ON GAMMA-IDEAL EXPANSIONS OF GAMMA-RINGS

Young Bae Jun\textsuperscript{1}, Mehmet Ali Öztürk\textsuperscript{2}, Mustafa Uçkun\textsuperscript{3}\textsuperscript{§}

\textsuperscript{1}Department of Mathematical Education
Gyeongsang National University
Chinju, 660-701, KOREA
e-mail: ybjun@gsnu.ac.kr

\textsuperscript{2}Department of Mathematics
Faculty of Arts and Sciences
Cumhuriyet University
Sivas, 58140, TURKEY
e-mail: maozturk@cumhuriyet.edu.tr

\textsuperscript{3}Department of Mathematics
Faculty of Arts and Sciences
İnönü University
Malatya, 44069, TURKEY
e-mail: muckun@inonu.edu.tr

Abstract: The notion of (intersection preserving, global) $\Gamma$-ideal expansion of a $\Gamma$-ring is introduced, and several properties are investigated.

AMS Subject Classification: 13A15

Key Words: $\Gamma$-ideal, intersection preserving, global $\Gamma$-ideal expansion, $\Gamma$-ring homomorphism

1. Introduction

Nobusawa \cite{4} introduced the notion of a $\Gamma$-ring, as more general than a ring. Barnes \cite{1} weakened slightly the conditions in the definition of the $\Gamma$-ring in the sense of Nobusawa. Barnes \cite{1}, Kyuno \cite{2} and Luh \cite{3} studied the structure of
\(\Gamma\)-rings and obtained various generalizations analogous to corresponding parts in ring theory. Prime ideals and primary ideals are two of the most important structures in ring theory. Zhao [5] investigated the possibility of a unified approach to studying such two ideals, and introduced the notion of \(\delta\)-primary ideals for a mapping \(\delta\) that assigns to each ideal \(I\) an ideal \(\delta(I)\) of the same ring. Such \(\delta\)-primary ideals unify the prime and primary ideals under one frame. The purpose of this paper is to apply the Zhao's idea in ring theory to a \(\Gamma\)-ring which is a generalization of a ring. We introduce the notion of (intersection preserving, global) \(\Gamma\)-ideal expansions in \(\Gamma\)-rings, and investigate several properties.

2. Preliminaries

Let \(M\) and \(\Gamma\) be two Abelian groups. If for all \(x, y, z \in M\) and all \(\alpha, \beta \in \Gamma\) the conditions:

1. \(x\alpha y \in M\);
2. \((x + y)\alpha z = x\alpha z + y\alpha z, \ x(\alpha + \beta)z = x\alpha z + x\beta z, \ x\alpha(y + z) = x\alpha y + x\alpha z;\)
3. \((x\alpha y)\beta z = x\alpha(y\beta z);\)

are satisfied, then we call \(M\) a \(\Gamma\)-ring. By a right (resp. left) \(\Gamma\)-ideal of a \(\Gamma\)-ring \(M\) we mean an additive subgroup \(U\) of \(M\) such that \(U\Gamma M \subseteq U\) (resp. \(M\Gamma U \subseteq U\)). If \(U\) is both a right and a left \(\Gamma\)-ideal, then we say that \(U\) is a \(\Gamma\)-ideal of \(M\). A \(\Gamma\)-ideal \(I\) of \(M\) is said to be prime if for any ideals \(U\) and \(V\) of \(M\), \(U\Gamma V \subseteq I\) implies \(U \subseteq I\) or \(V \subseteq I\). We note from [1] that a proper \(\Gamma\)-ideal \(I\) of \(M\) is prime if \(a\Gamma b \subseteq I\) implies \(a \in I\) or \(b \in I\) for all \(a, b \in M\). A mapping \(\sigma : M \rightarrow M'\) of \(\Gamma\)-rings is called a \(\Gamma\)-ring homomorphism if it satisfies:

1. \(\sigma(a + b) = \sigma(a) + \sigma(b)\) for all \(a, b \in M\);
2. \(\sigma(a\gamma b) = \sigma(a)\gamma \sigma(b)\) for all \(a, b \in M\) and \(\gamma \in \Gamma\).

3. \(\Gamma\)-ideal Expansions

In what follows let \(M\) denote a \(\Gamma\)-ring unless otherwise specified.

**Definition 1.** A \(\Gamma\)-ideal \(I\) of \(M\) is said to be primary if it satisfies:

\[(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I, a \notin I \Rightarrow b \in \sqrt{I}),\]

where \(\sqrt{I} := \{x \in M \mid (x\gamma)^{n-1}x \in I\}\) for some \(n \in \mathbb{N}\) and \(\gamma \in \Gamma\), and \((x\gamma)^{n-1}x = x\) when \(n = 1\).
Denote by $\mathcal{J}(M)$ the set of all $\Gamma$-ideals of $M$.

