, which is also equivalent by Lemma 5 to the condition $\lim_{v_1} {v_1}^{-1} (I - T)^{k-1} T^{v_1} = 0$. Therefore, the characterization given in [7] of valued operators satisfying the condition $E - k$ is a special case of our result.

Proposition 7. *Let $T \in L(X)^n$ be a c.m. operator satisfying the condition $(E-k)$. Then $a\big((I-T_j)|\bigcap_{1\le i\le n,\ i\ne j} \ker(I-T_i)\big) \le k_j$ for each $j \in \{1,\dots,n\}$.*

Proof. Let $x \in \big(\bigcap_{1\le i\le n,\ i\ne j} \ker(I-T_i)\big) \cap \ker(I-T_j)^{k_j+1}$. Then $(I-T_j)^{k_j}T_jx = (I-T_j)^{k_j}x$ and $T_ix = x$ for $i \ne j$. This implies that $(I-T_j)^{k_j}T_j^m x = (I-T_j)^{k_j}x$ for $m = 1,\dots,v_j$ and $\prod_{i\ne j} M_{v_i}(T_i)x = x$. Therefore, it follows that $(I-T_j)^{k_j}M_{v_j}(T_j)x = (I-T_j)^{k_j}x$ and $\prod_{i\ne j} M_{v_i}(T_i)x = x$. Therefore,

$$(I-T_j)^{k_j}M_v(T)x = (I-T_j)^{k_j}x.$$

Taking the limit and using the fact that T satisfies the condition $(E-k)$, we obtain $(I-T_j)^{k_j}x = 0$. i.e., $x \in \ker(I-T_j)^{k_j}$. Thus, we achieve the proof. □

Let us remark that our result given in the Proposition 7 is a generalized version of the result established in (Proposition 4, [7]).

Proposition 8. *Let $T \in L(X)^n$ be a c.m. operator satisfying the condition $(E-k)$. Then*

$$\Big[\sum_{j=1}^n (I-T_j)^{k_j}X\Big] \bigcap \Big[\bigcap_{i=1}^n \ker(I-T_i)\Big] = \{0\}.$$

Proof. Let $x \in \sum_{j=1}^n (I-T_j)^{k_j}X$ such that $x = T_ix$ for every $i = 1,\dots,n$. Then $x = \sum_{j=1}^n (I-T_j)^{k_j}x_j$. Hence, $M_v(T)x = \sum_{j=1}^n (I-T_j)^{k_j}M_v(T)x_j$. On the other hand, we have $x = T_ix$ $(i = 1,\dots,n)$. Consequently, $M_v(T)x = x$, which implies, since T satisfies the condition $(E-k)$ that

$$x = \lim_v M_v(T)x = \lim_v \sum_{j=1}^n (I-T_j)^{k_j}M_v(T)x_j = 0. \qquad \square$$

Theorem 9. (see Mbekhta-Vasilescu, [8]) *Let $T \in L(X)^n$ be a c.m. Assume that T is uniformly ergodic and $\lim_v R_{v,e,j}(T) = 0$. Then $X = \sum_{j=1}^n (I-T_j)X \bigoplus \bigcap_{j=1}^n \ker(I-T_j)$.*

By Lemma 5, an immediate consequence of the above theorem is that if T is an uniformly ergodic c.m. satisfying the condition $(E-e)$, then $X = \sum_{j=1}^n (I-T_j)X \bigoplus \bigcap_{j=1}^n \ker(I-T_j)$.

Let us give now sufficient conditions for uniform ergodicity of a commuting multioperator satisfying the condition $(E-k)$.

Theorem 10. *Let $k \in Z_+^n$ and let $T \in L(X)^n$ be a Cesàro quasi-bounded c.m. operator satisfying the condition s-$(E-k)$. Assume that*

$$X = [\sum_{j=1}^{n}(I-T_j)]X \bigoplus \bigcap_{j=1}^{n} \ker(I-T_j)$$

and

$$N\left(\sum_{j=1}^{n}(I-T_j)\right) \text{has a topological complement.}$$

Then T is uniformly ergodic.

Proof. Since $N\left(\sum\limits_{j=1}^{n}(I-T_j)\right)$ has a topological complement, then by Proposition 3, [6], there exists an unique range topology $\|.\|_{R(\sum\limits_{j=1}^{n}(I-T_j))}$ on $R(\sum\limits_{j=1}^{n}(I-T_j))$ such that $(R(\sum\limits_{j=1}^{n}(I-T_j)), \|.\|_{R(\sum\limits_{j=1}^{n}(I-T_j))})$ is a Banach space continuously embedded in X.

Put $r = \max\limits_{1\le j\le n} k_j$. Then it is clear that

$$X = [\sum_{j=1}^{n}(I-T_j)]^{r+1} X \bigoplus \bigcap_{j=1}^{n} \ker(I-T_j).$$

Let $\|.\|_1$ the norm defined on X by: $\|x\|_1 = \|x_1\|_{R(\sum\limits_{j=1}^{n}(I-T_j))} + \|x_2\|_X$, where $\|.\|_X$ is the usual norm of X, $x_1 \in R(\sum\limits_{j=1}^{n}(I-T_j))$, $x_2 \in \bigcap\limits_{j=1}^{n} \ker(I-T_j)$. Such a decomposition is unique. So the norm $\|.\|_1$ is well defined and $(X, \|.\|_1)$ is a Banach space. Therefore, $\|.\|_X$ and $\|.\|_1$ are equivalent. Let B a bounded subset of $(X, \|.\|_X)$. Then B is also bounded in $(X, \|.\|_1)$. Therefore, using the above properties of the range topology of $\sum\limits_{j=1}^{n}(I-T_j)$, we obtain that there exists a bounded subset B_1 of X such that for every $x \in X$ there exists $x_1 \in B_1$ and there exists $x_2 \in \bigcap\limits_{j=1}^{n} \ker(I-T_j)$ such that $x = [\sum\limits_{j=1}^{n}(I-T_j)]^r x_1 + x_2$. Consequently, $M_v(T)x = M_v(T)[\sum\limits_{j=1}^{n}(I-T_j)]^r x_1 + x_2$. Thus, using the fact that T satisfies the condition s-$(E-k)$, we deduce the result. $\square$

References

[1] P.A. Fillmore, Williams, On operator ranges, *Advances in Math.*, **7** (1971), 254-281.

[2] A. Goldberg, *Unbounded Linear Operator*, N.Y. Mc Graw Hill, New York (1966).

[3] H.G. Heuser, *Functional Analysis*, Willey, Chichester (1982).

[4] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Anal. Math.*, **6** (1958), 261-322.

[5] U. Krengel, *Ergodic Theorems*, De Gruyter, Berlin-New York (1985).

[6] S. Lahrech, A. Ouahab, A. Benbrik, A. Mbarki, On the construction of operator range topology and topological uniform descent of kato operators in banach spaces, *International Journal of Pure and Applied Mathematics*, **23**, No. 2 (2005).

[7] K.B. Laursen, M. Mbekhta, Operators with finite chain length and the ergodic theorem, *Proc. Amer. Math. Soc.* (1995), 3443-3448.

[8] M. Mbekhta, F.-H. Vasilescu, Uniformly ergodic multioperators, *Trans. Amer. Math. Soc.* (1995), 1847-1854.