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# UNIFORMLY ERGODIC THEOREM FOR SEMIGROUPS WITH $k$-DECOMPOSABLE KATO INFINITESIMAL GENERATORS

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**Abstract:** In this paper we shall extend the technical assumption $(E-k)$ to semigroups. We prove that if $T = (T(t), t \geq 0)$ is $C_0$-semigroup of operators in $L(X)$ with $k$-decomposable Kato infinitesimal generator $A$ satisfying the condition $(E-k)$, then $T$ is uniformly ergodic. These results are of interest in view of recent activity in the ergodic theory and its applications.

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## 1. Preliminaries

Let $X$ be a complex Banach space, and let $L(X)$ be the algebra of linear continuous operators acting in $X$. Let $A$ be a closed operator in $X$ with domain $\mathcal{D}(A)$ and let $k$ a positive integer.

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**Definition 1.** We say that $A$ is $k$-decomposable if $\mathcal{D}(A^k) = \mathcal{D}(A^{2k})$ and the following algebraic decomposition holds:

$$X = R(A^k) \bigoplus N(A).$$

**Definition 2.** (Kato Operators, see [4]) If the null space $N(A)$ has a topological complement $L$ in $X$, $L$ is invariant by $A$ (i.e., $A(\mathcal{D}(A) \bigcap L) \subset L$), we say that $A$ is a Kato operator.

**Theorem 3.** (Construction of the Operator Range Topology of Powers of Kato Operators, see [4]) *Assume that $A$ is a Kato operator such that*

$$\forall n : 1 \leq n \leq k \quad A^n(\mathcal{D}(A^{n+1})) = R(A^n).$$

*Then, there exists an unique family of norms $\{\|.\|_{R(A^n)}\}_{1 \leq n \leq k+1}$ such that $\forall n :\ 1 \leq n \leq k$ the following conditions hold:*

*(a) $R(A^n)$ under the norm $\|.\|_{R(A^n)}$ becomes a Banach space continuously embedded in $(X, \|.\|_X)$;*

*(b) $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n), \|.\|_{R(A^n)})$;*

*(c) The operator $\dot{A}_n$ inducted by $A$ from $R(A^n)$ into $R(A^{n+1})$ is bounded and bijective.*

*The norm $\|.\|_{R(A^n)}$ is called the range topology of $R(A^n)$.*

**Theorem 4.** *Assume that $A$ is a Kato operator such that $\mathcal{D}(A^k) = \mathcal{D}(A^{2k})$ and*

$$\forall n : 1 \leq n \leq k\,, \quad A^n(\mathcal{D}(A^{n+1})) = R(A^n).$$

*Assume also that there exists a positive integer $s$ such that*

$$1 \leq s \leq k \text{ and } X = R(A^k) + N(A^s).$$

*Then $A$ is $k$-decomposable.*

*Proof.* Since $A$ is a Kato operator such that

$$\forall n : 1 \leq n \leq k \quad A^n(\mathcal{D}(A^{n+1})) = R(A^n),$$

then using the (Proposition 7, [4]), we deduce the result. □

## 2. Semigroups and Uniform Ergodicity

Let $T = (T(u))_{u\geq 0}$ be a strongly continuous semigroup of bounded operators on a Banach space $X$. Then the (Cèsaro) mean values of $T$ (or the averages associated to $T$) are given by:

$$M_t(T)x = t^{-1}\int_0^t T(u)xdu, \quad x \in X,\ t > 0. \tag{1}$$

**Definition 5.** A semigroup $T$ is said to be Cèsaro-quasi-bounded if the sequence $(M_t(T))_t$ is bounded in $(L(X), \|.\|_{L(X)})$.

A semigroup $T$ is said to be uniformly ergodic if the limit

$$\lim_{t\to\infty} M_t(T) \tag{2}$$

exists in the uniform topology of $L(X)$.

**Remark 6.** If the semigroup $T$ is uniformly ergodic then it is Cèsaro-quasi-bounded.

### 2.1. Uniformly Ergodic Theorem

Let $k$ a positive integer. Throughout this section we consider a Banach space $(X, \|.\|_X)$ and a Kato operator $A : \mathcal{D}(A) \subset X \longrightarrow X$ satisfying the condition:

$$\forall n : 1 \leq n \leq k\,, \quad A^n(\mathcal{D}(A^{n+1})) = R(A^n).$$

By Theorem 3, there exists an unique family of norms $\{\|.\|_{R(A^n)}\}_{1\leq n\leq k+1}$ such that $\forall n :\ 1 \leq n \leq k$ the following conditions hold:

(a) $R(A^n)$ under the norm $\|.\|_{R(A^n)}$ becomes a Banach space continuously embedded in $(X, \|.\|_X)$;

(b) $(R(A^{n+1}), \|.\|_{R(A^{n+1})})$ is continuously embedded in $(R(A^n), \|.\|_{R(A^n)})$;

(c) The operator $\dot{A}_n$ inducted by $A$ from $R(A^n)$ into $R(A^{n+1})$ is bounded and bijective.

Assume that $A$ is an infinitesimal generator of a $C_0$-semigroup $T = (T(u))_{u\geq 0}$.

