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INTUITIONISTIC FUZZY QUASI-IDEALS OF A SEMIGROUP

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Abstract: We define an intuitionistic fuzzy quasi-ideal of a semigroup and use their properties to characterize regular and intraregular semigroups.

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1. Introduction

The theory of fuzzy sets proposed by Zadeh [22] in 1965 has achieved a greate success in various fields. Since then, some autors [1, 19, 20] applied this concept to group and ring theory. In particular, Ahsan and Latif [1] investigated fuzzy quasi-ideals in a semigroup.

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With the research of fuzzy sets, in 1986, Atanassov [2] presented intuitionistic fuzzy sets which are very effective to deal with vaguenes. The concept of the intuitionistic fuzzy sets is a generalization of one of the fuzzy sets. Recently, Çoker and his colleagues [6, 7, 8], Hur and his colleagues [13] and Lee and Lee [18] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigated some of itis properties. In 1989, Biswas [4] introduced the concept of intuitionistic fuzzy subgroups and studied some of itis properties. In 2003, Banerjee and Basnet [2] investigated intuitionistic fuzzy subgroups and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. Also, Hur and his colleagues [9-12, 14, 15, 17] studied various properties of intuitionistic fuzzy subgroupoids, intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings and intuitionistic fuzzy topological groups. In particular, Hur and his colleagues [16] investigated intuitionistic fuzzy congruences on a lattice.

In this paper, we initiate the study of intuitionistic fuzzy quasi-ideal of a semigroup. In Section 2, we list some basic definitions and some results in the later sections. And we prove preliminary lemmas. In Section 3, we define intuitionistic fuzzy quasi-ideals and establish some of their basic properties. In Section 4, we obtain characterizations of regular and intra-regular semigroups using the machinery developed in the preceding sections.

2. Definitions and Preliminary Lemmas

We will list some concepts and some results needed in the later sections and we obtain some results.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I and for an ordinary subset of a set X, we will denote the characteristic function of A as χ_A.

Definition 1.1. (see [2, 6]) Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set X defined by $10_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2. (see [2]) Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then:
(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\ ]A = (\mu_A, 1 - \mu_A), <> A = (1 - \nu_A, \nu_A)$.

Definition 1.3. (see [6]) Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_1}, \nu_{A_i})$ for each $i \in J$. Then:
(1) $\cap A_i = (\wedge \mu_{A_i}, \vee \nu_{A_i})$. (2) $\cup A_i = (\vee \mu_{A_i}, \wedge \nu_{A_i})$.

Definition 1.4. (see [9]) Let $(X, \cdot)$ be a groupoid and let $A, B \in \text{IFS}(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$ is defined as follows: for each $x \in X$,

$$A \circ B(x) = \begin{cases} (\bigvee_{x=yz}[\mu_A(y) \wedge \mu_B(z), \bigwedge_{x=yz}[\nu_A(y) \vee \nu_B(z)]) & \text{if } x = yz, \\ (0, 1) & \text{if otherwise.} \end{cases}$$

It is clear that for any $A, B, C \in \text{IFS}(X)$, if $B \subset C$, then $A \circ B \subset A \circ C$ and $B \circ A \subset C \circ A$.

Result 1.A. (see [9, Proposition 2.3]) *Let $(S, \cdot)$ be a groupoid. If $\cdot$ is assosiative [resp. commutative], then so is $\circ$ in IFS(S).*

The following is the immediate result of Definition 1.2.

Lemma 1.5. *Let X be a set and let $A, B, C \in$ IFS(X). Then*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and} \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Lemma 1.6. *Let S be a groupoid and let $A, B, C \in$ IFS(S). Then:*
(1) $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$, $(B \cup C) \circ A = (B \circ A) \cup (C \circ A)$.
(2) $A \circ (B \cap C) \subset (A \circ B) \cap (A \circ C)$, $(B \cap C) \circ A \subset (B \circ A) \cap (C \circ A)$.

Proof. (1) Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $[A \circ (B \cup C)](x) = (0, 1) = [(A \circ B) \cup (A \circ C)](x)$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned}
[A \circ (B \cup C)](x) &= (\bigvee_{x=yz} [\mu_A(y) \wedge \mu_{B \cup C}(z)], \bigwedge_{x=yz} [\nu_A(y) \vee \nu_{B \cup C}(z)]) \\
&= (\bigvee_{x=yz} [\mu_A(y) \wedge (\mu_B(z) \vee \mu_C(z))], \bigwedge_{x=yz} [\nu_A(y) \vee (\nu_B(z) \wedge \nu_C(z))])
\end{aligned}$$

$$= (\bigvee_{x=yz} [(\mu_A(y) \wedge \mu_B(z)) \vee (\mu_A(y) \wedge \mu_C(z))],$$
$$\bigwedge_{x=yz} [(\nu_A(y) \vee \nu_B(z)) \wedge (\nu_A(y) \vee \nu_C(z))])$$
$$= (\bigvee_{x=yz} [\mu_A(y) \wedge \mu_B(z)] \vee \bigvee_{x=yz} [\mu_A(y) \wedge \mu_C(z)],$$
$$\bigwedge_{x=yz} [\nu_A(y) \vee \nu_B(z)] \wedge \bigwedge_{x=yz} [\nu_A(y) \vee \nu_C(z)])$$
$$= (\mu_{A\circ B}(x) \vee \mu_{A\circ C}(x), \nu_{A\circ B}(x) \wedge \nu_{A\circ C}(x)) = [(A \circ B) \cup (A \circ C)](x).$$

Hence, in all, $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$. By the similar arguments, we have $(B \cup C) \circ A = (B \circ A) \cup (C \circ A)$.

(2) Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $[A \circ (B \cap C)](x) = (0, 1) = [(A \circ B) \cap (A \circ C)](x)$. Suppose x is expressible as $x = yz$. Then

$$\mu_{A\circ(B\cap C)}(x) = \bigvee_{x=yz} [\mu_A(y) \wedge \mu_{B\cap C}(z)] = \bigvee_{x=yz} [\mu_A(y) \wedge (\mu_B(z) \wedge \mu_C(z))]$$
$$= \bigvee_{x=yz} [(\mu_A(y) \wedge \mu_B(z)) \wedge ((\mu_A(y) \wedge \mu_C(z))]$$
$$\leq \bigvee_{x=yz} [\mu_A(y) \wedge \mu_B(z)] \wedge \bigvee_{x=yz} [\mu_A(y) \wedge \mu_C(z)]$$
$$= \mu_{A\circ B}(x) \wedge \mu_{A\circ C}(x) = \mu_{(A\circ B)\cap(A\circ C)}(x)\,,$$

and

$$\nu_{A\circ(B\cap C)}(x) = \bigwedge_{x=yz} [\nu_A(y) \vee \nu_{B\cap C}(z)] = \bigwedge_{x=yz} [\nu_A(y) \vee (\nu_B(z) \vee \nu_C(z))]$$
$$= \bigwedge_{x=yz} [(\nu_A(y) \vee \nu_B(z)) \vee (\nu_A(y) \vee \nu_C(z))]$$
$$\geq \bigwedge_{x=yz} [\nu_A(y) \vee \nu_B(z)] \vee \bigwedge_{x=yz} [\nu_A(y) \vee \nu_C(z)]$$
$$= \nu_{A\circ B}(x) \vee \nu_{A\circ C}(x) = \nu_{(A\circ B)\cap(A\circ C)}(x).$$

Hence, in all, $A \circ (B \cap C) \subset (A \circ B) \cap (A \circ C)$. By the similar arguments, we have $(B \cap C) \circ A \subset (B \circ A) \cap (C \circ A)$. This completes the proof. □

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset of A of such that $A^2 \subset A$ and by a *left* [resp. *right*] *ideal* of S we mean a non-empty subset A of S such that $SA \subset A$ [resp. $AS \subset A$].

By *tow-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. We well denote the set of all left ideals [resp. right ideals and ideals] of S as LI(S) [resp. RI(S) and I(S)].

