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ON GAMMA-IDEAL EXPANSIONS OF GAMMA-RINGS

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Abstract: The notion of (intersection preserving, global) Γ-ideal expansion of a Γ-ring is introduced, and several properties are investigated.

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1. Introduction

Nobusawa [4] introduced the notion of a Γ-ring, as more general than a ring. Barnes [1] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes [1], Kyuno [2] and Luh [3] studied the structure of

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Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Prime ideals and primary ideals are two of the most important structures in ring theory. Zhao [5] investigated the possibility of a unified approach to studying such two ideals, and introduced the notion of δ-primary ideals for a mapping δ that assigns to each ideal I an ideal $\delta(I)$ of the same ring. Such δ-primary ideals unify the prime and primary ideals under one frame. The purpose of this paper is to apply the Zhao's idea in ring theory to a Γ-ring which is a generalization of a ring. We introduce the notion of (intersection preserving, global) Γ-ideal expansions in Γ-rings, and investigate several properties.

2. Preliminaries

Let M and Γ be two Abelian groups. If for all $x, y, z \in M$ and all α, $\beta \in \Gamma$ the conditions:

1. $x\alpha y \in M$;
2. $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$;
3. $(x\alpha y)\beta z = x\alpha(y\beta z)$;

are satisfied, then we call M a Γ-*ring*. By a *right* (resp. *left*) Γ-*ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left Γ-ideal, then we say that U is a Γ-*ideal* of M. A Γ-ideal I of M is said to be *prime* if for any ideals U and V of M, $U\Gamma V \subseteq I$ implies $U \subseteq I$ or $V \subseteq I$. We note from [1] that a proper Γ-ideal I of M is *prime* if $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$ for all $a, b \in M$. A mapping $\sigma : M \to M'$ of Γ-rings is called a Γ-*ring homomorphism* if it satisfies:

1. $\sigma(a+b) = \sigma(a) + \sigma(b)$ for all $a, b \in M$;
2. $\sigma(a\gamma b) = \sigma(a)\gamma\sigma(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

3. Γ-ideal Expansions

In what follows let M denote a Γ-ring unless otherwise specified.

Definition 1. A Γ-ideal I of M is said to be *primary* if it satisfies:

$$(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I,\ a \notin I \ \Rightarrow\ b \in \sqrt{I}),$$

where $\sqrt{I} := \{x \in M \mid (x\gamma)^{n-1}x \in I \text{ for some } n \in \mathbb{N} \text{ and } \gamma \in \Gamma\}$, and $(x\gamma)^{n-1}x = x$ when $n = 1$.

Denote by $\mathfrak{I}(M)$ the set of all Γ-ideals of M.

Definition 2. Let $\mathbb{O}(M)$ be a set of objects in M. An *expansion of objects* in M is defined to be a function $\sigma : \mathbb{O}(M) \to \mathbb{O}(M)$ such that:

(o1) $(\forall G \in \mathbb{O}(M))\,(G \subseteq \sigma(G))$;

(o2) $(\forall G, H \in \mathbb{O}(M))\,(G \subseteq H \Rightarrow \sigma(G) \subseteq \sigma(H))$.

If $\mathbb{O}(M) = \mathfrak{I}(M)$, we say that σ is an *expansion of* Γ*-ideals* or a Γ*-ideal expansion* of M.

Example 3. (1) The identity function $\mathbf{1} : \mathfrak{I}(M) \to \mathfrak{I}(M)$ is a Γ-ideal expansion of M.

(2) Denote $\mathfrak{M}(I) := \bigcap\{J \mid I \subseteq J \text{ and } J \text{ is a maximal Γ-ideal of } M\}$. A function $g : \mathfrak{I}(M) \to \mathfrak{I}(M)$ given by $g(I) = \mathfrak{M}(I)$ for all $I \in \mathfrak{I}(M)$ is a Γ-ideal expansion of M.

(3) The constant function $c : \mathfrak{I}(M) \to \mathfrak{I}(M)$, $I \mapsto M$, is a Γ-ideal expansion of M.

Definition 4. Given a Γ-ideal expansion f of M, a Γ-ideal $I \in \mathfrak{I}(M)$ is said to be *f-primary* if it satisfies:

$$(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I,\, a \notin I \Rightarrow b \in f(I)). \tag{1}$$

Example 5. Every Γ-ideal $I \in \mathfrak{I}(M)$ is c-primary, where c is a Γ-ideal expansion of M in Example 3(3).