**Definition 2.** Let $\mathcal{O}(M)$ be a set of objects in $M$. An *expansion of objects* in $M$ is defined to be a function $\sigma : \mathcal{O}(M) \rightarrow \mathcal{O}(M)$ such that:

(i) $(\forall G \in \mathcal{O}(M))(G \subseteq \sigma(G))$;

(ii) $(\forall G, H \in \mathcal{O}(M))(G \subseteq H \Rightarrow \sigma(G) \subseteq \sigma(H))$.

If $\mathcal{O}(M) = \mathcal{J}(M)$, we say that $\sigma$ is an *expansion of $\Gamma$-ideals* or a *$\Gamma$-ideal expansion* of $M$.

**Example 3.**
(1) The identity function $1 : \mathcal{J}(M) \rightarrow \mathcal{J}(M)$ is a $\Gamma$-ideal expansion of $M$.

(2) Denote $M(I) := \bigcap \{ J \mid I \subseteq J \text{ and } J \text{ is a maximal } \Gamma \text{-ideal of } M \}$. A function $g : \mathcal{J}(M) \rightarrow \mathcal{J}(M)$ given by $g(I) = M(I)$ for all $I \in \mathcal{J}(M)$ is a $\Gamma$-ideal expansion of $M$.

(3) The constant function $c : \mathcal{J}(M) \rightarrow \mathcal{J}(M)$, $I \mapsto M$, is a $\Gamma$-ideal expansion of $M$.

**Definition 4.** Given a $\Gamma$-ideal expansion $f$ of $M$, a $\Gamma$-ideal $I \in \mathcal{J}(M)$ is said to be $f$-*primary* if it satisfies:

$$(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I, a \notin I \Rightarrow b \in f(I)).$$

**Example 5.** Every $\Gamma$-ideal $I \in \mathcal{J}(M)$ is $c$-primary, where $c$ is a $\Gamma$-ideal expansion of $M$ in Example 3(3).

**Theorem 6.** Let $f$ and $g$ be $\Gamma$-ideal expansions of $M$. If $f(I) \subseteq g(I)$ for all $I \in \mathcal{J}(M)$, then every $f$-primary $\Gamma$-ideal is also $g$-primary.

**Proof.** Let $I$ be an $f$-primary $\Gamma$-ideal of $M$. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in I$ and $a \notin I$. Then $b \in f(I) \subseteq g(I)$ by assumption. Hence $I$ is a $g$-primary $\Gamma$-ideal of $M$. $\square$

**Theorem 7.** Let $f_1$ and $f_2$ be $\Gamma$-ideal expansions of $M$ and let $f$ be a self map of $\mathcal{J}(M)$ defined by $f(I) = f_1(I) \cap f_2(I)$ for all $I \in \mathcal{J}(M)$. Then $f$ is a $\Gamma$-ideal expansion of $M$.

**Proof.** For every $I \in \mathcal{J}(M)$, we have $I \subseteq f_1(I)$ and $I \subseteq f_2(I)$ by (o1), and so $I \subseteq f_1(I) \cap f_2(I) = f(I)$. Let $G, H \in \mathcal{J}(M)$ be such that $G \subseteq H$. Then $f_1(G) \subseteq f_1(H)$ and $f_2(G) \subseteq f_2(H)$ by (o2), which imply that

$$f(G) = f_1(G) \cap f_2(G) \subseteq f_1(H) \cap f_2(H) = f(H).$$

Hence $f$ is a $\Gamma$-ideal expansion of $M$. $\square$
Generally, if \( \{ f_i \mid i \in \Lambda \} \) is a collection of \( \Gamma \)-ideal expansions of \( M \), then the function \( \bigcap_{i \in \Lambda} f_i : \mathcal{I}(M) \to \mathcal{I}(M) \) given by \( (\bigcap_{i \in \Lambda} f_i)(I) = \bigcap_{i \in \Lambda} f_i(I) \) for all \( I \in \mathcal{I}(M) \) is a \( \Gamma \)-ideal expansion of \( M \).

**Theorem 8.** Let \( f \) be a \( \Gamma \)-ideal expansion of \( M \). For any subset \( S \) of \( M \), denote by \( \mathcal{I}_f(S) \) the intersection of all \( f \)-primary \( \Gamma \)-ideals of \( M \) containing \( S \). Then the function \( h : \mathcal{I}(M) \to \mathcal{I}(M) \) given by \( h(I) = \mathcal{I}_f(I) \) for all \( I \in \mathcal{I}(M) \) is a \( \Gamma \)-ideal expansion of \( M \).