Definition 1.7. (see [9]) Let S be a semigroup and let $0_\sim \neq A \in \mathrm{IFS}(S)$. Then A is called an:

(1) *intuitionistic fuzzy subsemigroup* (in short, IFSG) of S, if

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \ \text{ and } \ \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

for any $x, y \in S$,

(2) *intuitionistic fuzzy left ideal* (in short, IFLI) of S, if

$$\mu_A(xy) \geq \mu_A(y) \ \text{ and } \ \nu_A(xy) \leq \nu_A(y)$$

for any $x, y \in S$,

(3) *intuitionistic fuzzy right ideal* (in short, IFRI) of S, if

$$\mu_A(xy) \geq \mu_A(x) \ \text{ and } \ \nu_A(xy) \leq \nu_A(x)$$

for any $x, y \in S$,

(4) *intuitionistic fuzzy (two-sided) ideal* (in short, IFI) of S if is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

We well denote the set of all IFSGs [resp. IFLIs, IFRIs and IFIs] of S as IFSG(S) [resp. IFLI(S), IFRI(S) and IFI(S)]. It is clear that $A \in \mathrm{IFI}(S)$ if and only if $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ for any $x, y \in S$, and if $A \in \mathrm{IFLI}(S)$ [resp. IFRI(S) and IFI(S)], then $A \in \mathrm{IFSG}(S)$.

Result 1.B. (see [9, Proposition 3.2]) *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. Then $A \in IFSG(S)$ if and only if $A \circ A \subset A$.*

Result 1.C. (see [9, Proposition 3.8]) *Let A be a non-empty subset of a semigroup S.*

(1) A is a subsemigroup of S if and only if $(\chi_A, \chi_{A^C}) \in IFSG(S)$.

(2) $A \in LI(S)$ [resp. $RI(S)$ and $I(S)$] if and only if $(\chi_A, \chi_{A^C}) \in IFLI(S)$ [resp. $IFRI(S)$ and $IFI(S)$].

Result 1.D. (see [17, Lemma 1.6 and Lemma 1.6$'$]) *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. Then $A \in IFLI(S)$ [resp. $IFRI(S)$] if and only if $1_\sim \circ A \subset A$ [resp. $A \circ 1_\sim \subset A$].*

Lemma 1.8. *Let S be a semigroup and let $A, B, C \in IFS(S)$. If $A \subset B$, then $A \circ C \subset B \circ C$ and $C \circ A \subset C \circ B$.*

Proof. Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(A \circ C)(x) = (0,1) = (B \circ C)(x)$. Suppose x is expressible as $x = yz$. Then

$$\mu_{A\circ C}(x) = \bigvee_{x=yz} [\mu_A(y) \wedge \mu_C(z)] \leq \bigvee_{x=yz} [\mu_B(y) \wedge \mu_C(z)] = \mu_{B\circ C}(x)\,,$$

since $A \subset B$, and

$$\nu_{A\circ C}(x) = \bigwedge_{x=yz} [\nu_A(y) \vee \nu_C(z)] \geq \bigwedge_{x=yz} [\nu_B(y) \vee \nu_C(z)] = \nu_{B\circ C}(x).$$

Hence, in all, $A \circ C \subset B \circ C$. By the similar arguments, we have $C \circ A \subset C \circ B$. This completes the proof. □

Lemma 1.9. *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. Then $1_\sim \circ A \in IFLI(S)$ [resp. $A \circ 1_\sim \in IFRI(S)$].*

Proof. $1_\sim \circ (1_\sim \circ A) = (1_\sim \circ 1_\sim) \circ A \subset 1_\sim \circ A$, by Result 1.A, Result 1.D and Lemma 1.8, respectively.

Hence, by Result 1.D, $1_\sim \circ A \in \mathrm{IFLI}(S)$. Similarly, we can see that $A \circ 1_\sim \in \mathrm{IFRI}(S)$. This completes the proof. □

Lemma 1.10. *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. Then $A \cup (1_\sim \circ A) \in IFLI(S)$[resp. $A \cup (A \circ 1_\sim) \in IFRI(S)$].*

Proof. $1_\sim \circ [A \cup (1_\sim \circ A)] = (1_\sim \circ A) \cup [1_\sim \circ (1_\sim \circ A)] = (1_\sim \circ A) \cup [(1_\sim \circ 1_\sim) \circ A] \subset (1_\sim \circ A) \cup (1_\sim \circ A) = 1_\sim \circ A \subset A \cup (1_\sim \circ A)$, by Lemma 1.6 (1) and Result 1.A, respectively.

Hence, by Result 1.D, $A \cup (1_\sim \circ A) \in \mathrm{IFLI}(S)$. By the similar arguments, we can see that $A \cup (A \circ 1_\sim) \in \mathrm{IFRI}(S)$. This completes the proof. □

Lemma 1.11. *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. If $A \in IFRI(S)$ [resp. $IFLI(S)$], then $A \cup (1_\sim \circ A)$[resp. $A \cup (1_\sim \circ A)$] is an IFI of S.*

Proof. Suppose $A \in \mathrm{IFRI}(S)$. Then $1_\sim \circ [A \cup (1_\sim \circ A)] = (1_\sim \circ A) \cup [1_\sim \circ (1_\sim \circ A)] \subset (1_\sim \circ A) \cup (1_\sim \circ A) = 1_\sim \circ A \subset A \cup (1_\sim \circ A)$, by Lemma 1.6 (1), Result 1.D and Lemma 1.9.

Thus $A \cup (1_\sim \circ A) \in \mathrm{IFLI}(S)$. On the other hand, $[A \cup (1_\sim \circ A)] \circ 1_\sim = (A \circ 1_\sim) \cup [(1_\sim \circ A) \circ 1_\sim] = (A \circ 1_\sim) \cup [1_\sim \circ (A \circ 1_\sim)] \subset A \cup (1_\sim \circ A)$, by Lemma 1.6 (1), Result 1.A, and since $A \in \mathrm{IFRI}(S)$, by Result 1.D and Lemma 1.8).

Thus $A \cup (1_\sim \circ A) \in \mathrm{IFRI}(S)$. Hence $A \cup (1_\sim \circ A) \in \mathrm{IFI}(S)$.

Similarly, we can see that if $A \in \mathrm{IFLI}(S)$, then $A \cup (A \circ 1_\sim) \in \mathrm{IFL}(S)$. This completes the proof. □

3. Intuitionistic Fuzzy Quasi-Ideals

A nonempty subset A of a semigroup S is called a *quasi-ideal* of S [22] if $AS \cap SA \subset A$. We will denote the set of all quasi-ideals of S as $\mathrm{QI}(S)$.

Definition 2.1. Let S be a semigroup and let $0_\sim \neq A \in \mathrm{IFS}(S)$. Then A is called an *intuitionistic fuzzy quasi-ideal* (in short, $IFQI$) of S if $(1_\sim \circ A) \cap (A \circ 1_\sim) \subset A$.

Example 2.2. Let $S = \{a, b, c\}$ be any semigroup with the following multiplication table:

	a	b	c
a	a	a	a
b	a	b	b
c	a	a	b

We define a complex mapping $A : S \to I \times I$ as follows:

$$A(a) = (0.8, 0.1),\ A(b) = (0.8, 0.1),\ A(c) = (0.6, 0.3).$$

Then we can see that $A \in \mathrm{IFQI}(S)$.

Theorem 2.3. *Let A be a nonempty subset of a semigroup S. Then $A \in$ QI(S) if and only if $(\chi_A, \chi_{A^c}) \in$ IFQI(S).*

Proof. ($\Rightarrow$) Suppose $A \in \mathrm{QI}(S)$ and let $x \in S$. Suppose $x \in A$. Then clearly

$$\chi_A(x) = 1 \geq \mu_{[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x)$$

and

$$\chi_{A^c}(x) = 0 \leq \mu_{[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x).$$

Thus $[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim] \subset (\chi_A, \chi_{A^c})$. Suppose $x \notin A$. Then either x is expressible as $x = yz$ or not.