Theorem 6. *Let f and g be Γ-ideal expansions of M. If $f(I) \subseteq g(I)$ for all $I \in \mathfrak{I}(M)$, then every f-primary Γ-ideal is also g-primary.*

Proof. Let I be an f-primary Γ-ideal of M. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in I$ and $a \notin I$. Then $b \in f(I) \subseteq g(I)$ by assumption. Hence I is a g-primary Γ-ideal of M. □

Theorem 7. *Let f_1 and f_2 be Γ-ideal expansions of M and let f be a self map of $\mathfrak{I}(M)$ defined by $f(I) = f_1(I) \cap f_2(I)$ for all $I \in \mathfrak{I}(M)$. Then f is a Γ-ideal expansion of M.*

Proof. For every $I \in \mathfrak{I}(M)$, we have $I \subseteq f_1(I)$ and $I \subseteq f_2(I)$ by (o1), and so $I \subseteq f_1(I) \cap f_2(I) = f(I)$. Let $G, H \in \mathfrak{I}(M)$ be such that $G \subseteq H$. Then $f_1(G) \subseteq f_1(H)$ and $f_2(G) \subseteq f_2(H)$ by (o2), which imply that

$$f(G) = f_1(G) \cap f_2(G) \subseteq f_1(H) \cap f_2(H) = f(H).$$

Hence f is a Γ-ideal expansion of M. □

Generally, if $\{f_i \mid i \in \Lambda\}$ is a collection of Γ-ideal expansions of M, then the function $\bigcap_{i\in\Lambda} f_i : \mathfrak{I}(M) \to \mathfrak{I}(M)$ given by $(\bigcap_{i\in\Lambda} f_i)(I) = \bigcap_{i\in\Lambda} f_i(I)$ for all $I \in \mathfrak{I}(M)$ is a Γ-ideal expansion of M.

Theorem 8. *Let f be a Γ-ideal expansion of M. For any subset S of M, denote by $\mathfrak{I}_f(S)$ the intersection of all f-primary Γ-ideals of M containing S. Then the function $h : \mathfrak{I}(M) \to \mathfrak{I}(M)$ given by $h(I) = \mathfrak{I}_f(I)$ for all $I \in \mathfrak{I}(M)$ is a Γ-ideal expansion of M.*

Proof. Obviously, $I \subseteq \mathfrak{I}_f(I) = h(I)$ for all $I \in \mathfrak{I}(M)$. Let $I, J \in \mathfrak{I}(M)$ be such that $I \subseteq J$. Then

$$\begin{aligned} h(I) = \mathfrak{I}_f(I) &= \bigcap\{H \in \mathfrak{I}(M) \mid I \subseteq H \text{ and } H \text{ is } f\text{-primary}\} \\ &\subseteq \bigcap\{H \in \mathfrak{I}(M) \mid J \subseteq H \text{ and } H \text{ is } f\text{-primary}\} = \mathfrak{I}_f(J) = h(J). \end{aligned}$$

Hence h is a Γ-ideal expansion of M. □

Theorem 9. *Let f be a Γ-ideal expansion of M. If $\{J_i \mid i \in \Lambda\}$ is a directed collection of f-primary Γ-ideals of M, where Λ is an index set, then the Γ-ideal $J := \bigcup_{i\in\Lambda} J_i$ is f-primary.*

Proof. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in J$ and $a \notin J$. Then there exists J_i such that $a\gamma b \in J_i$ and $a \notin J_i$. Since J_i is f-primary and $J_i \subseteq J$, it follows that $b \in f(J_i) \subseteq f(J)$ so that J is f-primary. □

Theorem 10. *Let f be a Γ-ideal expansion of M. If P is an f-primary Γ-ideal of M, then*

$$(\forall I, J \in \mathfrak{I}(M))\,(I\,\Gamma J \subseteq P,\; I \nsubseteq P \;\Rightarrow\; J \subseteq f(P)).$$

Proof. Assume that P is an f-primary Γ-ideal of M and let $I, J \in \mathfrak{I}(M)$ be such that $I\,\Gamma J \subseteq P$ and $I \nsubseteq P$. Suppose that $J \nsubseteq f(P)$. Then there exist $a \in I \setminus P$ and $b \in J \setminus f(P)$, which imply that $a\gamma b \in I\,\Gamma J \subseteq P$. But $a \notin P$ and $b \notin f(P)$. This contradicts the assumption that P is f-primary. Consequently, the result is valid. □

Definition 11. A Γ-ideal expansion f is said to be *intersection preserving* if it satisfies:

$$(\forall I, J \in \mathfrak{I}(M))(f(I \cap J) = f(I) \cap f(J)).$$

A Γ-ideal expansion f is said to be *global* if for each Γ-ring homomorphism $\sigma : M \to M'$ of Γ-rings, the following holds:

$$(\forall I \in \mathfrak{I}(M'))(f(\sigma^{-1}(I)) = \sigma^{-1}(f(I))).$$

Note that the Γ-ideal expansion **1** of M in Example 3(1) is both intersection preserving and global.

Theorem 12. *For each $I \in \mathfrak{I}(M)$, let*

$$\mathfrak{P}(I) := \bigcap\{J \mid I \subseteq J \text{ and } J \text{ is a prime } \Gamma\text{-ideal of } M\}.$$

Then a function $f : \mathfrak{I}(M) \to \mathfrak{I}(M)$ given by $f(I) = \mathfrak{P}(I)$ for all $I \in \mathfrak{I}(M)$ is an intersection preserving Γ-ideal expansion of M.