**Proof.** Obviously, \( I \subseteq \mathcal{I}_f(I) = h(I) \) for all \( I \in \mathcal{I}(M) \). Let \( I, J \in \mathcal{I}(M) \) be such that \( I \subseteq J \). Then

\[
h(I) = \mathcal{I}_f(I) = \bigcap \{ H \in \mathcal{I}(M) \mid I \subseteq H \text{ and } H \text{ is } f \text{-primary} \}
\leq \bigcap \{ H \in \mathcal{I}(M) \mid J \subseteq H \text{ and } H \text{ is } f \text{-primary} \} = \mathcal{I}_f(J) = h(J).
\]

Hence \( h \) is a \( \Gamma \)-ideal expansion of \( M \).

**Theorem 9.** Let \( f \) be a \( \Gamma \)-ideal expansion of \( M \). If \( \{ J_i \mid i \in \Lambda \} \) is a directed collection of \( f \)-primary \( \Gamma \)-ideals of \( M \), where \( \Lambda \) is an index set, then the \( \Gamma \)-ideal \( J := \bigcup_{i \in \Lambda} J_i \) is \( f \)-primary.

**Proof.** Let \( a, b \in M \) and \( \gamma \in \Gamma \) be such that \( a \gamma b \in J \) and \( a \notin J \). Then there exists \( J_i \) such that \( a \gamma b \in J_i \) and \( a \notin J_i \). Since \( J_i \) is \( f \)-primary and \( J_i \subseteq J \), it follows that \( b \in f(J_i) \subseteq f(J) \) so that \( J \) is \( f \)-primary.

**Theorem 10.** Let \( f \) be a \( \Gamma \)-ideal expansion of \( M \). If \( P \) is an \( f \)-primary \( \Gamma \)-ideal of \( M \), then

\[
(\forall I, J \in \mathcal{I}(M)) (I \Gamma J \subseteq P, I \not\subseteq P \Rightarrow J \subseteq f(P)).
\]

**Proof.** Assume that \( P \) is an \( f \)-primary \( \Gamma \)-ideal of \( M \) and let \( I, J \in \mathcal{I}(M) \) be such that \( I \Gamma J \subseteq P \) and \( I \not\subseteq P \). Suppose that \( J \not\subseteq f(P) \). Then there exist \( a \in I \setminus P \) and \( b \in J \setminus f(P) \), which imply that \( a \gamma b \in I \Gamma J \subseteq P \). But \( a \notin P \) and \( b \notin f(P) \). This contradicts the assumption that \( P \) is \( f \)-primary. Consequently, the result is valid.

**Definition 11.** A \( \Gamma \)-ideal expansion \( f \) is said to be **intersection preserving** if it satisfies:

\[
(\forall I, J \in \mathcal{I}(M))(f(I \cap J) = f(I) \cap f(J)).
\]
A $\Gamma$-ideal expansion $f$ is said to be global if for each $\Gamma$-ring homomorphism $\sigma : M \to M'$ of $\Gamma$-rings, the following holds:

$$\forall I \in \mathcal{I}(M') \ (f(\sigma^{-1}(I)) = \sigma^{-1}(f(I))).$$

Note that the $\Gamma$-ideal expansion 1 of $M$ in Example 3(1) is both intersection preserving and global.

**Theorem 12.** For each $I \in \mathcal{I}(M)$, let

$$\Psi(I) := \bigcap\{J \mid I \subseteq J \text{ and } J \text{ is a prime } \Gamma \text{-ideal of } M\}.$$

Then a function $f : \mathcal{I}(M) \to \mathcal{I}(M)$ given by $f(I) = \Psi(I)$ for all $I \in \mathcal{I}(M)$ is an intersection preserving $\Gamma$-ideal expansion of $M$.

**Proof.** Obviously, $f$ is a $\Gamma$-ideal expansion of $M$. For every $I, J \in \mathcal{I}(M)$, let

$$\Psi_1 := \{P \mid I \cap J \subseteq P \text{ and } P \text{ is a prime } \Gamma \text{-ideal of } M\}$$

and

$$\Psi_2 := \{P \mid I \subseteq P \text{ or } J \subseteq P, \ P \text{ is a prime } \Gamma \text{-ideal of } M\}.$$

Then $\bigcap \Psi_1 = \Psi(I \cap J)$ and $\bigcap \Psi_2 = \Psi(I) \cap \Psi(J)$. Obviously, $\Psi_2 \subseteq \Psi_1$. If $P \in \Psi_1$, then $I \cap J \subseteq I \cap J \subseteq P$ and so $I \subseteq P$ or $J \subseteq P$ because $P$ is prime. Hence $P \in \Psi_2$, and thus $\Psi_1 = \Psi_2$. Therefore

$$f(I \cap J) = \Psi(I \cap J) = \bigcap \Psi_1 = \bigcap \Psi_2 = \Psi(I) \cap \Psi(J) = f(I) \cap f(J).$$

This completes the proof. □

**Theorem 13.** Let $f$ be a $\Gamma$-ideal expansion of $M$ which is intersection preserving. If $I_1, I_2, \cdots, I_n$ are $f$-primary $\Gamma$-ideals of $M$ and $J = f(I_k)$ for all $k = 1, 2, \cdots, n$, then $I := \bigcap_{k=1}^n I_k$ is an $f$-primary $\Gamma$-ideal of $M$.