Case (i). Suppose x is not expressible as $x = yz$. Then

$$([1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim])(x) = (0, 1) = (\chi_A, \chi_{A^c})(x).$$

Case (ii) Suppose x is expressible as $x = yz$. Since $x \notin A$, $y \notin A$ and $z \notin A$. If $y \in A$ and $z \notin A$, then there cannot be another expression of the form $x = ab$, where $a \notin A$ and $b \in A$ (assume that there exist $a \notin A$ and $b \in A$ such that $x = ab$. Then $x \in SA \cap AS \subset A$. Thus $x \in A$. This contradicts the fact that $x \notin A$). Thus either $[1_\sim \circ (\chi_A, \chi_{A^c})](x) = (0, 1)$ or $[(\chi_A, \chi_{A^c}) \circ 1_\sim](x) = (0, 1)$. So $([1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim])(x) = (0, 1)$.

Then $[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim] \subset (\chi_A, \chi_{A^c})$. Hence, in all, $(\chi_A, \chi_{A^c}) \in$ IFQI(S).

($\Leftarrow$) Suppose the necessary condition holds. Let $x \in SA \cap AS$. Then $x \in SA$ and $x \in AS$. Thus there exist $a, a' \in A$ and $s, s' \in S$ such that $x = sa$ and $x = a's'$. So

$$\begin{aligned}\mu_{[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x) &= \mu_{[1_\sim \circ (\chi_A, \chi_{A^c})]}(x) \wedge \mu_{[(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x) \\ &= \bigvee_{x=yz} [\mu_{1_\sim}(y) \wedge \chi_A(z)] \wedge \bigvee_{x=pq} [\chi_A(p) \wedge \mu_{1_\sim}(q)] \\ &\geq (\mu_{1_\sim} \wedge \chi_A(a)) \wedge (\chi_A(a') \wedge \mu_{1_\sim}) = 1\,,\end{aligned}$$

since $x = sa$ and $x = a's'$, and

$$\begin{aligned}\nu_{[1_\sim \circ (\chi_A, \chi_{A^c})] \cap [(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x) &= \nu_{[1_\sim \circ (\chi_A, \chi_{A^c})]}(x) \vee \nu_{[(\chi_A, \chi_{A^c}) \circ 1_\sim]}(x) \\ &= \bigwedge_{x=yz} [\nu_{1_\sim}(y) \vee \chi_{A^c}(z)] \vee \bigwedge_{x=pq} [\chi_{A^c}(p) \vee \nu_{1_\sim}(q)] \\ &\leq (\nu_{1_\sim} \vee \chi_{A^c}(a)) \vee (\chi_{A^c}(a') \vee \nu_{1_\sim}) = 0.\end{aligned}$$

Then, by the hypothesis, $\chi_A(x) = 1$ and $\chi_{A^c}(x) = 0$. Thus $x \in A$. So $SA \cap AS \subset A$. Hence $A \in$ QI(S). This completes the proof. □

Definition 2.4. (see [1]) A nonempty fuzzy set μ_A of a semigroup S is called a fuzzy quasi-ideal of S if $(\mu_S \circ \mu_A) \wedge (\mu_A \circ \mu_S) \leq \mu_A$, where μ_S is the whole fuzzy set defined by $\mu_S(x) = 1$ for each $x \in S$.

Remark 2.5. Let S be a semigroup.
(1) If μ_A is a fuzzy quasi-ideal of S, then $A = (\mu_A, \mu_{A^c}) \in$ IFQI(S).
(2) If $A \in$ IFQI(S), then μ_A and ν_{A^c} are fuzzy quasi-ideal of S.
(3) If $A \in$ IFQI(S), then $[\ \ ]A$, $\langle\ \ \rangle A \in$ IFQI(S).

Proposition 2.6. *Let S be a semigroup. Then IFQI$(S) \subset$ IFSG(S).*

Proof. Let $A \in$ IFQI(S). Since $A \subset 1_\sim$, by Lemma 1.8, $A \circ A \subset 1_\sim \circ A$ and $A \circ A \subset A \circ 1_\sim$. Then $A \circ A \subset (1_\sim \circ A) \cap (A \circ 1_\sim)$. Since $A \in$ IFQI(S), $(1_\sim \circ A) \cap (A \circ 1_\sim) \subset A$. Thus $A \circ A \subset A$. Hence, by Result 1.B, $A \in$ IFSG(S). □

Proposition 2.7. *Let S be a semigroup. Then IFLI$(S) \subset$ IFQI(S) and IFRI$(S) \subset$ IFQI(S).*

Proof. Let $A \in$ IFLI(S). Then, by Result 1.D, $1_\sim \circ A \subset A$. Thus $(1_\sim \circ A) \cap (A \circ 1_\sim) \subset 1_\sim \circ A \subset A$. Hence $A \in$ IFQI(S). Similarly, we can see that IFRI$(S) \subset$ IFQI(S). □

Proposition 2.8. *Let S be a semigroup, let $A \in$ IFLI(S) and let $B \in$ IFRI(S). Then $A \cap B \in$ IFQI(S).*

Proof. Let $A \in$ IFLI(S) and let $B \in$ IFRI(S). Then, by Result 1.D, $A \circ 1_\sim \subset A$ and $1_\sim \circ B \subset B$. Thus

$$\begin{aligned} &[1_\sim \circ (A \cap B)] \cap [(A \cap B) \circ 1_\sim] \\ &\subset [(1_\sim \circ A) \cap (1_\sim \circ B)] \cap [(A \circ 1_\sim) \cap (B \circ 1_\sim)] \subset [(1_\sim \circ A) \cap B] \cap [A \circ \cap (B \circ 1_\sim)] \\ &= [(1_\sim \circ A) \cap (B \circ 1_\sim)] \cap (A \cap B) \subset A \cap B\,, \end{aligned}$$

by Lemma 1.6 (2), Result 1.D and Lemma 1.8.

Hence $A \cap B \in$ IFQI(S). □

The following is the immediate result of Lemma 1.10 and Proposition 2.8.

Corollary 2.8. *Let S be a semigroup and let $0_\sim \neq A \in$ IFS(S). Then $[A \cup (1_\sim \circ A)] \cap [A \cup (A \circ 1_\sim)] \in$ IFQI(S).*

Proposition 2.9. *Let S be a semigroup and let $A \in$ IFQI(S). Then*

$$A = [A \cup (1_\sim \circ A)] \cap [A \cup (A \circ 1_\sim)].$$

Proof. It is clear that $[A \cup (1_\sim \circ A)] \in$ IFLI(S) and $[A \cup (A \circ 1_\sim)] \in$ IFRI(S), from Lemma 1.10. Also from Proposition 2.8, it is clear that $[A \cup (1_\sim \circ A)] \cap [A \cup (A \circ 1_\sim)] \in$ IFQI(S). Then it is sufficient to show that the equality holds.

$$\begin{aligned} &A \subset [A \cup (1_\sim \circ A)] \cap [A \cup (A \circ 1_\sim)] \\ &= ([A \cup (1_\sim \circ A)] \cap A) \cup ([A \cup (1_\sim \circ A)] \cap (A \circ 1_\sim)) \subset A \cup ([A \cup (1_\sim \circ A)] \cap (A \circ 1_\sim)) \\ &= A \cup [A \cap (A \circ 1_\sim)] \cup [(1_\sim \circ A) \cap (A \circ 1_\sim)] \subset A \cup [A \cap (A \circ 1_\sim)] \cup A \\ &\subset A \cup A \cup A(A \circ 1_\sim) \subset A) = A. \end{aligned}$$

Hence the equality holds. □

The following is the immediate result of Proposition 2.8 and Proposition 2.9.

