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INTUITIONISTIC H-FUZZY RELATIONS ON INTUITIONISTIC H-FUZZY SETS

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Abstract: We study intuitionistic H-fuzzy relations on an intuitionistic H-fuzzy sets in a categorical point of view. In particular, we show that the category **IFRel**(H) is (co)topological, catesian closed and final episinks in **IFRel**(H) are preserved by pullbacks. Also, we show that the subcategory $\mathbf{IFRel_R}(H)$ of **IFRel**(H) has the same properties as **IFRel**(H).

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1. Introduction

Rosenfeld [17] introduced a notion of fuzzy relation on fuzzy sets. This concept has been applied to various areas including fuzzified graphic theory in [2, 3, 18]. Hur [7-9] investigated the categories **Set**(H), **Rel**(H), $\mathbf{Rel_R}(H)$ and some subcategories of $\mathbf{Rel_R}(H)$ in the sense of a topological universe. Moreover,

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Hur and his colleagues [11-14] studied the categorical structures of the categories **ISet**(H), **IRel**(H) and $\mathbf{IRel_R}(H)$ and some subcategories of $\mathbf{IRel_R}(H)$ in a topological universe view point. In particular, Hur [8] introduced the category **FRel**(H) of H-fuzzy relations on a H-fuzzy set and showed that it is (co)topological Cartesian closed and final episinks in **FRel**(H) are preserved by pullbacks.

In this paper, we introduce the categories **IRel**(H) of intuitionistic H-fuzzy relations on an intuitionistic H-fuzzy set and show that it has the same properties as **FRel**(H). Also, we study the categorical structures of the subcategory $\mathbf{IFRel_R}(H)$ in a topological viewpoint.

It is well-known [11-13] that the category **ISet**(H) (resp. **IRel**(H) and $\mathbf{IRel_R}(H)$) is (co)topological over **Set** and final episinks in **ISet**(H) (resp. **IRel**(H) and $\mathbf{IRel_R}(H)$) are preserved by pullbacks. As a matter of fact, the category **IFRel**(H) will be understand as a reflective and coreflective subcategory of the mixture of **ISet**(H) and **IRel**(H).

For a general background for lattice theory, we refer to [1] and for a general categorical background to [4, 5, 15].

2. Preliminaries

We will list some well-known results, needed in the later sections.

Result 1.A. (see [15, Theorem 2.4; 5, Proposition 36.10 and Proposition 36.11]) *Let* **A** *be a well-powered and co-(well-powered) topological category and let* **B** *be a subcategory of* **A**. *Then the following are equivalent:*

(1) **B** *is epireflective in* **A**.

(2) **B** *is closed under the formation of initial monosources.*

(3) **B** *is closed under the formation of products and pullbacks in* **A**.

Result 1.B. (see [15, Theorem 2.5]) *Let* **A** *be a well-powered and co-(well-powered) topological category and let* **B** *be a subcategory of* **A**. *Then the following are equivalent:*

(1) **B** *is bireflective in* **A**.

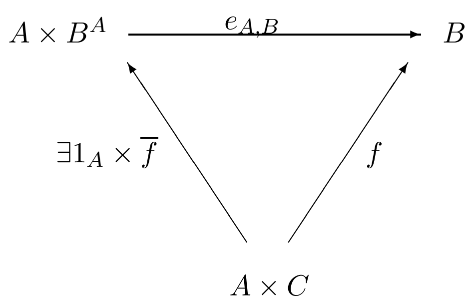
(2) **B** *is closed under the formation of initial sources.*

Result 1.C. (see [15, Theorem 2.6]) *If* **A** *is a (property fibred, resp.) topological category and* **B** *is a bireflective subcategory of* **A**, *then* **B** *is also a (property fibred, resp.) topological category. Moreover, every source in* **B** *which is initial in* **A** *is initial in* **B**.

Definition 1.1. (see [4]) A category **A** is called *Cartesian closed* provided that the following conditions hold:

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential exist in **A**, i.e., for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any **A**-object B, there exists an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes.

Definition 1.2. (see [16]) A category **A** is called a *topological universe over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, i.e.: (i) **A** is a concrete category; (ii) **A** has the fibre-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Throughout this paper, we use H as a complete Heyting algebra.

