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TOTALLY UMBILICAL LIGHTLIKE HYPERSURFACES IN SEMI-RIEMANNIAN MANIFOLD WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract: In this paper, we study totally umbilical lightlike hypersurfaces of a semi-Riemannian manifold admitting semi-symmetric metric connection. We give the equations of Gauss and Codazzi. Then we obtain some results on Ricci tensor of a totally umbilical lightlike hypersurface with respect to semi-symmetric connection to be symmetric.

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1. Introduction

Hayden [6] introduced a semi-symmetric metric connection on a Riemannian manifold in 1932. In 1972, Imai [8] gave basic properties of a hypersurface of a Riemannian manifold with semi-symmetric metric connection and got confor-

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mal equations of Gauss and Codazzi.

Duggal and Sharma [3] studied semi-symmetric connections in a semi-Riemannian manifold in 1986. In this work, they showed that there exists an interplay between the Riemannian and semi-Riemannian geometry with respect to semi-symmetric metric connection.

The geometry of totally umbilical lightlike submanifolds have been studied by Ioan [9], Duggal and Jın [4] on semi-Riemannian manifold. In [4] the authors obtained some results for totally umbilical lightlike submanifolds and their structure equations. Besides, they got conditions for the induced Ricci curvature tensor of this submanifold to be symmetric.

The aim of this paper is to study totally umbilical lightlike hypersurface of a semi-Riemannian manifold admitting the semi-symmetric metric connection. It is well known that lightlike hypersurfaces are of metrics with vanishing determinants and this degeneracy of these metrics leads to several difficulties: Firstly, the contravariant metric cannot immediately be defined, so the connection cannot be specified uniquely in the normal way. Secondly, the normal is a lightlike vector lying in the tangent plane, which makes it necessary to look for some other vector to rig the hypersurface, and makes it impossible to normalise the normal in the usual way. Since these objects are considered, to study the general theory of lightlike hypersurfaces is very important topic. Several papers have been written on lightlike hypersurfaces in recent years [1], [2], [5], [10].

In the present paper, we have proved that on totally umbilical lightlike hypersurfaces the connection induced from semi-symmetric metric connection is semi-symmetric but not metric, and also on the screen distribution the connection induced from that connection is metric. For a totally umbilical lightlike hypersurface and screen distribution we define the induced geometrical objects with respect to semi-symmetric connection such as second fundamental form, shape operator, etc,... Then we give equations of Gauss and Codazzi. Finally, we obtain some results on Ricci tensor of a totally umbilical lightlike hypersurface with respect to semi-symmetric connection to be symmetric.

2. Preliminaries

Let M be a hypersurface of a $(n+1)-$dimensional, $n>1$, semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\tilde{g}$ of index $1 \leq \nu \leq n$. We consider

$$T_xM^{\perp} = \left\{ Y_x \in T_x\widetilde{M} \mid \tilde{g}_x\left(Y_x, X_x\right) = 0, \forall X_x \in T_xM \right\}$$

for any $x \in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface* of $\widetilde{M}$ or equivalently, the immersion

$$i : M \to \widetilde{M}$$

of M in $\widetilde{M}$ is *lightlike (null, degenerate)* if $T_xM \cap T_xM^{\perp} \neq \{0\}$ at any $x \in M$. Henceforth we identify $i\,(M)$ with M and we denote the differentional di, immersing a vector field X in M to a vector field ϕX in $\widetilde{M}$, by ϕ. Thus the induced metric tensor $g = \tilde{g}_{|M}$ is defined by

$$g\,(X, Y) = \tilde{g}(\phi X, \phi Y), \forall X, Y \in \Gamma\,(TM)\,.$$

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM called the *screen distribution* on M and denoted $S\,(TM)$. We have the following splitting into orthogonal direct sum:

$$TM = S\,(TM) \perp TM^{\perp}. \tag{2.1}$$

The subbundle $S\,(TM)$ is non-degenerate, so is $S\,(TM)^{\perp}$, and the following holds:

$$T\widetilde{M} = S\,(TM) \perp S\,(TM)^{\perp}\,, \tag{2.2}$$

where $S\,(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S\,(TM)$ in $T\widetilde{M}\Big|_{M}$.