**Definition 7.** Let $1 \leq r \leq k$. Assume that $R(A^r) \subset \mathcal{D}(A^r)$. We say that $T$ satisfies the condition $(E - r)$ if

$$A^r M_t(T) \text{converges uniformly to 0 with respect to the range topology of } R(A^r) \text{ on each bounded subset of } (R(A^r), \|.\|_{R(A^r)}) \text{ as } t \to \infty.$$

**Remark 8.**

$$\forall x \in \mathcal{D}(A^r)\ M_t(T)x \in \mathcal{D}(A^r) \text{ and } A^r M_t(T) = M_t(T)A^r \text{ on } \mathcal{D}(A^r).$$

**Proposition 9.** *Let $1 \leq r \leq k$ such that $R(A^r) \subset \mathcal{D}(A^r)$. Then:*

*$T$ satisfies the condition $(E-r)$ if and only if*
*$t^{-1}A^{r-1}(T(t)-I)$ converges uniformly to $0$ with respect to the range topology of $R(A^r)$ on each bounded subset of $(R(A^r), \|.\|_{R(A^r)})$ as $t \to \infty$.*

*Proof.* By [Theorem 2.4, 5], we have

$$A\Big(\int_0^t T(u)du\Big) = T(t) - I.$$

It follows then that

$$A^r M_t(T) = t^{-1}A^{r-1}(T(t)-I) \text{ on } \mathcal{D}(A^{r-1}) \text{ and hence on } \mathcal{D}(A^r). \qquad (*)$$

Letting $t \to \infty$ in $(*)$ and using the fact that $R(A^r) \subset \mathcal{D}(A^r)$, we deduce the result. □

**Proposition 10.** *Let $r_1$ and $r_2$ two integers such that $1 \leq r_1 \leq r_2 \leq k$ and $R(A^{r_1}) \subset \mathcal{D}(A^{r_1})$, $R(A^{r_1+1}) \subset \mathcal{D}(A^{r_1+1})$,........, $R(A^{r_2}) \subset \mathcal{D}(A^{r_2})$. Assume that $T$ satisfies the condition $(E-r_1)$. Then $T$ satisfies the condition $(E-r_2)$.*

*Proof.* By Theorem 3, the operator $A$ is bounded from $R(A^{r_1})$ into $R(A^{r_1+1})$ and $(R(A^{r_1+1}), \|.\|_{R(A^{r_1+1})})$ is continuously embedded in $(R(A^{r_1}), \|.\|_{R(A^{r_1})})$. Therefore, $T$ satisfies the condition $(E-r_1+1)$. Repeating the same argument until $r_2$, we get the result. □

**Theorem 11.** *Assume that $A$ is $k$-decomposable, $T$ satisfies the condition $(E-k)$. Assume also that for every $t_n \downarrow 0$ and every bounded sequence $(x_n)_n$ in $(N(A), \|.\|_X)$*

$$t_n^{-1}\int_0^{t_n} T(u)x_n du \to 0 \text{ in } (X, \|.\|_X).$$

*Then $T$ is uniformly ergodic. More precisely, $M_t(T) \to 0$ in $L(X)$ as $t \to \infty$.*

*Proof.* Let $\|.\|'_X$ be the norm defined on $X$ by

$$\forall x \in X \quad \|x\|'_X = \|x_1\|_{R(A^k)} + \|x_2\|_X,$$

where $x_1$ and $x_2$ are such that $x_1 \in R(A^k)$, $x_2 \in N(A)$ and $x = x_1 + x_2$. Such a decomposition is unique and therefore, the norm $\|.\|'_X$ is well defined.

It is clear that $(X, \|.\|'_X)$ is a Banach space and the norm $\|.\|'_X$ is equivalent to the norm $\|.\|_X$.

Let us remark that

$$t^{-1}\int_0^t T(u)du \text{ converges uniformly to 0 on each bounded subset of } (N(A), \|.\|_X).$$

Indeed, assume the contrary. Then there exists a bounded subset $C$ in $(N(A), \|.\|_X)$ and there exists $\varepsilon > 0$ such that

$$\forall k \in N \ \exists \text{ a strictly increasing sequence } (n_k)_k \ \exists \{x_{n_k}\} \subset C \text{ such that } \|t_{n_k}^{-1}\int_0^{t_{n_k}} T(u)x_{n_k}du\|_X > \varepsilon.$$

Let $l \in C$. Put $x_n = l$ for $n \notin \bigcup\limits_{k=1}^{\infty} \{n_k\}$. Then $\{x_n\} \subset C$. Therefore, $(x_n)_n$ is bounded in $(N(A), \|.\|_X)$. Thus,

$$t_n^{-1}\int_0^{t_n} T(u)x_n du \to 0 \text{ in } (X, \|.\|_X) \text{ as } n \to \infty.$$

This implies that

$$t_{n_k}^{-1}\int_0^{t_{n_k}} T(u)x_{n_k} du \to 0 \text{ in } (X, \|.\|_X) \text{ as } k \to \infty.$$

But this contradicts the fact that $\|t_{n_k}^{-1}\int_0^{t_{n_k}} T(u)x_{n_k}du\|_X > \varepsilon$. Hence,

$$M_t(T) \to 0 \text{ as } t \to \infty \text{ in } L((N(A), \|.\|_X), (X, \|.\|_X)),$$

where $L((N(A), \|.\|_X), (X, \|.\|_X))$ is the space of all linear continuous mappings acting from $(N(A), \|.\|_X)$ into $(X, \|.\|_X)$.

Let $B$ a bounded subset of $(X, \|.\|_X)$. Since $\|.\|'_X$ is equivalent to the norm $\|.\|_X$, then $B$ is also bounded in $(X, \|.\|'_X)$. Therefore, using the fact that $A$ is $k$-decomposable, $T$ satisfies the condition $(E-k)$, $\|.\|_{R(A^k)}$ is continuously embedded in $(X, \|.\|_X)$ and

$$M_t(T) \to 0 \text{ in } L((N(A), \|.\|_X), (X, \|.\|_X)) \text{ as } t \to \infty,$$

we deduce the result. □

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