Definition 1.3. (see [11]) Let X be a set. A triple (X, μ, ν) is called an *intuitionistic H-fuzzy set* (in short, *IHFS*) on X if the following conditions hold:

(i) $\mu, \nu \in H^X$, i.e., μ and ν are H-fuzzy sets.

(ii) $\mu \leq N(\nu)$, i.e., $\mu(x) \leq N(\nu(x))$ for each $x \in X$, where $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

Definition 1.4. (see [11]) Let (X, μ_X, ν_X) and (Y, μ_Y, ν_Y) be IHFSs. A mapping $f : X \to Y$ is called a *morphism* if $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$.

From Definition 1.3 and Definition 1.4, we can form a concrete category **ISet**(H) consisting of all IHFSs and morphisms between them. In this case, each **ISet**(H)-morphism will be called an **ISet**(H)*-mapping*.

Definition 1.5. (see [12]) Let X be a set. A pair $R = (\mu_R, \nu_R)$ is called an *intuitionistic H-fuzzy relation* (in shot, *IHFR*) on X if it satisfies the following conditions:

(i) $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ are mappings, where μ_R and ν_R denote the *degree of membership* (namely $\mu_R(x, y)$) and the *degree of nonmembership* (namely $\nu_R(x, y)$) of each $(x, y) \in X \times X$ to R.

(ii) $\mu_R \leq N(\nu_R)$, i.e., $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$.

In this case, (X, R) or (X, μ_R, ν_R) is called an *intuitionistic H-fuzzy relatinal space* (in short, *IHFRS*).

Definition 1.6. (see [12]) Let (X, R_X) and (Y, R_Y) be an IHFRSs. A mapping $f : X \to Y$ is called a *relation preserving mapping* if $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$.

From Definition 1.5 and Definition 1.6, we can form a concrete category **IRel**(H) consisting of all IHFRSs and relation preserving mappings between them. Every **IRel**(H)-morphism will be called an **IRel**(H)*-mapping*. Moreover, it is clear that if $f : (X, R_X) \to (Y, R_Y)$ is an **IRel**(H)-mapping, then $f : (X, \mu_{R_X}) \to (Y, \mu_{R_Y})$ is a **Rel**(H)-mapping.

3. Intuitionistic H-Fuzzy Relations on Intuitionistic H-Fuzzy Sets

Definition 2.1. Let (X, μ, ν) be any IHFS. Then R is called an *intuitionistic H-fuzzy relation on* (X, μ, ν) if it is an IHFS on $X \times X$ satisfying the following conditions: for any $x, y \in X$,

$$\mu_R(x, y) \leq \mu(x) \wedge \mu(y) \text{ and } \nu_R(x, y) \geq \nu(x) \vee \nu(y).$$

In this case, the tetrad (X, μ, ν, R) is called an *intuitionistic H-fuzzy relational space* over (X, μ, ν).

Definition 2.2. A mapping $f : (X, \mu_X, \nu_X, R_X) \to (Y, \mu_Y, \nu_Y, R_Y)$ is called a relation preserving mapping if $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ is an **ISet**(H)-mapping and $f : (X, R_X) \to (Y, R_Y)$ is an **IRel**(H)-mapping.

We denote the category of all intuitionistic H-fuzzy relational spaces over IHFSs and relation preserving mappings between them by **IFRel**(H), and the mixture of the categories **ISet**(H) and **IRel**(H) by **IFRel**$(H) \wedge$ **ISet**(H) (cf. [6]). Since **ISet**(H) and **IRel**(H) are topological categories over **Set**, so is the mixture **ISet**$(H) \wedge$ **IRel**(H) with natural structures by Proposition 2.2 in [6].

Lemma 2.3. **IFRel**(H) *is a bi(co)reflective subcategory of* **ISet**$(H) \wedge$ **IRel**(H).