Let $tr(TM)$ denote the complementary vector bundle of $TM^{\perp}$ in $S\,(TM)^{\perp}$. Then we have

$$S\,(TM)^{\perp} = TM^{\perp} \oplus tr(TM). \tag{2.3}$$

Let U be a coordinate neighborhood in M and ξ be a basis of $\Gamma(TM^{\perp}|_{U})$. Then there exists a basis N of $tr\,(TM)|_{U}$ satisfying the following conditions:

$$\tilde{g}(N, \xi) = 1,$$

and

$$\tilde{g}\,(N, N) = \tilde{g}(W, \xi) = 0, \quad \forall W \in \Gamma(S\,(TM)|_{U}).$$

The subbundle $tr\,(TM)$ is called a *lightlike transversal vector bundle* of M. We note that $tr\,(TM)$ is never orthogonal to TM. From (2.1), (2.2) and (2.3) we have

$$T\widetilde{M}\Big|_{M} = S\,(TM) \perp \Big(TM^{\perp} \oplus tr\,(TM)\Big) = TM \oplus tr\,(TM) \;\; [2]. \tag{2.4}$$

3. Semi-Symmetric Metric Connection

Let $\widetilde{M}$ be an $(n+1)$ −dimensional, $n > 1$, differentiable manifold of class C^∞ and $\tilde{\nabla}$ a lineer connection in $\widetilde{M}$. Then the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is given by

$$\tilde{T}(\tilde{X},\tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X},\tilde{Y}], \quad \forall \tilde{X},\tilde{Y} \in \Gamma(T\widetilde{M})$$

and is of type $(1,2)$. When the torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{X},\tilde{Y}) = \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{\pi}(\tilde{X})\tilde{Y}$$

for a 1−form $\tilde{\pi}$, the connection $\tilde{\nabla}$ is said to be *semi-symmetric*, see [13].

Let there be given a semi-Riemannian metric $\tilde{g}$ of index ν with $1 \leq \nu \leq n$ in $\widetilde{M}$ and $\tilde{\nabla}$ satisfy

$$\tilde{\nabla}\tilde{g} = 0\,,$$

then such a lineer connection is called a *metric connection*, see [12].

We now suppose that the semi-Riemannian manifold $\widetilde{M}$ admits a semi-symmetric metric connection given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{g}(\tilde{X},\tilde{Y})\tilde{Q} \tag{3.1}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\tilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\tilde{g}$, $\tilde{\pi}$ is a 1−form and $\tilde{Q}$ is the vector field defined by

$$\tilde{g}(\tilde{Q},\tilde{X}) = \tilde{\pi}(\tilde{X})$$

for an arbitrary vector field $\tilde{X}$ of $\widetilde{M}$. From (2.4), we can write

$$\tilde{Q} = \phi Q + \mu N\,, \tag{3.2}$$

where Q is a vector field and μ is a function in M.

Denoting by $\overset{\circ}{\nabla}$ the symmetric linear connection induced on the lightlike hypersurface from $\overset{\circ}{\tilde{\nabla}}$ Levi-Civita connection, we have

$$\overset{\circ}{\tilde{\nabla}}_{\phi X}\phi Y = \phi(\overset{\circ}{\nabla}_X Y) + B\,(X,Y)\,N \tag{3.3}$$

for arbitrary vector fields X and Y of M, where B is the local second fundamental form of M. Denoting by ∇ the connection induced on the lightlike hypersurface from $\tilde{\nabla}$ the semi-symmetric metric connection, we have

$$\tilde{\nabla}_{\phi X}\phi Y = \phi\,(\nabla_X Y) + m\,(X,Y)\,N \tag{3.4}$$

for arbitrary vector fields X and Y of M, where m is a tensor of type $(0,2)$ of the lightlike hypersurface of M and we call (3.4) the *equations of Gauss* with respect to the induced connection ∇.