Proof. Obviously, f is a Γ-ideal expansion of M. For every $I, J \in \mathfrak{I}(M)$, let

$$\mathfrak{P}_1 := \{P \mid I \cap J \subseteq P \text{ and } P \text{ is a prime } \Gamma\text{-ideal of } M\}$$

and

$$\mathfrak{P}_2 := \{P \mid I \subseteq P \text{ or } J \subseteq P, \quad P \text{ is a prime } \Gamma\text{-ideal of } M\}.$$

Then $\bigcap \mathfrak{P}_1 = \mathfrak{P}(I \cap J)$ and $\bigcap \mathfrak{P}_2 = \mathfrak{P}(I) \cap \mathfrak{P}(J)$. Obviously, $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$. If $P \in \mathfrak{P}_1$, then $I\,\Gamma J \subseteq I \cap J \subseteq P$ and so $I \subseteq P$ or $J \subseteq P$ because P is prime. Hence $P \in \mathfrak{P}_2$, and thus $\mathfrak{P}_1 = \mathfrak{P}_2$. Therefore

$$f(I \cap J) = \mathfrak{P}(I \cap J) = \bigcap \mathfrak{P}_1 = \bigcap \mathfrak{P}_2 = \mathfrak{P}(I) \cap \mathfrak{P}(J) = f(I) \cap f(J).$$

This completes the proof. □

Theorem 13. *Let f be a Γ-ideal expansion of M which is intersection preserving. If $I_1, I_2, \cdots, I_n$ are f-primary Γ-ideals of M and $J = f(I_k)$ for all $k = 1, 2, \cdots, n$, then $I := \bigcap_{k=1}^{n} I_k$ is an f-primary Γ-ideal of M.*

Proof. Obviously, $I := \bigcap_{k=1}^{n} I_k$ is a Γ-ideal of M. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in I$ and $a \notin I$. Then $a \notin I_k$ for some $k \in \{1, 2, \cdots, n\}$. But $a\gamma b \in I \subseteq I_k$ and I_k is f-primary, which imply that $b \in f(I_k)$. Since f is intersection preserving, we have

$$f(I) = f(\bigcap_{k=1}^{n} I_k) = \bigcap_{k=1}^{n} f(I_k) = J = f(I_k),$$

and so $b \in f(I)$. Therefore I is an f-primary Γ-ideal of M. □

Let $\sigma : M \to M'$ be a Γ-ring homomorphism of Γ-rings. Note that if J is a Γ-ideal of M', then $\sigma^{-1}(J)$ is a Γ-ideal of M, and that if σ is surjective and I is a Γ-ideal of M, then $\sigma(I)$ is a Γ-ideal of M'.

Theorem 14. *Let f be a Γ-ideal expansion which is global and let $\sigma : M \to M'$ be a Γ-ring homomorphism of Γ-rings. If J is an f-primary Γ-ideal of M', then $\sigma^{-1}(J)$ is an f-primary Γ-ideal of M.*

Proof. Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in \sigma^{-1}(J)$ and $a \notin \sigma^{-1}(J)$. Then $\sigma(a)\gamma\sigma(b) = \sigma(a\gamma b) \in J$ and $\sigma(a) \notin J$, which imply from (1) that $\sigma(b) \in f(J)$. Since f is global, it follows that $b \in \sigma^{-1}(f(J)) = f(\sigma^{-1}(J))$. Hence $\sigma^{-1}(J)$ is f-primary. □

It can be easily verified that if $\sigma : M \to M'$ is a Γ-ring homomorphism of Γ-rings, then $\sigma^{-1}(\sigma(I)) = I$ for any $I \in \mathfrak{I}(M)$ that contains $\ker(\sigma)$.

Theorem 15. *Let $\sigma : M \to M'$ be a surjective Γ-ring homomorphism of Γ-rings and let I be a Γ-ideal of M that contains $\ker(\sigma)$. Then I is f-primary if and only if $\sigma(I)$ is an f-primary Γ-ideal of M', where f is a global Γ-ideal expansion.*

Proof. If $\sigma(I)$ is an f-primary Γ-ideal of M', then I is f-primary by $I = \sigma^{-1}(\sigma(I))$ and Theorem 14. Suppose that I is f-primary. Let $x, y \in M'$ and $\gamma \in \Gamma$ be such that $x\gamma y \in \sigma(I)$ and $x \notin \sigma(I)$. Since σ is surjective, we have $\sigma(a) = x$ and $\sigma(b) = y$ for some $a, b \in M$. Then $\sigma(a\gamma b) = \sigma(a)\gamma\sigma(b) = x\gamma y \in \sigma(I)$ and $\sigma(a) = x \notin \sigma(I)$, which imply that $a\gamma b \in \sigma^{-1}(\sigma(I)) = I$ and $a \notin \sigma^{-1}(\sigma(I)) = I$. Since I is f-primary, it follows that $b \in f(I)$ so that $y = \sigma(b) \in \sigma(f(I))$. Using the fact that f is global, we have

$$f(I) = f(\sigma^{-1}(\sigma(I))) = \sigma^{-1}(f(\sigma(I))),$$

and so $\sigma(f(I)) = \sigma(\sigma^{-1}(f(\sigma(I)))) = f(\sigma(I))$ since σ is surjective. Therefore $\sigma(I)$ is f-primary. This completes the proof. □

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