Proof. Let (X, μ, ν, R) be any object in **ISet**$(H) \wedge$ **IRel**(H). Define two mappings $\overline{\mu} : X \to H$ and $\overline{\nu} : X \to H$ as follows: for each $x \in X$,

$$\overline{\mu} = \mu(x) \vee [\bigvee_{y \in X} \mu_R(x,y)]$$

and

$$\overline{\nu} = \nu(x) \wedge [\bigwedge_{y \in X} \nu_R(x,y)].$$

Then it is easy to check that $1_X : (X, \mu, \nu, R) \to (X, \overline{\mu}, \overline{\nu}, R)$ is an $\mathbf{IFRel}(H)$-reflection of (X, μ, ν, R). Now define two mappings $\mu_{R_X} : X \times X \to H$ and $\nu_{R_X} : X \times X \to H$ by

$$\mu_{R_X}(x,y) : \mu_R(x,y) \wedge \mu(x) \wedge \nu(x) \text{ and } \nu_{R_X}(x,y) : \nu_R(x,y) \vee \mu(x) \vee \nu(x).$$

Then $1_X : (X, \mu, \nu, R) \to (X, \mu, \nu, R)$ is an $\mathbf{IFRel}(H)$-coreflection of (X, μ, ν, R). This completes the proof. □

By Lemma 2.3, Theorem 2.6 and Theorem 2.8 in [6], we have the following result.

Theorem 2.4. *(1)* $\mathbf{IFRel}(H)$ *is topological over* **Set***.*
(2) Final episinks in $\mathbf{IFRel}(H)$ *are preserved by pullbacks.*

Remark 2.5. (1) Let X be a set and let $(f_\alpha : X \to (X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha)_\Gamma$ any source, where $(X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha) \in \mathbf{IFRel}(H)$ for each $\alpha \in \Gamma$. Define four mappings $\mu : X \to H$, $\nu : X \to H$ and $\mu_R : X \times X \to H$, $\nu_R : X \times X$ as follows: for any $x, y \in X$, $\mu(x) = \bigwedge_\Gamma \mu_\alpha \circ f(x)$, $\nu(x) = \bigvee_\Gamma \nu_\alpha \circ f(x)$ and $\mu_R(x,y) = \bigwedge_\Gamma \mu_{R_\alpha} \circ f_\alpha^2(x,y)$, $\nu_R(x,y) = \bigvee_\Gamma \nu_{R_\alpha} \circ f_\alpha^2(x,y)$. Then (X, μ, ν, R) is equipped with the initial structure with respect to $(f_\alpha)_\Gamma$ in $\mathbf{IRel}(H)$.

(2) Let X be a set and let $(f_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha) \to X)_\Gamma$ any sink, where $(X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha) \in \mathbf{IFRel}$ for each $\alpha \in \Gamma$. Define four mappings $\mu : X \to H$, $\nu : X \to H$ and $\mu_R : X \times X \to H$, $\nu_R : X \times X \to H$ as follows: for any $x, y \in X$,

$$\mu(x) = \bigvee_\Gamma \bigvee_{x_\alpha \in f_\alpha^{-1}(x)} \mu_\alpha(x_\alpha), \quad \nu(x) = \bigwedge_\Gamma \bigwedge_{x_\alpha \in f_\alpha^{-1}(x)} \nu_\alpha(x_\alpha)$$

and

$$\mu_R(x,y) = \bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in {f_\alpha^{-1}}^2(x,y)} \mu_{R_\alpha}(x_\alpha, y_\alpha),$$

$$\nu_R(x,y) = \bigwedge_\Gamma \bigwedge_{(x_\alpha, y_\alpha) \in {f_\alpha^{-1}}^2(x,y)} \nu_{R_\alpha}(x_\alpha, y_\alpha).$$

Then (X, μ, ν, R) is equipped with the final structure with respect to $(f_\alpha)_\Gamma$ in **IRel**(H).

