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RADO AND POPOVICIU TYPE INEQUALITIES FOR PSEUDO ARITHMETIC AND GEOMETRIC MEANS

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Abstract: In this paper we prove Rado and Popoviciu type inequalities for pseudo arithmetic and geometric means a_n and g_n, defined by

$$a_n = \frac{P_n}{p_1} - \frac{1}{p_1}\sum_{i=2}^{n} p_i x_i \quad \text{and} \quad g_n = x_1^{P_n/p_1} \Big/ \prod_{i=2}^{n} x_i^{p_i/p_1},$$

where x_i and p_i $(i = 1, 2, \ldots, n)$ are positive real number and $P_n = \sum_{i=1}^{n} p_i$.

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1. Introduction

The classical inequality between the weighted arithmetic and geometric means

$$G_n = G_n(\mathbf{y};\mathbf{q}) = \prod_{i=1}^{n} y_i^{q_i/Q_n} \leq \frac{1}{Q_n}\sum_{i=1}^{n} q_i y_i = A_n(\mathbf{y};\mathbf{q}) = A_n \tag{1.1}$$

is valid for all positive real numbers y_i and q_i $(i = 1, 2, \ldots, n)$ with $Q_n = \sum_{i=1}^{n} q_i$. Equality holds in (1.1) if and only if $y_1 = y_2 = \cdots = y_n$.

For this inequality, which is probably the most important inequality, many

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proofs, extensions, refinements and variants are known (see [3], [5], [7], [11]).

In this paper we denote by $a_n(\mathbf{x;p})$ and $g_n(\mathbf{x;p})$ the following expressions which are closely connected to A_n and G_n. For positive real numbers x_i and $p_i (i = 1, 2, \ldots, n)$ with $P_n = \sum_{i=1}^{n} p_i$, we define

$$a_n = a_n(\mathbf{x;p}) = \frac{P_n}{p_1} x_1 - \frac{1}{p_1} \sum_{i=2}^{n} p_i x_i \tag{1.2}$$

and

$$g_n = g_n(\mathbf{x;p}) = x_1^{P_n/p_1} / \prod_{i=2}^{n} x_i^{p_i/P_n} . \tag{1.3}$$

Although there is no general agreement in literature what constitutes a means value, most authors consider the intermediate property as the main feature. Since a_n and g_n do not satisfy this condition, this means the double inequalities

$$\min_{1\le i\le n} x_i \le a_n \le \max_{1\le i\le n} x_i \text{ and } \min_{1\le i\le n} x_i \le g_n \le \max_{1\le i\le n} x_i$$

are not true for all positive x_i $(i = 1, 2, \ldots, n)$, we call a_n and g_n pseudo arithmetic and geometric means (see [2]).

In 1990, H. Alzer [2] published the following companion of inequality (1.1):

$$a_n = a_n(\mathbf{x;p}) \le g_n(\mathbf{x;p}) = g_n \tag{1.4}$$

with equality holding only if $x_1 = x_2 = \cdots = x_n$. For the special case $p_1 = p_2 = \cdots = p_n$ the inequality (1.4) was proved by S. Iwamoto, R.J. Tomkins and C.L. Wang [8].

We note that inequality (1.4) is an example of so-called reverse inequality. One of the first reverse inequalities was published in 1956 by J. Aczél [1], who proved the following intriguing variant of the Cuchy-Schwarz inequality:

If x_i and $y_i (i = 1, 2, \ldots, n)$ are real numbers with $x_1^2 > \sum_{i=2}^{n} x_i^2$ and $y_1^2 > \sum_{i=2}^{n} y_i^2$, then

$$(x_1 y_1 - \sum_{i=2}^{n} x_i y_i)^2 \ge (x_1^2 - \sum_{i=2}^{n} x_i^2)(y_1^2 - \sum_{i=2}^{n} y_i^2) . \tag{1.5}$$

Equality holds in (1.5) if and only if $x_i = c y_i$ $(i = 1, 2, \ldots, n)$ where c is a constant.

Further interesting reverse inequalities were given in [4], [6], [8], [9], [11], [12], [13].

The aim of this paper is to prove Rado and Popoviciu type inequalities for the pseudo arithmetic and geometric means, which have different weighted.

2. Inequalities Involving a_n(x;q) and g_n(x,p)

Two well-known extensions of the arithmetic mean-geometric mean inequality (1.1) are the following inequalities of Rado and Popoviciu:

$$Q_n(A_n(\mathbf{y;q}) - G_n(\mathbf{y,q})) \geq Q_{n-1}(A_{n-1}(\mathbf{y;q}) - G_{n-1}(\mathbf{y,q})) \tag{2.1}$$

and

$$\Big(A_n(\mathbf{y;q})/G_n(\mathbf{y;q})\Big)^{Q_n} \geq \Big(A_{n-1}(\mathbf{y;q})/G_{n-1}(\mathbf{y;q})\Big)^{Q_{n-1}}. \tag{2.2}$$

Equality holds in (2.1) if and only if $y_n = G_{n-1}$ and in (2.2) if and only if $y_n = A_{n-1}$; see [5], [11].