**Proof.** Obviously, $I := \bigcap_{k=1}^n I_k$ is a $\Gamma$-ideal of $M$. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in I$ and $a \notin I$. Then $a \notin I_k$ for some $k \in \{1, 2, \cdots, n\}$. But $a\gamma b \in I \subseteq I_k$ and $I_k$ is $f$-primary, which imply that $b \in f(I_k)$. Since $f$ is intersection preserving, we have

$$f(I) = f(\bigcap_{k=1}^n I_k) = \bigcap_{k=1}^n f(I_k) = J = f(I_k),$$

and so $b \in f(I)$. Therefore $I$ is an $f$-primary $\Gamma$-ideal of $M$. □
Let $\sigma : M \to M'$ be a $\Gamma$-ring homomorphism of $\Gamma$-rings. Note that if $J$ is a $\Gamma$-ideal of $M'$, then $\sigma^{-1}(J)$ is a $\Gamma$-ideal of $M$, and that if $\sigma$ is surjective and $I$ is a $\Gamma$-ideal of $M$, then $\sigma(I)$ is a $\Gamma$-ideal of $M'$.

**Theorem 14.** Let $f$ be a $\Gamma$-ideal expansion which is global and let $\sigma : M \to M'$ be a $\Gamma$-ring homomorphism of $\Gamma$-rings. If $J$ is an $f$-primary $\Gamma$-ideal of $M'$, then $\sigma^{-1}(J)$ is an $f$-primary $\Gamma$-ideal of $M$.

**Proof.** Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in \sigma^{-1}(J)$ and $a \notin \sigma^{-1}(J)$. Then $\sigma(a)\gamma\sigma(b) = \sigma(a\gamma b) \in J$ and $\sigma(a) \notin J$, which imply from (1) that $\sigma(b) \in f(J)$. Since $f$ is global, it follows that $b \in \sigma^{-1}(f(J)) = f(\sigma^{-1}(J))$. Hence $\sigma^{-1}(J)$ is $f$-primary.

It can be easily verified that if $\sigma : M \to M'$ is a $\Gamma$-ring homomorphism of $\Gamma$-rings, then $\sigma^{-1}(\sigma(I)) = I$ for any $I \in \mathcal{I}(M)$ that contains $\ker(\sigma)$.

**Theorem 15.** Let $\sigma : M \to M'$ be a surjective $\Gamma$-ring homomorphism of $\Gamma$-rings and let $I$ be a $\Gamma$-ideal of $M$ that contains $\ker(\sigma)$. Then $I$ is $f$-primary if and only if $\sigma(I)$ is an $f$-primary $\Gamma$-ideal of $M'$, where $f$ is a global $\Gamma$-ideal expansion.

**Proof.** If $\sigma(I)$ is an $f$-primary $\Gamma$-ideal of $M'$, then $I$ is $f$-primary by $I = \sigma^{-1}(\sigma(I))$ and Theorem 14. Suppose that $I$ is $f$-primary. Let $x, y \in M'$ and $\gamma \in \Gamma$ be such that $x\gamma y \in \sigma(I)$ and $x \notin \sigma(I)$. Since $\sigma$ is surjective, we have $\sigma(a) = x$ and $\sigma(b) = y$ for some $a, b \in M$. Then $\sigma(a\gamma b) = \sigma(a)\gamma\sigma(b) = x\gamma y \in \sigma(I)$ and $\sigma(a) = x \notin \sigma(I)$, which imply that $a\gamma b \in \sigma^{-1}(\sigma(I)) = I$ and $a \notin \sigma^{-1}(\sigma(I)) = I$. Since $I$ is $f$-primary, it follows that $b \in f(I)$ so that $y = \sigma(b) \in \sigma(f(I))$. Using the fact that $f$ is global, we have

$$f(I) = f(\sigma^{-1}(\sigma(I))) = \sigma^{-1}(f(\sigma(I))) ,$$

and so $\sigma(f(I)) = \sigma(\sigma^{-1}(f(\sigma(I)))) = f(\sigma(I))$ since $\sigma$ is surjective. Therefore $\sigma(I)$ is $f$-primary. This completes the proof.

**Acknowledgements**

The first author was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).
References