Theorem 2.10. *Let S be a semigroup and let $0_\sim \neq A \in$ IFS(S). Then $A \in$ IFQI(S) if and only if there exist $B \in$ IFRI(S) and $C \in$ IFLI(S) such that $A = B \cap C$.*

Proposition 2.11. *Let S be a semigroup and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset$ IFQI(S). Then either $\bigcap_{\alpha \in \Gamma} A_\alpha = 0_\sim$ or $\bigcap_{\alpha \in \Gamma} A_\alpha \in$ IFQI(S).*

Proof. Let $\{A_\alpha\}_{\alpha \in \Gamma} \subset$ IFQI(S) and let $A = \bigcap_{\alpha \in \Gamma} A_\alpha$. Suppose $A \neq 0_\sim$. Then

$$(1_\sim \circ A) \cap (A \circ 1_\sim) = (1_\sim \circ \bigcap_{\alpha\in\Gamma} A_\alpha) \cap (\bigcap_{\alpha\in\Gamma} A_\alpha \circ 1_\sim)$$
$$\subset \bigcap_{\alpha\in\Gamma}(1_\sim \circ A_\alpha) \cap \bigcap_{\alpha\in\Gamma}(A_\alpha \circ 1_\sim) \subset \bigcap_{\alpha\in\Gamma}[(1_\sim \circ A_\alpha) \cap (A_\alpha \circ 1_\sim)]$$
$$\subset \bigcap_{\alpha\in\Gamma} A_\alpha = A.$$

Hence $A = \bigcap_{\alpha\in\Gamma} A_\alpha \in \text{IFQI}(S)$. □

A nonempty subset A of a semigroup S is called a *bi-ideal* [24] of S if $A^2 \subset A$ and $ASA \subset A$. We will denote the set of all bi-ideal of S as $\text{BI}(S)$.

Definition 2.12. (see [14]) Let S be a semigroup and let $0_\sim \neq A \in \text{IFS}(S)$. Then A is called an *intuitionistic fuzzy bi-ideal* (in short, $IFBI$) of S if it satisfies the following conditions: for any $x, y, z \in S$.

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$

(ii) $\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z)$ and $\nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z)$.

We will denote the set of all IFBIs of S as $\text{IFBI}(S)$.

Result 2.A. (see [14, Proposition 2.5]) *Let A be a nonempty subset of a semigroup. Then $A \in BI(S)$ if and only if $(\chi_A, \chi_{A^c}) \in IFBI(S)$.*

Theorem 2.13. *Let S be a semigroup and let $0_\sim \neq A \in IFS(S)$. Then $A \in IFBI(S)$ if and only if $A \circ A \subset A$ and $A \circ 1_\sim \circ A \subset A$.*

Proof. (⇒) Suppose $A \in \text{IFBI}(S)$. By Result 1.B, $A \circ A \subset A$. Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(A \circ 1_\sim \circ A)(x) = (0, 1)$. Thus $A \circ 1_\sim \circ A \subset A$. Suppose x is expressible as $x = yz$. Then $(A \circ 1_\sim \circ A)(x) \neq (0, 1)$. Thus

$$\mu_{A\circ 1_\sim \circ A}(x) = \bigvee_{x=yz} [\mu_A(y) \wedge \mu_{1_\sim \circ A}(z)] > 0$$

and

$$\nu_{A\circ 1_\sim \circ A}(x) = \bigwedge_{x=yz} [\nu_A(y) \vee \nu_{1_\sim \circ A}(z)] < 1.$$

So, $\mu_{1_\sim \circ A}(z) > 0$ and $\nu_{1_\sim \circ A}(z) < 1$. Then there exist $p, q \in S$ with $z = pq$ such that

$$\mu_{1_\sim \circ A}(z) = \bigvee_{z=pq} [\mu_{1_\sim}(p) \wedge \mu_A(q)] = \bigvee_{z=pq} \mu_A(q)$$

and

$$\nu_{1_\sim \circ A}(z) = \bigwedge_{z=pq} [\nu_{1_\sim}(p) \vee \nu_A(q)] = \bigwedge_{z=pq} \nu_A(q).$$

Since $A \in \text{IFBI}(S)$,

$$\mu_A(x) = \mu_A(ypq) \geq \mu_A(y) \wedge \mu_A(q)$$

and

$$\nu_A(x) = \nu_A(ypq) \leq \nu_A(y) \vee \nu_A(q).$$

Then

$$\mu_A(x) \geq \bigvee_{x=yz} [\mu_A(y) \wedge \bigvee_{z=pq} \mu_A(q)] = \mu_{A \circ 1_\sim \circ A}(x)$$

and

$$\nu_A(x) \leq \bigwedge_{x=yz} [\nu_A(y) \vee \bigwedge_{z=pq} \nu_A(q)] = \nu_{A \circ 1_\sim \circ A}(x).$$

Hence, in all, $A \circ 1_\sim \circ A \subset A$.

($\Leftarrow$) Suppose the necessary conditions hold. Since $A \circ A \subset A$

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \text{ for any } x, y \in S.$$

Let $x, y, z \in S$ and let $p = xyz$. Then

$$\begin{aligned}\mu_A(xyz) = \mu_A(p) \geq \mu_{A \circ 1_\sim \circ A}(p) &= \bigvee_{p=st} [\mu_A(s) \wedge \mu_{1_\sim \circ A}(t)] \\ &\geq \mu_A(x) \wedge \mu_{1_\sim \circ A}(yz) = \mu_A(x) \wedge (\bigvee_{yz=ab} [\mu_{1_\sim}(a) \wedge \mu_A(b)]) \\ &\geq \mu_A(x) \wedge \mu_{1_\sim}(y) \wedge \mu_A(z) = \mu_A(x) \wedge \mu_A(z)\end{aligned}$$

by the hypothesis and since $p = x(yz)$. Moreover

$$\begin{aligned}\nu_A(xyz) = \nu_A(p) \leq \nu_{A \circ 1_\sim \circ A}(p) &= \bigwedge_{p=st} [\nu_A(s) \vee \nu_{1_\sim \circ A}(t)] \\ &\leq \nu_A(x) \vee \nu_{1_\sim \circ A}(yz)] = \nu_A(x) \vee (\bigwedge_{yz=ab} [\nu_{1_\sim}(a) \vee \nu_A(b)]) \\ &\leq \nu_A(x) \vee \nu_{1_\sim}(y) \vee \nu_A(z) = \nu_A(x) \vee \nu_A(z),\end{aligned}$$

since $p = x(yz)$. Hence $A \in \text{IFBI}(S)$. This completes the proof. □

Proposition 2.14. *Let S be a semigroup. Then IFQI(S) $\subset$ IFBI(S).*

Proof. Let $A \in \text{IFQI}(S)$. Then, by Proposition 2.6, $A \in \text{IFSG}(S)$. Thus

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \text{ for any } x, y \in S.$$

So, from Result 1.B, $A \circ A \subset A$. It is clear that $A \circ 1_\sim \subset 1_\sim$ and $1_\sim \circ A \subset 1_\sim$. Then, by Lemma 1.8, $A \circ 1_\sim \circ A \subset 1_\sim \circ A$ and $A \circ 1_\sim \circ A \subset A \circ 1_\sim$. Since $A \in$ IFQI(S),

$$A \circ 1_\sim \circ A \subset (1_\sim \circ A) \cap (A \circ 1_\sim) \subset A.$$

Hence, by Theorem 2.13, $A \in$ IFBI(S). □

The converse inclusion of Proposition 2.14 is not generally true.

Example 2.15. Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

We define a complex mapping $A : S \to I \times I$ as follows:

$$A(a) = (1, 0),\ A(b) = (0.6, 0.3),\ A(c) = (0.7, 0.2) \text{ and } A(d) = (0.5, 0.5).$$

Then we can see that $A \notin$ IFQI(S) but $A \in$ IFBI(S).

The product of two quasi-ideals need not be a quasi-ideal. So the intuitionistic fuzzy product of two IFQIs need not be an IFQI.