(3) Let $(g_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha) \to (Y, \mu_Y, \nu_Y, R_Y))_\Gamma$ be any final episink in **IHRel**(H) and let $(f : (W, \mu_W, \nu_W, R_W) \to (Y, \mu_Y, \nu_Y, R_Y)$ any **IHRel**(H)-mapping. For each $\alpha \in \Gamma$, let $U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\}$ and let us four mappings $\mu_{U_\alpha} : U_\alpha \to H$, $\nu_{U_\alpha} : U_\alpha \to H$ and $\mu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H$, $\nu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H$ as follows: for any $(w, x_\alpha), (w', x'_\alpha) \in U_\alpha$,

$$\mu_{U_\alpha}(w, x_\alpha) = (\mu_w \times \mu_\alpha)(w, x_\alpha) = \mu_w(x) \wedge \mu_\alpha(x_\alpha),$$

$$\nu_{U_\alpha}(w, x_\alpha) = (\nu_w \times \nu_\alpha)(w, x_\alpha) = \nu_w(x) \vee \nu_\alpha(x_\alpha),$$

and

$$\begin{aligned}\mu_{R_{U_\alpha}}&((w, x_\alpha), (w', x'_\alpha)) \\ &= \mu_{R_w \times R_\alpha}((w, x_\alpha), (w', x'_\alpha)) = \mu_{R_w}(w, w') \wedge \mu_{R_\alpha}(x_\alpha, x'_\alpha),\end{aligned}$$

$$\begin{aligned}\nu_{R_{U_\alpha}}&((w, x_\alpha), (w', x'_\alpha)) \\ &= \nu_{R_w \times R_\alpha}((w, x_\alpha), (w', x'_\alpha)) = \nu_{R_w}(w, w') \vee \nu_{R_\alpha}(x_\alpha, x'_\alpha).\end{aligned}$$

Then for each $\alpha \in \Gamma$, $e_\alpha : (U_\alpha, \mu_{U_\alpha}, \nu_{U_\alpha}, R_{U_\alpha}) \to (W, \mu_W, \nu_W, R_W)$ is the full-back of g_α along f in **IFRel**(H), where e_α denotes the usual projection of U_α. Moreover, $(e_\alpha : (U_\alpha, \mu_{U_\alpha}, \nu_{U_\alpha}, R_{U_\alpha}) \to (W, \mu_W, \nu_W, R_W))_\Gamma$ is a final episink in **IFRel**(H).

(4) We note that **IFRel**(H) is not properly fibred, since both IHFS structures and IHFR structures on a singleton set are not unique.

Remark 2.6. (1) **IFRel**(H) is topological over **ISet**(H): Let $(X, \mu, \nu) \in$ **ISet**(H) and let $((X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha))_\Gamma$ be any family of intuitionistic H-fuzzy relational spaces indexed by a class Γ. Let $(f_\alpha : (X, \mu, \nu) \to (X_\alpha, \mu_\alpha, \nu_\alpha))_\Gamma$ be any source of **ISet**(H)-mappings. Define two mappings $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ by for any $x, y \in X$

$$\mu_R(x, y) = (\bigwedge_\Gamma \mu_{R_\alpha} \circ f^2(x, y)) \wedge (\mu(x) \wedge \mu(y))$$

and

$$\nu_R(x, y) = (\bigvee_\Gamma \nu_{R_\alpha} \circ f^2(x, y)) \vee (\nu(x) \vee \nu(y)).$$

Then R is the initial structure on (X, μ, ν) with respect to $(f_\alpha)_\Gamma$.

(2) **IFRel**(H) is cotopological over **ISet**(H): Let $(X, \nu, \nu) \in$ **ISet**(H) and let $((X_\alpha, \mu_\alpha, \nu_\alpha, R_\alpha))_\Gamma$ be any family of intuitionistic H-fuzzy relational spaces indexed by a class Γ. Let $(f_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha) \to (X, \mu, \nu))_\Gamma$ be any sink of **ISet**(H)-mappings. Define two mappings $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ by for any $x, y \in X$

$$\mu_R(x,y) = (\bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in f_\alpha^{-1\,2}(x,y)} \mu_{R_\alpha}(x_\alpha, y_\alpha)$$

and

$$\nu_R(x,y) = (\bigwedge_\Gamma \bigwedge_{(x_\alpha, y_\alpha) \in f_\alpha^{-1\,2}(x,y)} \nu_{R_\alpha}(x_\alpha, y_\alpha).$$

Then R is the final structure on (X, μ, ν) w.r.t. $(f_\alpha)_\Gamma$.

Theorem 2.7. **IFRel**(H) *is Cartesian closed over* **Set**.