From (3.1), we obtain

$$\tilde{\nabla}_{\phi X}\phi Y = \mathring{\tilde{\nabla}}_{\phi X}\phi Y + \tilde{\pi}(\phi Y)\phi X - \tilde{g}(\phi X, \phi Y)\tilde{Q},$$

and hence, using (3.3) and (3.4), we can also write

$$\phi\left(\nabla_X Y\right) + m\left(X,Y\right)N = \phi(\mathring{\nabla}_X Y) + B\left(X,Y\right)N + \tilde{\pi}(\phi Y)\phi X - \tilde{g}(\phi X, \phi Y)\tilde{Q}. \quad (3.5)$$

Substituting (3.2) into (3.5), we get

$$\phi\left(\nabla_X Y\right) + m\left(X,Y\right)N = \phi(\mathring{\nabla}_X Y + \pi\left(Y\right)X - g\left(X,Y\right)Q) + \{B\left(X,Y\right) - \mu g\left(X,Y\right)\}N,$$

from this equation, we have

$$\nabla_X Y = \mathring{\nabla}_X Y + \pi\left(Y\right)X - g\left(X,Y\right)Q, \quad (3.6)$$

where $\pi\left(X\right) = \tilde{\pi}(\phi X)$ and

$$m\left(X,Y\right) = B\left(X,Y\right) - \mu g\left(X,Y\right). \quad (3.7)$$

Definition 1. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n+1)-$dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ admitting a semi-symmetric metric connection. Then we call M a totally umbilical lightlike hypersurface if

$$m(X,Y) = \rho\widetilde{g}(X,Y), \quad (3.8)$$

for any $X, Y \in \Gamma(TM)$, where m is the second fundamental form with respect to a semi-symmetric connection of M and $\rho \in \mathbb{R}$.

From (3.6), (3.7) and (3.8), we have

$$\left(\nabla_X g\right)(Y,Z) = (\rho + \mu)\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}, \quad (3.9)$$

where

$$\eta\left(Z\right) = g(Z,N).$$

From (3.6), we also have

$$T(X,Y) = \pi\left(Y\right)X - \pi\left(X\right)Y. \quad (3.10)$$

From (3.9) and (3.10), we have the following proposition.

Proposition 1. *The connection induced on a totally umbilical lightlike hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection is semi-symmetric, but not a metric connection.*

Corollary 1. *The connection induced on a totally umbilical lightlike hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection is semi-symmetric metric if and only if*

$$\rho = -\mu.$$

The equation of Weingarten with respect to the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ is

$$\overset{\circ}{\tilde{\nabla}}_{\phi X} N = -\phi(A_N X) + \tau(X) N \tag{3.11}$$

for any vector field X in M, where A_N is the shape operator of M and τ is the $1-$form [2].

On the other hand, we denote P as the projection of TM on $S(TM)$ with respect to the decomposition (2.1) and using (3.1), we get

$$\tilde{\nabla}_{\phi X} N = \overset{\circ}{\tilde{\nabla}}_{\phi X} N + \lambda \phi X - \lambda' \phi Q - \mu \lambda' N,$$

because of

$$\tilde{\pi}(N) = \lambda \text{ and } \tilde{g}(\phi X, N) = \lambda'.$$

Thus using (3.10), we find

$$\tilde{\nabla}_{\phi X} N = \phi((-A_N + \lambda I) X - \lambda' Q) + (\tau(X) - \mu \lambda') N, \tag{3.12}$$

where I is the unit tensor. (3.11) is called *the equation of Weingarten* with respect to the semi-symmetric connection.

Proposition 2. *Let $S(TM)$ and $S(TM)'$ be two screen distributions on M and m and m' be the second fundamental forms with respect to $tr(TM)$ and $tr(TM)'$, respectively. Then the local second fundamental form of M on U is independent of the choice of screen distribution.*

Proof. The proof follows from (3.4) for both screen distributions $S(TM)$ and $S(TM)'$. In fact, we have

$$\begin{aligned} m(X,Y) = \tilde{g}(\tilde{\nabla}_{\phi X} \phi Y - \phi(\nabla_X Y), \xi) &= \tilde{g}(\tilde{\nabla}_{\phi X} \phi Y, \xi) - g(\phi(\nabla_X Y), \xi) \\ &= \tilde{g}(\tilde{\nabla}_{\phi X} \phi Y, \xi) = m'(X,Y) \end{aligned}$$

for $\forall X,\, Y \in \Gamma(TM|_U)$. □

Thus we have the following result.