Proposition 2.16. *Let S be a semigroup, let $A \in$ IFQI(S) and let $0_\sim \neq B \in$ IFS(S). Then $A \circ B, B \circ A \in$ IFBI(S).*

Proof. Let $A \in$ IFQI(S) and let $0_\sim \neq B \in$ IFS(S). Thus, by Proposition 2.14, $A \in$ IFBI(S). Then, by Theorem 2.13, $A \circ 1_\sim \circ A \subset A$ and $A \circ A \subset A$. So

$$\begin{aligned}
&(A \circ B) \circ (A \circ B) \subset (A \circ 1_\sim) \circ (A \circ B) && (\text{Since } A \circ B \subset A \circ 1_\sim)\\
&= (A \circ 1_\sim \circ A) \circ B && (\text{By Result 1.A})\\
&\subset A \circ B.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(A \circ B) \circ 1_\sim \circ (A \circ B) \subset (A \circ 1_\sim) \circ 1_\sim \circ (A \circ B) && (\text{Since } A \circ B \subset A \circ 1_\sim)\\
&= A \circ (1_\sim \circ 1_\sim) \circ (A \circ B) && (\text{By Result 1.A})\\
&\subset A \circ 1_\sim \circ (A \circ B) = (A \circ 1_\sim \circ A) \circ B && (\text{By Result 1.A})\\
&\subset A \circ B.
\end{aligned}$$

Hence, by Theorem 2.13, $A \circ B \in$ IFBI(S). Similarly, we can see that $B \circ A \in$ IFBI(S). This completes the proof. □

The following is the immediate result of Proposition 2.16.

Corollary 2.16. *Let S be a semigroup and let $A, B \in$ IFQI(S). Then $A \circ B \in$ IFBI(S).*

4. Regular Semigroups

A semigroup S is said to be *regular* if for each $a \in S$ there exists $x \in S$ such that $a = axa$.

Theorem 3.1. *Let S be a semigroup. Then the following are equivalent:*
(1) S is regular.
(2) For each $A \in$ IFRI(S) and each $B \in$ IFLI(S), $A \circ B = A \cap B$.
(3) For each $A \in$ IFRI(S) and each $B \in$ IFLI(S),
(a) $A^2 = A \circ A = A$.
(b) $B^2 = B \circ B = B$.
(c) $A \circ B \in$ IFQI(S).
(4) (IFQI$(S), \circ$) is a regular semigroup.
(5) Every IFQI A of S has the form $A = A \circ 1_\sim \circ A$.

Proof. (1)⇒(2) Suppose S is regular. Let $A \in$ IFRI(S) and let $B \in$ IFLI(S). Then, by Result 1.D, $A \circ B \subset A \circ 1_\sim \subset A$ and $A \circ B \subset 1_\sim \circ B \subset B$. Thus

$$A \circ B \subset A \cap B.$$

Now let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then

$$\mu_{A \circ B}(a) = \bigvee_{a=yz} [\mu_A(y) \wedge \mu_B(z)] \geq \mu_A(ax) \wedge \mu_B(a) \qquad \text{(Since } a = axa\text{)}$$
$$\geq \mu_A(a) \wedge \mu_B(a) \qquad \text{(Since } A \in \text{IFRI}(S)\text{)}$$
$$= \mu_{A \cap B}(a)$$

and

$$\nu_{A \circ B}(a) = \bigwedge_{a=yz} [\nu_A(y) \vee \nu_B(z)] \leq \nu_A(ax) \vee \nu_B(a) \leq \nu_A(a) \vee \nu_B(a) = \mu_{A \cap B}(a).$$

Thus $A \circ B \supset A \cap B$. Hence $A \circ B = A \cap B$

(2)⇒(3) Suppose the condition (2) holds.

(a) Let $A \in$ IFRI(S). Then, by Lemma 1.11, $A \cup (1_\sim \circ A) \in$ IFI(S). By the hypothesis,

$$A = A \cap [A \cup (1_\sim \circ A)] = A \circ [A \cup (1_\sim \circ A)]$$

$$\begin{aligned}
&= (A \circ A) \cup [A \circ (1_\sim \circ A)] && \text{(By Lemma 1.6(1))}\\
&= (A \circ A) \cup [(A \circ 1_\sim) \circ A] && \text{(By Result 1.A)}\\
&\subset (A \circ A) \cup (A \circ A) && \text{(Since } A \in \mathrm{IFRI}(S))\\
&= (A \circ A).
\end{aligned}$$

Thus $A \subset A \circ A$. On the other hand, $A \circ A \subset A \circ 1_\sim \subset A$. Hence $A \circ A = A$.

(b) Let $B \in \mathrm{IFLI}(S)$. Then, by the similar arguments of the proof of (a), we can see that $B \circ B = B$.

(c) Let $A \in \mathrm{IFRI}(S)$ and let $B \in \mathrm{IFLI}(S)$. Then, by the hypothesis, $A \circ B = A \cap B$. By Proposition 2.8, $A \cap B \in \mathrm{IFQI}(S)$. Hence $A \circ B \in \mathrm{IFQI}(S)$.

(3)⇒(4) Suppose the condition (3) holds. Let $A \in \mathrm{IFQI}(S)$. Then, by Lemma 1.4, $A \cup (1_\sim \circ A) \in \mathrm{IFLI}(S)$. Thus

$$\begin{aligned}
&A \subset A \cup (1_\sim \circ A)\\
&= [A \cup (1_\sim \circ A)] \circ [A \cup (1_\sim \circ A)]\\
&\qquad \text{(By the condition (3) (b))}\\
&= [(A \cup 1_\sim \circ A) \circ A] \cup [(A \cup (1_\sim \circ A)) \circ ((\chi_S, \chi_{S^c}) \circ A)]\\
&\qquad \text{(By Lemma 1.6(1))}\\
&= [(A \circ A) \cup \{(1_\sim \circ A) \circ A\}] \cup [(A \circ (1_\sim \circ A)) \cup (1_\sim \circ A)^2]\\
&\qquad \text{(By Lemma 1.6(1))}\\
&= [(A \circ A) \cup \{1_\sim \circ (A \circ A)\}] \cup [(A \circ (1_\sim \circ A)) \cup (1_\sim \circ A)^2]\\
&\qquad \text{(By Result 1.A)}\\
&\subset [(1_\sim \circ A) \cup (1_\sim \circ A)] \cup [(1_\sim \circ (1_\sim \circ A)) \cup (1_\sim \circ A)^2]\\
&\qquad \text{(Since } A \circ A \subset A \text{ and } A \subset 1_\sim)\\
&= [(1_\sim \circ A) \cup (1_\sim \circ A)] \cup [(1_\sim \circ (1_\sim \circ A)) \cup (1_\sim \circ A)]\\
&\qquad \text{(By the condition (3) (b))}\\
&\subset (1_\sim \circ A) \cup (1_\sim \circ A) \cup (1_\sim \circ A)\\
&\qquad \text{(Since } 1_\sim \circ A \in \mathrm{IFLI}(S))\\
&= 1_\sim \circ A.
\end{aligned}$$

So, $A \subset 1_\sim \circ A$. By the similar arguments, we can see that $A \subset A \circ 1_\sim$. Then $A \subset (1_\sim \circ A) \cap (A \circ 1_\sim)$. Since $A \in \mathrm{IFQI}(S)$, $(1_\sim \circ A) \cap (A \circ 1_\sim) \subset A$. So,

$$A = (1_\sim \circ A) \cap (A \circ 1_\sim). \qquad (*)$$

Let $C \in \mathrm{IFRI}(S)$ and let $D \in \mathrm{IFLI}(S)$. Then, by the condition (3)(c), $C \circ D \in \mathrm{IFQI}(S)$. Thus, by (*),

$$C \circ D = [1_\sim \circ (C \circ D)] \cap [(C \circ D) \circ 1_\sim]. \qquad (**)$$

Now let $A, B \in$ IFQI(S). Then, by Lemma 2.2, $1_\sim \circ A \circ B \in$ IFLI(S) and $A \circ B \circ 1_\sim \in$ IFRI(S). By Proposition 2.7, $1_\sim \circ A \circ B$, $A \circ B \circ 1_\sim \in$ IFQI(S). Thus

$$\begin{aligned} & 1_\sim \circ A \circ B \\ &= [1_\sim \circ (1_\sim \circ A \circ B)] \cap [(1_\sim \circ A \circ B) \circ 1_\sim] && \text{(By (*))} \\ &= (1_\sim \circ A \circ B) \circ (1_\sim \circ A \circ B) && \text{(By (**))} \\ &= [(1_\sim \circ A \circ B) \circ 1_\sim] \circ [1_\sim \circ (1_\sim \circ A \circ B)] \\ &= (1_\sim \circ A \circ B) \circ (1_\sim \circ 1_\sim) \circ (1_\sim \circ A \circ B) \\ &= (1_\sim \circ A \circ B) \circ 1_\sim \circ (1_\sim \circ A \circ B) && \text{(By the condition (3)(b)).} \end{aligned}$$