Proof. Since **IFRel**(H) has finite products by Theorem 2.4, it is enough to show that **IFRel**(H) has exponential objects.

For any $\mathbf{X} = (X, \mu_X, \nu_X, R_X)$, $\mathbf{Y} = (Y, \mu_Y, \nu_Y, R_Y) \in$ **IFRel**(H), let Y^X be the set of all morphisms from X into Y in **IFRel**(H). We define four mappings $\mu : Y^X \to H$, $\nu : Y^X \to H$ and $\mu_R : Y^X \times Y^X \to H$, $\nu_R : Y^X \times Y^X \to H$ by for any $f, g \in Y^X$

$$\mu(f) = \bigvee\{h \in H : \mu_X(x) \wedge h \leq \mu_Y \circ f(x) \ \text{ for each } \ x\},$$

$$\nu(f) = \bigwedge\{h \in H : \nu_X(x) \vee h \geq \nu_Y \circ f(x) \ \text{ for each } \ x\},$$

$$\mu_R(f,g) = (\bigvee\{h \in H : \mu_{R_X}(x,y) \wedge h \leq \mu_{R_Y}(f(x), g(y)) \ \text{ for any } \ x, y \in X\})$$
$$\wedge(\mu(f) \wedge \mu(y)),$$

$$\nu_R(f,g) = (\bigwedge\{h \in H : \mu_{R_X}(x,y) \vee h \geq \nu_{R_Y}(f(x), g(y)) \ \text{ for any } \ x, y \in X\})$$
$$\vee(\nu(f) \vee \nu(y)).$$

Then clearly $(Y^X, \mu, \nu) \in$ **ISet**(H) and $(Y^X, \mu, \nu, R) \in$ **IFRel**(H). Let $\mathbf{Y^X} = (Y^X, \mu, \nu, R)$. We define a mapping $e_{X,Y} : X \times Y^X \to Y$ by for any $(x, f) \in X \times Y^X$, $e_{X,Y}(x, f) = f(x)$. Then clearly $e_{X,Y} : (X \times Y^X, \mu_X \times \mu, \nu_X \times \nu) \to (Y, \mu_Y, \nu_Y)$ is an **ISet**(H)-mapping. By the process of the proof of Theorem 2.7 in [8], $\mu_{R_X \times R} \leq \mu_{R_Y} \circ e^2_{X,Y}$. Let $(x, f), (y, g) \in X \times Y^X$. Then

$$\nu_{R_X \times R}((x,f),(y,g)) = \nu_{R_X}(x,y) \vee \nu_R(f,g) \geq \nu_{R_Y}(f(x), g(y))$$

$$= \nu_{R_Y}(e_{X,Y}(x,f), e_{X,Y}(y,g)) = \nu_{R_Y} \circ e^2_{X,Y}((x,f),(y,g)).$$

So, $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an $\mathbf{IFRel}(H)$-mapping.

For any $\mathbf{Z} = (Z, \mu_Z, \nu_Z, R_Z)$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be any $\mathbf{IFRel}(H)$-mapping, where $\mathbf{X} \times \mathbf{Z} = (X \times Z, \mu_X \times \mu_Z, \nu_X \times \nu_Z, R_X \times R_Z)$. We define a mapping $\overline{h} : Z \to Y^X$ by $[\overline{h}(z)](x) = h(x,z)$ for each $z \in Z$ and each $x \in X$. By the process of the proof of Theorem 2.7 in [8], $\overline{h} : (Z, \mu_Z, \nu_Z) \to (Y^X, \mu, \nu)$ is an $\mathbf{ISet}(H)$-mapping and $\mu_{R_Z} \leq \mu_R \circ \overline{h}^2$. Now let $(x, x') \in X \times X$ and let $(z, z') \in Z \times Z$. Since $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is an $\mathbf{IFRel}(H)$-mapping,

$$\begin{aligned}\nu_{R_X \times R_Z}((x,z),(x',z')) &= \nu_{R_X}(x,x') \vee \nu_{R_Z}(z,z') \\ &\geq \nu_{R_Y} \circ h^2((x,z),(x',z')) = \nu_{R_Y}(h(x,z), h(x',z')) \\ &= \nu_{R_Y}([\overline{h}(z)](x), [\overline{h}(z')](x')).\end{aligned}$$