Corollary 2. *The second fundamental form of a totally umbilical lightlike hypersurface with a semi-symmetric connection is degenerate.*

Proof. Taking into account that $\tilde{\nabla}$ is a semi-symmetric metric connection and (3.8), from Proposition 3.2 it follows that

$$m(X,\xi) = \rho g(X,\xi) = 0, \quad \forall X \in \Gamma(TM)$$

which proves the assertion. □

Since we denote P as the projection of TM on $S(TM)$ with respect to the decomposition (2.1), we obtain

$$\overset{\circ}{\nabla}_{\phi X} P\phi Y = \phi(\overset{\circ}{\overset{*}{\nabla}}_X PY) + C(X,PY)\,\xi \tag{3.13}$$

and

$$\nabla_{\phi X} P\phi Y = \phi(\overset{*}{\nabla}_X PY) + D(X,PY)\,\xi\,, \tag{3.14}$$

where $\phi(\overset{\circ}{\overset{*}{\nabla}}_X PY)$ and $\phi(\overset{*}{\nabla}_X PY)$ belong to $S(TM)$, and C, D are $1-$forms on M. From (3.6) , we obtain

$$\nabla_{\phi X} P\phi Y = \overset{\circ}{\nabla}_{\phi X} P\phi Y + \pi(P\phi Y)\phi X - g(\phi X, \phi PY)\phi Q\,,$$

where ϕQ is a vector field defined by

$$\phi Q = PQ + \lambda\xi.$$

Thus, using (3.13) and (3.14), we have

$$\begin{aligned}\phi(\overset{*}{\nabla}_X PY) + D(X,PY)\,\xi = {}& \phi(\overset{\circ}{\overset{*}{\nabla}}_X PY) + C(X,PY)\,\xi \\ & + \pi(P\phi Y)\phi X - g(\phi X, \phi PY)(\phi PQ + \lambda\xi)\,,\end{aligned}$$

from this equation, we get

$$D(X,PY) = C(X,PY) - \lambda g(X,PY) \tag{3.15}$$

and

$$\overset{*}{\nabla}_X PY = \overset{\circ}{\overset{*}{\nabla}}_X PY + \pi(PY)\,X - g(X,PY)\,PQ\,, \tag{3.16}$$

where $\pi(X) = \tilde{\pi}(\phi X)$.

By using (3.16), we find

$$\overset{*}{\nabla}_{\phi X}(g(\phi PY,\phi PZ)) = (\overset{*}{\nabla}_{\phi X}g)(P\phi Y,P\phi Z) + \overset{\circ}{\overset{*}{\nabla}}_{\phi X}(g(P\phi Y,P\phi Z)) - (\overset{\circ}{\overset{*}{\nabla}}_{\phi X}g)(P\phi Y,P\phi Z)$$

from this, we obtain

$$(\overset{*}{\nabla}_{\phi X}g)(P\phi Y,P\phi Z) = 0. \tag{3.17}$$

We also have from (3.16),

$$\overset{*}{T}(\phi X,\phi Y) = \pi\left(PY\right)X - \pi\left(PX\right)Y. \tag{3.18}$$

From (3.17) and (3.18), we have the following proposition.

Proposition 3. *The connection $\overset{*}{\nabla}$ induced on a screen distribution of totally umbilical lightlike hypersurface is a semi-symmetric metric connection.*

4. Equations of Gauss and Codazzi

We denote by

$$\overset{\circ}{\tilde{K}}(\tilde{X},\tilde{Y})\tilde{Z} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\overset{\circ}{\tilde{\nabla}}_{\tilde{Y}}\tilde{Z} - \overset{\circ}{\tilde{\nabla}}_{\tilde{Y}}\overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\tilde{Z} - \overset{\circ}{\tilde{\nabla}}_{[\tilde{X},\tilde{Y}]}\tilde{Z}$$

the curvature tensor of $\widetilde{M}$ with respect to Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ and by

$$\overset{\circ}{K}\left(X,Y\right)Z = \overset{\circ}{\nabla}_X\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y\overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]}Z$$

that of M with respect to induced connection $\overset{\circ}{\nabla}$. Then the *Gauss-Codazzi equations of lightlike hypersurface* are given by

$$\widetilde{g}(\overset{\circ}{\tilde{K}}(\phi X,\phi Y,\phi Z),P\phi W) = \phi(g(\overset{\circ}{K}\left(X,Y\right)Z,PW)) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW)\,, \tag{4.1}$$

$$\widetilde{g}(\overset{\circ}{\tilde{\tilde{K}}}(\phi X,\phi Y,\phi Z),\xi)=(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z) + B(Y,Z)\tau(X)-B(X,Z)\tau(Y)\,, \quad (4.2)$$