So $1_\sim \circ A \circ B = (1_\sim \circ A \circ B) \circ 1_\sim \circ (1_\sim \circ A \circ B)$. Similarly, we have $A \circ B \circ 1_\sim = (A \circ B \circ 1_\sim) \circ 1_\sim) \circ (A \circ B \circ 1_\sim)$. Then

$$\begin{aligned} & [1_\sim \circ A \circ B] \cap [A \circ B \circ 1_\sim] \\ &= [(1_\sim \circ A \circ B) \circ 1_\sim \circ (1_\sim \circ A \circ B)] \\ &\quad \cap [(A \circ B \circ 1_\sim) \circ 1_\sim \circ (A \circ B \circ 1_\sim)] \\ &= [1_\sim \circ (A \circ B \circ 1_\sim) \circ (1_\sim \circ A \circ B)] \\ &\quad \cap [(A \circ B \circ 1_\sim) \circ (1_\sim \circ A \circ B) \circ 1_\sim] \\ &= (A \circ B \circ 1_\sim) \circ (1_\sim \circ A \circ B) && \text{(By (**))} \\ &\subset (A \circ B \circ 1_\sim) \circ 1_\sim \circ (1_\sim \circ B) && \text{(Since } A \subset 1_\sim) \\ &= (A \circ B \circ 1_\sim) \circ (1_\sim \circ 1_\sim) \circ B \\ &= (A \circ B \circ 1_\sim) \circ 1_\sim \circ B && \text{(Since } 1_\sim \circ 1_\sim = 1_\sim) \\ &= (A \circ B) \circ (1_\sim \circ 1_\sim) \circ B \\ &= A \circ (B \circ 1_\sim \circ B) \\ &\subset A \circ B. && \text{(Since } B \circ 1_\sim \circ B \subset B) \end{aligned}$$

So $A \circ B \in$ IFQI(S). Since " $\circ$ " is associative, $(IFQI(S), \circ)$ is a semigroup. Let $A \in$ IFQI(S). Then

$$\begin{aligned} A &= (A \circ 1_\sim) \cap (1_\sim \circ A) && \text{(By (*))} \\ &= (A \circ 1_\sim) \circ (1_\sim \circ A) && \text{(By the condition (c))} \\ &= A \circ 1_\sim \circ A. \end{aligned}$$

It is clear that $1_\sim \in$ IFQI(S). So A is a regular element of IFQI(S). Hence (IFQI(S), $\circ$) is a regular semigroup.

(4)⇒(5) Suppose the condition (4) holds. Let $A \in \text{IFQI}(S)$. Then, by the hypothesis, there exists $B \in \text{IFQI}(S)$ such that $A = ABA$. Thus

$$\begin{aligned} A = ABA &= A \circ 1_\sim \circ A \qquad\qquad (\text{Since } B \subset 1_\sim) \\ &\subset (A \circ 1_\sim) \cap (1_\sim \circ A) \subset A. \end{aligned}$$

Hence $A = A \circ 1_\sim \circ A$.

(5)⇒(1): Suppose the condition (5) holds. Let $x \in S$ and let $A = \{x\} \cup (Sx \cap xS)$ be the quasi-ideal of S generated by x. Then, by Theorem 2.3, $(\chi_A, \chi_{A^c}) \in \text{IFQI}(S)$. Thus, by the hypothesis, $(\chi_A, \chi_{A^c}) = 1_\sim \circ 1_\sim \circ (\chi_A, \chi_{A^c})$. So

$$1 = \chi_A(x) = \mu_{(\chi_A,\chi_{A^c})\circ 1_\sim \circ (\chi_A,\chi_{A^c})}(x) = \bigvee_{x=yz} [\chi_A(y) \wedge \mu_{1_\sim \circ (\chi_A,\chi_{A^c})}(z)]$$

and

$$0 = \chi_{A^c}(x) = \nu_{(\chi_A,\chi_{A^c})\circ 1_\sim \circ (\chi_A,\chi_{A^c})}(x) = \bigwedge_{x=yz} [\chi_{A^c}(y) \vee \nu_{1_\sim \circ (\chi_A,\chi_{A^c})}(z)].$$

Then there exist $p, q \in S$ with $x = pq$ such that

$$\chi_A(p) = 1, \quad \chi_{A^c}(p) = 0 \text{ and } \mu_{1_\sim \circ (\chi_A,\chi_{A^c})}(q) = 1, \quad \nu_{1_\sim \circ (\chi_A,\chi_{A^c})}(q) = 0.$$

Since $\chi_A(p) = 1$ and $\chi_{A^c}(p) = 0$, $p \in A$, i.e., $p = x$ or $p = xs$, where $s \in S$. Since $\mu_{1_\sim \circ (\chi_A,\chi_{A^c})}(q) = 1$ and $\nu_{1_\sim \circ (\chi_A,\chi_{A^c})}(q) = 0$,

$$\bigvee_{q=st} [\mu_{1_\sim}(s) \wedge \chi_A(t)] = 1 \text{ and } \bigwedge_{q=st} [\nu_{1_\sim}(s) \vee \chi_{A^c}(t)] = 0.$$

Then there exist $a, b \in S$ with $q = ab$ such that

$$\chi_A(b) = 1 \text{ and } \chi_{A^c}(b) = 0.$$

Thus $b \in A$, i.e., either $b = x$ or $b = s_1 x$, where $s_1 \in S$. So $x = pq = xcx$, where $c \in S$. Then x is a regular element of S. Hence S is regular. This completes the proof. □

Theorem 3.2. *Let S be a regular semigroup and let $0_\sim \neq A \in IFS(S)$. Then $A \in IFQI(S)$ if and only if there exist $B \in IFRI(S)$ and $C \in IFLI(S)$ such that $A = B \circ C$.*

Proof. (⇒) Suppose $A \in \text{IFQI}(S)$. Then

$$A = A \circ 1_\sim \circ A = A \circ (1_\sim \circ 1_\sim) \circ A = (A \circ 1_\sim) \circ (1_\sim \circ A),$$

by Theorem 3.1.

Let $A \circ 1_\sim = B$ and let $C = 1_\sim \circ A$. Then, by Lemma 2.2, $B \in$ IFRI(S) and $C \in$ IFLI(S). Hence the necessary condition holds.

($\Leftarrow$) Suppose the necessary condition holds. Let $A \in$ IFS(S). Then there exist $B \in$ IFRI(S) and $C \in$ IFLI(S) such that $A = B \circ C$. Since S is regular, by Theorem 3.1, $B \circ C \in$ IFQI(S). Hence $A \in$ IFQI(S). This completes the proof. □

Theorem 3.3. *Let S be a regular semigroup. Then the following hold:*
(1) If $A \in$ IFQI(S), then $A^2 = A^3$.
(2) $A \in$ IFQI(S) if and only if $A \in$ IFBI(S).

Proof. (1) Suppose $A \in$ IFQI(S). Since $A \in$ IFSG(S), $A \circ A \subset A$. Thus $A \circ (A \circ A) \subset A \circ A$. So $A^3 \subset A^2$. Since S is regular, by Theorem 3.1, $A \circ A \in$ IFQI(S). Since IFQI(S) is regular, there exists $B \in$ IFQI(S) such that $A^2 = A^2 \circ B \circ A^2$. On the other hand, $A^2 \circ B \circ A^2 \subset A^2 \circ 1_\sim \circ A^2 = A \circ (A \circ 1_\sim \circ A) \circ A = A \circ A \circ A = A^3$. Thus $A^2 \subset A^3$. Hence $A^2 = A^3$.

(2) ($\Rightarrow$) It is clear from Proposition 2.14.