Thus $\nu_{R_Z}(z,z') \geq \bigwedge\{h \in H : \nu_{R_X}(x,x') \vee h \geq \nu_{R_Y}([\overline{h}(z)](x), [\overline{h}(z')](x'))$ for all $x, x' \in X\}$. On the other hand, $\nu_{R_Z}(z,z') \geq \nu(z) \vee \nu(z') \geq \nu(\overline{h}(z)) \vee \nu(\overline{h}(z'))$. Thus

$$\begin{aligned}\nu_{R_Z}(z,z') \geq (\bigwedge\{&h \in H : \nu_{R_X}(x,x') \vee h \\ &\geq \nu_{R_Y}([\overline{h}(z)](x), [\overline{h}(z')](x')) \text{ for all } x, x' \in X\}) \vee \nu(\overline{h}(z)) \vee (\overline{h}(z')).\end{aligned}$$

So $\nu_{R_Z}(z,z') \geq \nu_R(\overline{h}(z), (\overline{h}(z'))$ i.e., $\nu_{R_Z} \geq \nu_R \circ \overline{h}^2$ by the definition of R. In all, $\overline{h} : \mathbf{Z} \to \mathbf{Y^X}$ is an $\mathbf{IFRel}(H)$-mapping. Moreover, $\overline{h}$ is unique and $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

Remark 2.8. Define a mapping $E : \mathbf{ISet}(H) \to \mathbf{IFRel}(H)$ by $E((X, \mu, \nu)) = (X, \mu, \nu, R)$, where $\mu_R(x,y) = \mu(x) \wedge \mu(y)$, $\nu_R(x,y) = \nu(x) \vee \nu(y)$ for each $(x,y) \in X \times X$ and $E(f) = f$. Then E is a full and embedding functor. Hence we may consider $\mathbf{ISet}(H)$ as a subcategory of $\mathbf{IFRel}(H)$. Moreover, it can be shown, by routine work, that $\mathbf{ISet}(H)$ is a bireflective subcategory of $\mathbf{IFRel}(H)$. In a similar way, we can show that $\mathbf{ISet}(H)$ is a bireflective subcategory of $\mathbf{IFRel}(H)$ with the functor $G(X, R_X) = (X, 1, 0, R_X)$ and $G(f) = f$.

4. The Category $IFRel_R(H)$

Definition 3.1. Let R be an IFR on IFS (X, μ, ν). Then R is said to be *reflexive* if $\mu_R(x,x) = \mu(x)$ and $\nu_R(x,x) = \nu(x)$ for each $x \in X$.

The class of all intuitionistic fuzzy reflexive relational spaces over IFSs and $\mathbf{IFRel}(H)$-mappings between them forms a subcategory of $\mathbf{IFRel}(H)$ and is denote by $\mathbf{IFRel_R}(H)$.

Lemma 3.2. $\mathbf{IFRel_R}(H)$ *is closed under the formation of initial sources in* $\mathbf{IFRel}(H)$.

Proof. Let $(f_\alpha : (X,\mu,\nu,R) \to (X_\alpha,\mu_\alpha,\nu_\alpha,R_\alpha))_\Gamma$ be an initial source in $\mathbf{IFRel}(H)$ such that $(X_\alpha,\mu_\alpha,\nu_\alpha,R_\alpha) \in \mathbf{IFRel_R}(H)$ for each $\alpha \in \Gamma$. Let $x \in X$. Since R_α is reflexive on $(X_\alpha,\mu_\alpha,\nu_\alpha)$ for each $\alpha \in \Gamma$, $\mu_{R_\alpha} \circ f_\alpha^2(x,x) = \mu_\alpha \circ f_\alpha(x)$ and $\nu_{R_\alpha} \circ f_\alpha^2(x,x) = \nu_\alpha \circ f_\alpha(x)$ for each $\alpha \in \Gamma$. Thus $\bigwedge_\Gamma \mu_{R_\alpha} \circ f_\alpha^2(x,x) = \bigwedge_\Gamma \mu_\alpha \circ f_\alpha(x)$ and $\bigvee_\Gamma \nu_{R_\alpha} \circ f_\alpha^2(x,x) = \bigvee_\Gamma \nu_\alpha \circ f_\alpha(x)$. So, by the hypothesis, $\mu_R(x,x) = \mu(x)$ and $\nu_R(x,x) = \nu(x)$ for each $x \in X$. Hence $(X,\mu,\nu,R) \in \mathbf{IFRel_R}(H)$. This completes the proof. □

Hence, by Result 1.B and Result 1.C, we obtain the following theorem.