$$\widetilde{g}(\overset{\circ}{\tilde{\tilde{K}}}(\phi X,\phi Y,\phi Z),N)=\widetilde{g}(\overset{\circ}{K}(X,Y)\,Z,N) \quad (4.3)$$

for any $X,\,Y,\,Z,\,W\in\Gamma(TM)$ [2], where

$$\overset{\circ}{\tilde{K}}(\phi X,\phi Y,\phi Z,\phi PW)=\tilde{g}(\tilde{K}(\phi X,\phi Y)\phi Z,\phi PW)$$

and

$$K(X,Y,Z,PW)=g(K\,(X,Y)\,Z,PW).$$

Now, we shall find the equations of Gauss and Codazzi of totally umbilical lightlike hypersurface with a semi-symmetric connection. The curvature tensor of $\widetilde{M}$ with respect to a semi-symmetric metric connection $\tilde{\nabla}$ is

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}=\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}-\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z}-\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}.$$

Putting $\tilde{X}=\phi X$, $\tilde{Y}=\phi Y$, $\tilde{Z}=\phi Z$, we get

$$\tilde{R}(\phi X,\phi Y)\phi Z=\tilde{\nabla}_{\phi X}\tilde{\nabla}_{\phi Y}\phi Z-\tilde{\nabla}_{\phi Y}\tilde{\nabla}_{\phi X}\phi Z-\tilde{\nabla}_{\phi[X,Y]}\phi Z.$$

Thus, using (3.4), (3.8), (3.10) and (3.12), we have

$$\begin{aligned}\tilde{R}(\phi X,\phi Y)\phi Z&=\phi\left(R\left(X,Y\right)Z\right)+\rho g\left(Y,Z\right)\left(-A_N+\lambda I\right)\phi X-\lambda^{'}\rho g\left(Y,Z\right)\phi Q\\&\quad-\rho g\left(X,Z\right)\left(-A_N+\lambda I\right)\phi Y+\lambda^{'}\rho g\left(X,Z\right)\phi Q\\&\quad+\rho\left\{g\left(\pi\left(Y\right)X-\pi\left(X\right)Y,Z\right)+\left(\nabla_X g\right)\left(Y,Z\right)-\left(\nabla_Y g\right)\left(X,Z\right)\right.\\&\quad\left.+g\left(Y,Z\right)\left(\tau\left(X\right)-\mu\lambda^{'}\right)-g\left(X,Z\right)\left(\tau\left(Y\right)-\mu\lambda^{'}\right)\right\}N\,, \quad (4.4)\end{aligned}$$

where $R\,(X,Y)\,Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$ is the curvature tensor of a totally umbilical lightlike hypersurface with a semi-symmetric connection ∇.

Putting now

$$\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{\xi})=\tilde{g}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{\xi}),\quad R(X,Y,Z,\xi)=g(R\,(X,Y)\,Z,\xi)$$

we obtain from (4.4)

$$\begin{aligned}&\tilde{R}(\phi X,\phi Y,\phi Z,\phi PW)\\&\quad=\phi(g(R\,(X,Y)\,Z),PW)+\rho g\,(Y,Z)\,(-A_N+\lambda I)\,\tilde{g}(X,PW)\end{aligned}$$

$$- \rho g\,(X,Z)\,(-A_N+\lambda I)\,g(Y,PW)+\lambda^{'}\rho g\,(X,Z)\,g(Q,PW) \\ - \lambda^{'}\rho g\,(Y,Z)\,g(Q,PW), \quad (4.5)$$

$$\tilde{R}(\phi X,\phi Y,\phi Z,\phi\xi) \\ = \rho g\,(\pi\,(Y)\,X-\pi\,(X)\,Y,Z)+\rho(\nabla_X g)\,(Y,Z)-\rho(\nabla_Y g)\,(X,Z) \\ + \rho g\,(Y,Z)\,(\tau\,(X)-\mu\lambda^{'})-\rho g\,(X,Z)\,(\tau\,(Y)-\mu\lambda^{'}), \quad (4.6)$$

$$\tilde{R}(\phi X,\phi Y,\phi Z,N) \\ = g(\phi\,(R\,(X,Y)\,Z)\,,N)+\rho g\,(Y,Z)\,(-A_N+\lambda I)\,g(X,N) \\ - \rho g\,(X,Z)\,(-A_N+\lambda I)\,g(Y,N)+\lambda^{'}\rho g\,(X,Z)\,g(Q,N) \\ - \lambda^{'}\rho g\,(Y,Z)\,g(Q,N). \quad (4.7)$$

We call equations (4.5)-(4.7) the *Gauss-Codazzi equations* of the totally umbilical lightlike hypersurface with a semi-symmetric connection.