($\Leftarrow$) Suppose $A \in$ IFBI(S). Then $1_\sim \circ A \in$ IFLI(S) and $A \circ 1_\sim \in$ IFRI(S). Thus

$$
\begin{aligned}
&(A \circ 1_\sim) \cap (1_\sim \circ A) = (A \circ 1_\sim) \circ (1_\sim \circ A) && \text{(By Theorem 3.1)}\\
&= A \circ (1_\sim \circ 1_\sim) \circ A && \text{(By Result 1.A)}\\
&= A \circ 1_\sim \circ A \subset A && \text{(By the hypothesis)}
\end{aligned}
$$

Hence $A \in$ IFQI(S). This completes the proof. □

A semigroup S is called a *band* (see works of J.M. Howie) if for each $a \in S$, $aa = a$.

Theorem 3.4. *Let S be a semigroup. Then the following are equivalent:*
(1) For each $A \in$ IFRI(S) and each $B \in$ IFLI(S), $A \circ B = A \cap B \subset B \circ A$.
(2) (IFQI(S), $\circ$) is a band.
(3) For each $A \in$ IFQI(S), $A \circ A = A$.

Proof. (1)$\Rightarrow$(2) Suppose the condition (1) holds. Then, by Theorem 3.1, S is regular. Thus, by Theorem 3.1, (IFQI(S), $\circ$) is a regular semigroup. Let $A \in$ IFQI(S). Then

$$
\begin{aligned}
A &= A \circ 1_\sim \circ A && \text{(By Theorem 3.1)}\\
&= (A \circ 1_\sim \circ A) \circ 1_\sim \circ (A \circ 1_\sim \circ A) && \text{(By Theorem 3.1)}\\
&= (A \circ 1_\sim) \circ (A \circ 1_\sim) \circ (1_\sim \circ A) \circ (1_\sim \circ A) && \text{(Since } 1_\sim \circ 1_\sim = 1_\sim)\\
&\subset (A \circ 1_\sim) \circ (1_\sim \circ A) \circ (A \circ 1_\sim) \circ (1_\sim \circ A) && \text{(By the hypothesis)}
\end{aligned}
$$

$$
\begin{aligned}
&= A \circ (1_\sim \circ 1_\sim) \circ A \circ A \circ (1_\sim \circ 1_\sim) \circ A && \text{(By Result 1.A)}\\
&= (A \circ 1_\sim \circ A) \circ (A \circ 1_\sim \circ A) && \text{(Since } 1_\sim \circ 1_\sim = 1_\sim)\\
&= A \circ A && \text{(Since } A \circ 1_\sim \circ A = A)\\
&\subset A. && \text{(Since } A \circ A \subset A)
\end{aligned}
$$

Thus $A \circ A = A$. So A is an idempotent element of IFQI(S). Hence $(IFQI(S), \circ)$ is a band.

(2)⇒(3) It is clear.

(3)⇒(1) Suppose the condition (3) holds. Let $A \in$ IFRI(S) and let $B \in$ IFLI(S). Then, by Proposition 2.15, $A \cap B \in$ IFQI(S). Thus

$$
\begin{aligned}
A \cap B &= (A \cap B) \circ (A \cap B) && \text{(By the hypothesis)}\\
&\subset (A \circ (A \cap B)) \cap (B \circ (A \cap B)) && \text{(By Lemma 1.2(2))}\\
&\subset (A \circ A) \cap (A \circ B) \cap (B \circ A) \cap (B \circ B). && \text{(By Lemma 1.2(2))}
\end{aligned}
$$

So

$$A \cap B \subset A \circ B \text{ and } (A \cap B) \subset B \circ A. \qquad (***)$$

On the other hand, $A \circ B \subset A \circ 1_\sim \subset A$ and $A \circ B \subset 1_\sim \circ B \subset B$. Thus $A \circ B \subset A \cap B$. Hence, by (***), $A \circ B = A \cap B \subset B \circ A$. This completes the proof. □

A semigroup S is said to be *intra-regular* (see works of O. Steinifild) if for each $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$.

Theorem 3.5. *Let S be a semigroup. Then S is intra-regular if and only if for each $A \in$ IFRI(S) and each $B \in$ IFLI(S), $A \cap B \subset B \circ A$.*

Proof. (⇒) Suppose S is intra-regular. Let $A \in$ IFRI(S), let $B \in$ IFLI(S) and let $a \in S$. Then. by the hypothesis, there exists $x, y \in S$ such that $a = xa^2y$. Thus

$$
\begin{aligned}
\mu_{B\circ A}(a) &= \bigvee_{a=st} [\mu_B(s) \wedge \mu_A(t)]\\
&\geq \mu_B(xa) \wedge \mu_A(ay) && \text{(Since } a = (xa)(ay))\\
&\geq \mu_B(a) \wedge \mu_A(a) && \text{(Since } B \in IFLI(S) \text{ and } A \in IFRI(S))\\
&= \mu_{A\cap B}(a)
\end{aligned}
$$

and

$$\nu_{B\circ A}(a) = \bigwedge_{a=st} [\nu_B(s) \vee \nu_A(t)] \leq [\nu_B(xa) \vee \nu_A(ay)] \leq \nu_B(a) \vee \nu_A(a)$$

$$= \nu_{A\cap B}(a).$$

Hence $A \cap B \subset B \circ A$.

($\Leftarrow$) Suppose the necessary condition holds and let $x \in S$. Let $L = \{x\} \cup Sx$ and $R = \{x\} \cup xS$ be the left and right ideals of S generated by x, respectively. Then, by result 1, $(\chi_L, \chi_{L^c}) \in$ IFLI(S) and $(\chi_R, \chi_{R^c}) \in$ IFRI(S). Thus, by the hypothesis,

$$(\chi_R, \chi_{R^c}) \cap (\chi_L, \chi_{L^c}) \subset (\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c}).$$

So

$$\mu_{(\chi_L,\chi_{L^c})\circ(\chi_R,\chi_{R^c})}(x) = \bigvee_{x=yz} [\chi_L(y) \wedge \chi_R(z)] \geq \chi_L(x) \wedge \chi_R(x) = 1$$

and

$$\nu_{(\chi_L,\chi_{L^c})\circ(\chi_R,\chi_{R^c})}(x) = \bigwedge_{x=yz} [\chi_{L^c}(y) \vee \chi_{R^c}(z)] \leq \chi_{L^c}(x) \vee \chi_{R^c}(x) = 0.$$

Then there exist $p, q \in S$ with $x = pq$ such that

$$\chi_L(p) = 1, \quad \chi_{L^c}(p) = 0 \text{ and } \chi_R(q) = 1, \quad \chi_{R^c}(q) = 0.$$

Thus $p \in L$ and $q \in R$. So $p = x$ or $p = sx$ and $q = x$ or $q = xs$, where $s \in S$. In any case, $x = pq = ax^2b$, where $a, b \in S$. Hence S is intra-regular. This completes the proof. □

The following is the immediate result of Theorem 3.4 and Theorem 3.5.

Theorem 3.6. *Let S be a semigroup. Then the following are equivalent:*
(1) S is regular and intra-regular.
(2) For each $A \in$ IFRI(S) and each $B \in$ IFLI(S), $A \circ B = A \cap B \subset B \circ A$.
(3) $(IFQI(S), \circ)$ is a band.
(4) For each $A \in$ IFQI(S), $A \circ A = A$.

Example 3.7. Let $S = \{a, b, c, d, e\}$ be the semigroup with the following multiplication table:

	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

Then clearly S is a non-commutative semigroup which is not intra-regular. We define two non-empty complex mapping $A : S \to I \times I$ and $B : S \to S$ as follows, respectively:

$$\mu_A(a) \geq \mu_A(x), \ \nu_A(a) \leq \nu_A(x) \quad \text{for each } x \in S,$$

$$A(b) = A(d), \quad A(c) = A(e),$$

and

$$\mu_B(a) \geq \mu_B(x),\ \nu_B(a) \leq \nu_B(x) \ \text{ for each } x \in S,$$
$$B(b) = B(c), \quad B(d) = B(e).$$

Then we can easily see that $0_\sim \neq A \in \text{IFLI}(S)$ and $0_\sim \neq B \in \text{IFRI}(S)$. Moreover, we can check that $A \circ A = A$, $B \circ B = B$ and $B \circ A = B \cap A$.