Theorem 3.3. *(1)* $\mathbf{IFRel_R}(H)$ *is a bireflective subcategory of* $\mathbf{IFRel}(H)$.
(2) $\mathbf{IFRel_R}(H)$ *is topological over* $\mathbf{ISet}$.

Remark 3.4. (1) $\mathbf{IFRel_R}(H)$ is topological over **Set**: Let X be any set and let $(X_\alpha,\mu_\alpha,\nu_\alpha,R_\alpha)_\Gamma$ any family of objects of $\mathbf{IFRel_R}(H)$ indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be a sink of mappings. Define four mappings $\mu,\nu : X \to H$ and $\mu_R,\nu_R : X \times X \to H$ as follows: for any $x,y \in X$,

$$\mu(x) = \bigvee_\Gamma \bigvee_{x_\alpha \in f_\alpha^{-1}(x)} \mu_\alpha(x_\alpha), \quad \nu(x) = \bigwedge_\Gamma \bigwedge_{x_\alpha \in f_\alpha^{-1}(x)} \nu_\alpha(x_\alpha)$$

and

$$\mu_R(x,y) = \begin{cases} (\bigvee_\Gamma \bigvee_{(x_\alpha,y_\alpha)\in f_\alpha^{-1\,2}(x,y)} \mu_{R_\alpha}(x_\alpha,y_\alpha), & \text{if } (x,y) \in (X \times X - \Delta_X), \\ \mu(x), & \text{if } (x,y) \in \Delta_X, \end{cases}$$

$$\nu_R(x,y) = \begin{cases} (\bigwedge_\Gamma \bigwedge_{(x_\alpha,y_\alpha)\in f_\alpha^{-1\,2}(x,y)} \nu_{R_\alpha}(x_\alpha,y_\alpha), & \text{if } (x,y) \in (X \times X - \Delta_X), \\ \nu(x), & \text{if } (x,y) \in \Delta_X. \end{cases}$$

Then we can easily see that $(f_\alpha : (X_\alpha,\mu_\alpha,\nu_\alpha,R_\alpha) \to (X,\mu,\nu,R))_\Gamma$ is a final sink in $\mathbf{IFRel_R}(H)$.

(2) Final episinks in $\mathbf{IFRel_R}(H)$ are preserved by pullbacks: Let $(g_\alpha : (X_\alpha,\mu_\alpha,\nu_\alpha,R_\alpha) \to (Y,\mu_Y,\nu_Y,R_Y))_\Gamma$ be any final episink in $\mathbf{IFRel_R}(H)$ and let $f : (W,\mu_W,\nu_W,R_W) \to (Y,\mu_Y,\nu_Y,R_Y)$ and $\mathbf{IFRel_R}(H)$-mapping. For each $\alpha \in \Gamma$, let us take $U_\alpha, \mu_{U_\alpha}, \nu_{U_\alpha}, R_{U_\alpha}, e_\alpha$ and p_α as in Remark 2.5(3). Then we can easily check that $(e_\alpha : (U_\alpha,\mu_{U_\alpha},\nu_{U_\alpha},R_{U_\alpha}) \to (W,\mu_W,\nu_W,R_W))_\Gamma$ is a final episink in $\mathbf{IFRel_R}(H)$.

(3) $\mathbf{IFRel_R}(H)$ is not properly fibred over **Set**, since the intuitionistic fuzzy reflexive relational space (X,μ,ν,R) is not unique for any singleton set X.

Hence, by Theorem 3.3(2) and Remark 3.4(2), we obtain the following theorem.

Theorem 3.5. $\mathbf{IFRel_R}(H)$ *satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

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