5. The Ricci Tensor of Totally Umbilical Lightlike Hypersurface with Semi-Symmetric Connection

The Ricci tensor with respect to semi-symmetric connection of a totally umbilical lightlike hypersurface is defined by

$$Ric(\phi X,\phi Y)=\text{trace}\,\{\phi Z\longrightarrow R(\phi X,\phi Z)\phi Y\}\,,\forall\phi X,\phi Y,\phi Z\in\Gamma\,(TM)\,. \quad (5.1)$$

Locally, Ricci tensor with respect to semi-symmetric connection of a totally umbilical lightlike hypersurface is given by

$$Ric(\phi X,\phi Y)=\sum_{i=1}^{n-1}\varepsilon_i g(R(\phi X,\phi w_i)\phi Y,\phi w_i)+g(R(\phi X,\xi)\phi Y,N), \quad (5.2)$$

where $\{w_i,...,w_{n-1}\}$ is a local orthonormal frames field on M, see [2].

Thus, by using (3.8), (3.14), (4.4) and (5.2) we obtain

$$Ric(\phi X,\phi Y)-Ric(\phi Y,\phi X)=\{\rho+\mu\}g(X,PY)\eta(\phi Y) \\ - \{\rho+\mu\}g(Y,PX)\eta(\phi X)+2d\tau(\phi X,\phi Y), \quad (5.3)$$

for any $\phi X,\phi Y\in\Gamma\,(TM)$.

From (5.3), we have the following results.

Proposition 4. *Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then Ricci tensor with respect to semi-symmetric connection of a totally umbilical lightlike hypersurface is symmetric if and only if the $1-$ form τ is closed and*

$$\rho = -\mu.$$

Proposition 5. *Let M be a totally umbilical lightlike hypersurface of an $(n+1)-$dimensional semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric metric connection. Then we get*

$$\mathrm{Ric}(\phi X, \phi Y) = (n-1)\, cg(X,Y) + (n-2)\, \varepsilon_i \rho g\,(X,Y)\,(-A_N + \lambda I). \tag{5.4}$$

Proof. By definition of totally umbilical lightlike hypersurface, we have

$$\mathrm{Ric}(\phi X, \phi Y) = \sum_{i=1}^{n-1} \varepsilon_i g(R(\phi X, \phi w_i)\phi Y, \phi w_i) + g(R(\phi X, \xi)\phi Y, N)\,.$$

Thus, from (3.8), (5.2) and (4.4) we obtain (5.4). □

Corollary 3. *Let M be a totally umbilical lightlike hypersurface of an $(n+1)-$dimensional semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric metric connection. Then the Ricci tensor of a totally umbilical lightlike hypersurface with a semi-symmetric connection is symmetric.*

Proposition 6. *Let M be a totally umbilical lightlike hypersurface of an $(n+1)-$dimensional semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric metric connection. If the induced semi-symmetric connection on M is metric, then the Ricci tensor of M is parallel $(n \geq 3)$.*

Proof. At first, we compute the derivative of Ricci tensor. Then, we get

$$\begin{aligned}&(\nabla_{\phi Z} Ric)(\phi X, \phi Y)\\ &\qquad = \nabla_{\phi Z} Ric(\phi X, \phi Y) - Ric(\nabla_{\phi Z}\phi X, \phi Z) - Ric(\phi X, \nabla_{\phi Z}\phi Y)\end{aligned}$$

Hence, from (3.7), (3.8) and (5.4) we have

$$\begin{aligned}&(\nabla_{\phi Z} Ric)(\phi X, \phi Y) = (n-1)\, c\{\rho + \mu\} g(X,Z)\eta(Y)\\ &\quad + (n-1)\, c\{\rho + \mu\} g(Z,Y)\eta(X) + (n-2)\, \rho \varepsilon_i (\nabla_Z g)\,(X,Y)\,(-A_N + \lambda I)\,,\end{aligned}$$

which proves the assertion of proposition. □

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