The next propositions provide analogues of (2.1) and (2.2) for pseudo arithmetic and geometric means [2].

Proposition 1. *For all positive real numbers x_i $(i = 1, 2, \dots, n; n \geq 2)$ we have*

$$g_n(\mathbf{x,p}) - a_n(\mathbf{x,p}) \geq g_{n-1}(\mathbf{x,p}) - a_{n-1}(\mathbf{x;p}), \tag{2.3}$$

with equality holding if and only if $x_1 = x_n$.

Proposition 2. *Let x_i $(i = 1, 2, \dots, n; n \geq 2)$ be positive real numbers such that $a_n(\mathbf{x;p}) > 0$ and $a_{n-1}(\mathbf{x;p}) > 0$. Then we have*

$$g_n(\mathbf{x;p})/a_n(\mathbf{x,p}) \geq g_{n-1}(\mathbf{x;p})/a_{n-1}(\mathbf{x;p}), \tag{2.4}$$

with equality holding if and only if $x_1 = x_n$.

The most obvious extension is to allow the means in the Rado and Popoviciu inequalities to have different weights [5]:

$$Q_nA_n(\mathbf{y;q}) - \frac{q_n}{p_n}P_nG_n(\mathbf{y;p}) \geq Q_{n-1}A_{n-1}(\mathbf{y;q}) - \frac{q_n}{p_n}P_{n-1}G_{n-1},(\mathbf{y;p}), \tag{2.5}$$

$$(A_n(\mathbf{y;q}))^{Q_n/q_n}/(G_n(\mathbf{y;p}))^{P_n/p_n} \geq (A_{n-1}(\mathbf{y;q}))^{Q_{n-1}/q_n}/(G_{n-1}(\mathbf{y;p}))^{P_{n-1}/p_n}. \tag{2.6}$$

Using this inequalities we obtain generalization of the inequalities (2.3) and (2.4) (see also [10]).

Theorem 3. *For all positive real numbers x_i $(i = 1, 2, \dots, n; n \geq 2)$ we have*

$$g_n(\mathbf{x};\mathbf{p}) - a_n(\mathbf{x},\mathbf{q}) \geq g_{n-1}(\mathbf{x},\mathbf{p}) - a_{n-1}(\mathbf{x};\mathbf{q}). \tag{2.7}$$

Proof. If we put in (2.5) $y_1 = g_n(\mathbf{x};\mathbf{p})$, $y_i = x_i$ $(i = 2, \dots, n)$ then we obtain

$$Q_n A_n(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n} P_n G_n(\mathbf{y};\mathbf{p}) = q_1(g_n(\mathbf{x};\mathbf{p}) - a_n(\mathbf{x};\mathbf{q})) + x_1(Q_n - \frac{q_n}{p_n} P_n)$$

and

$$Q_{n-1} A_{n-1}(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n} P_{n-1} G_{n-1}(\mathbf{y};\mathbf{p}) \\ = q_1(g_{n-1}(\mathbf{x};\mathbf{p}) - a_{n-1}(\mathbf{x};\mathbf{q})) + x_1(Q_{n-1} - \frac{q_n}{p_n} P_{n-1})$$

which leads to inequality (2.7) because

$$Q_n - \frac{q_n}{p_n} P_n = Q_{n-1} - \frac{q_n}{p_n} P_{n-1}. \qquad \square$$

Theorem 4. *Let x_i $(i = 1, 2, \dots, n; n \geq 2)$ be positive real numbers such that $a_n(\mathbf{x};\mathbf{q}) > 0$ and $a_{n-1}(\mathbf{x};\mathbf{q}) > 0$. Then we have*

$$g_n(\mathbf{x};\mathbf{p})/a_n(\mathbf{x};\mathbf{q}) \geq g_{n-1}(\mathbf{x};\mathbf{p})/a_{n-1}(\mathbf{x},\mathbf{q}). \tag{2.8}$$

Proof. If we set in (2.6) $y_1 = a_n(\mathbf{x},\mathbf{p}), y_i = x_i (i = 2, \dots, n)$ then we have

$$(A_n(\mathbf{y},\mathbf{q}))^{Q_n/q_n} / (G_n(\mathbf{y},\mathbf{p}))^{P_n/p_n} = (g_n(\mathbf{x};\mathbf{p})/a_n(\mathbf{x};\mathbf{q}))^{p_1/p_n} x_1^{Q_n/q_n - P_n/p_n}$$

and

$$(A_{n-1}(\mathbf{y};\mathbf{q}))^{Q_{n-1}/q_n} / (G_{n-1}(\mathbf{y};\mathbf{p}))^{P_{n-1}/p_n} \\ = (g_{n-1}(\mathbf{x};\mathbf{p})/a_{n-1}(\mathbf{x};\mathbf{q}))^{p_1/p_n} x_1^{Q_{n-1}/q_n - P_{n-1}/p_n},$$

which leads to inequality (2.8) because

$$Q_n/q_n - P_n/p_n = Q_{n-1}/q_n - P_{n-1}/p_n. \qquad \square$$

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