Now we define a complex mapping $C : S \to I \times I$ as follows,

$$C(a) = (1, 0),\ C(b) = C(c) = C(d) = (0.4, 0.5),\ \text{and } C(e) = (0.5, 0.5).$$

Then we can see that $A \in \text{IFQI}(S)$ and $A \neq A \circ A = A \circ A \circ A$.

Example 3.8. Let $S = \{0, a, 1\}$ be the semigroup with the following multiplication table:

	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Then S is a commutative semigroup which is regular and intra-regular. Let $A \in \text{IFQI}(S)$ and let $x \in S$. Then

$$\mu_A(x) \geq \mu_{A\circ A}(x) = \bigvee_{x=yz} [\mu_A(y) \wedge \mu_A(z)] = \bigvee_{x=yz} \mu_A(y)\,,$$

since $A \in \text{IFSG}(S)$ and S is commutative, and

$$\nu_A(x) \leq \nu_{A\circ A}(x) = \bigwedge_{x=yz} [\nu_A(y) \vee \nu_A(z)] = \bigwedge_{x=yz} \nu_A(y).$$

Thus

$$\mu_A(0) = \bigvee_{0=yz} \mu_A(y) \geq \mu_A(x), \quad \nu_A(0) = \bigwedge_{0=yz} \nu_A(y) \leq \nu_A(x)$$

for each $x \in S$ and

$$\mu_A(a) = \mu_A(a) \vee \mu_A(1) \geq \mu_A(1), \quad \nu_A(a) = \nu_A(a) \vee \nu_A(1) \leq \nu_A(1).$$

So, $\mu_A(0) \geq \mu_A(a) \geq \mu_A(1)$ and $\nu_A(0) \leq \nu_A(a) \leq \nu_A(1)$. Then

$$\mu_{A\circ A}(0) = \bigvee_{0=xy} [\mu_A(x) \wedge \mu_A(y)] = \mu_A(0),$$

$$\nu_{A\circ A}(0) = \bigwedge_{0=xy} [\nu_A(x) \vee \nu_A(y)] = \nu_A(0),$$
$$\mu_{A\circ A}(a) = [\mu_A(a) \wedge \mu_A(a)] \vee [\mu_A(a) \wedge \mu_A(1)] = \mu_A(a),$$
$$\nu_{A\circ A}(a) = [\nu_A(a) \vee \nu_A(a)] \wedge [\nu_A(a) \vee \nu_A(1)] = \nu_A(a),$$

and

$$\mu_{A\circ A}(1) = \mu_A(1) \wedge \mu_A(1) = \mu_A(1), \quad \nu_{A\circ A}(1) = \nu_A(1) \vee \nu_A(1) = \nu_A(1).$$

Hence $A^2 = A$.

Example 3.9. Let $S = \{a, b, c\}$ be the semigroup with the following multiplication table:

	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is not commutative but it is regular and intra-regular. We can easily see that $A \in \text{IFQI}(S)$ for each $0_\sim \neq A \in \text{IFS}(S)$. Let $A \in \text{IFQI}(S)$ and let $a \in S$. Then

$$\mu_{A\circ A}(a) = \bigvee_{a=xy} [\mu_A(x) \wedge \mu_A(y)]$$
$$= (\mu_A(a) \wedge \mu_A(a)) \vee (\mu_A(a) \wedge \mu_A(b)) \vee (\mu_A(a) \wedge \mu_A(c)) = \mu_A(a),$$

$$\nu_{A\circ A}(a) = \bigwedge_{a=xy} [\nu_A(x) \vee \nu_A(y)]$$
$$= (\nu_A(a) \vee \nu_A(a)) \wedge (\nu_A(a) \vee \nu_A(b)) \wedge (\nu_A(a) \vee \nu_A(c)) = \nu_A(a),$$

$$\mu_{A\circ A}(b) = (\mu_A(b) \wedge \mu_A(a)) \vee (\mu_A(b) \wedge \mu_A(b)) \vee (\mu_A(b) \wedge \mu_A(c)) = \mu_A(b),$$

$$\nu_{A\circ A}(b) = (\nu_A(b) \vee \nu_A(a)) \wedge (\nu_A(b) \vee \nu_A(b)) \wedge (\nu_A(b) \vee \nu_A(c)) = \nu_A(b),$$

and

$$\mu_{A\circ A}(c) = (\mu_A(c) \wedge \mu_A(a)) \vee (\mu_A(c) \wedge \mu_A(b)) \vee (\mu_A(c) \wedge \mu_A(c)) = \mu_A(c),$$

$$\nu_{A\circ A}(c) = (\nu_A(c) \vee \nu_A(a)) \wedge (\nu_A(c) \vee \nu_A(b)) \wedge (\nu_A(c) \vee \nu_A(c)) = \nu_A(c).$$

So, $A \circ A = A$. Hence each $A \in \text{IFQI}(S)$ is idempotent.

References

[1] J. Ahsan, R.M. Latif, Fuzzy quasi-ideals in semigroups, *J. Fuzzy Math.* **9**, No. 2 (2001), 259-270.

[2] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20** (1986), 87-96.

[3] B. Banerjee, D.Kr. Basnet, Intuitionistic fuzzy subrings and ideals, *J. Fuzzy Math.*, **11**, No. 1 (2003), 139-155.

[4] R. Biswas, Intuitionistic fuzzy subgroups, *Mathematical Forum*, **X** (1989), 37-46.

[5] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroup*, Volume 1, AMS, Soc. Providence (1961).

[6] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems*, **88** (1997), 81-89.

[7] D. Çoker, A. Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, *J. Fuzzy Math.*, **3** (1995), 899-909.

[8] H. Gurcay, D. Çoker, A. Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, *J. Fuzzy Math.*, **5** (1997), 365-378.

[9] K. Hur, S.Y. Jang, H.W. Kang, Intuitionistic fuzzy subgroupoids, *International Journal of Fuzzy Logic and Intelligent Systems*, **3**, No. 1 (2003), 72-77.

[10] K. Hur, H.W. Kang, H.K. Song, Intuitionistic fuzzy subgroups and subrings, *Honam Mathematical J.*, **25**, No. 2 (2003), 19-41.

[11] K. Hur, S.Y. Jang, H.W. Kang, Intuitionistic fuzzy subgroups and cosets, *Honam Math. J.*, **26**, No. 1 (2004), 17-41.

[12] K. Hur, Y.B. Jun, J.H. Ryou, Intuitionistic fuzzy topological groups, *Honam Math. J.*, **26**, No. 2 (2004), 163-192.

[13] K. Hur, J.H. Kim, J.H. Ryou, Intuitionistic fuzzy topological spaces, *J. Korea Soc. Math. Edu. Ser. B: Pure Appl. Math.*, **11**, No. 3 (2004), 243-265.

[14] K. Hur, K.J. Kim, H.K. Song, Intuitionistic fuzzy ideals and bi-ideals, *Honam Math. J.*, **26**, No. 3 (2004), 309-330.

[15] K. Hur, S.Y. Jang, H.W. Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, *Honam Math. J.*, **26**, No. 4 (2004), 559-587.

[16] K. Hur, S.Y. Jang, H.W. Kang, Intuitionistic fuzzy congruences on a lattice, *J. Appl. Math. and Computing*, **18**, No. 2 (2005), 465-486.

[17] K. Hur, H.W. Kang, J.H. Ryou, Intuitionistic fuzzy generalized bi-ideals of a semigroup, *Submited*.

[18] S.J. Lee, E.P. Lee, The category of intuitionistic fuzzy topological spaces, *Bull. Korean Math. Soc.*, **37**, No. 1 (2000), 63-76.

[19] Wang-jin Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems*, **8** (1982), 133-139.

[20] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512-517.

[21] Otto Steinfeld, *Quasi-ideals in Rings and Semigroups*, Akademiakiado, Budapest (1978).

[22] L.A. Zadeh, Fuzzy sets, *Inoform. and Centrol*, **8** (1965), 338-353.