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ON THE GRACEFULNESS OF THE DIGRAPHS $n \cdot \vec{C}_m$

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Abstract: A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of $D(V, E)$, while f' is called the induced edge's graceful labeling of D. In this paper we discuss the gracefulness of the digraph $n \cdot \vec{C}_m$ and prove that $n \cdot \vec{C}_m$ is a graceful digraph for $m = 9, 11, 13$ and even n.

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1. Introduction

A graph $G(V,E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u,v) = |f(u) - f(v)|$ for every edge (u,v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of G, while $f^{'}$ is called the induced edge's graceful labeling of G. A digraph $D(V,E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u,v) = [f(v) - f(u)]$ (mod $|E| + 1$) for every directed edge (u,v) is a bijection, where $[v]$ (mod u) denotes the least positive residue of v modulo n. In this case, f is called a graceful labeling (graceful numbering) of D and $f^{'}$ is called the induced edge's graceful labeling of D (see [3]).

Let C_m and $\vec{C}_m$ denote the cycle and directed cycle on m vertices, respectively, $n \cdot C_n(\, n - C_m)$ denote the graphs obtained from any n copies of C_m which have just one common vertex (edge). At the same time, let $n \cdot \vec{C}_m$ $(n - \vec{C}_m)$ denote the digraphs obtained from any n copies of the directed cycle $\vec{C}_m$ which have just one common vertex (edge).

As to the gracefulness of $n \cdot \vec{C}_m$ we know the following results: Ma proved in [3] that the gracefulness of $n \cdot \vec{C}_3$ implies that n is even, at same times he conjectured that the condition that n is even was also sufficient for $n \cdot \vec{C}_3$ to be graceful. In [5], the first author of this paper has showed this conjecture. It was showed that $n \cdot \vec{C}_{2k}$ is graceful for every integer $n \geq 1$ and $k \geq 1$ have made a conjecture that the $n \cdot \vec{C}_m (m = 2k+1)$ be a graceful for even n in [6], and $n \cdot \vec{C}_m$ is graceful for n even and $m = 5, 7$ in [7].

In this paper, we will further discuss the gracefulness of the digraph $n \cdot \vec{C}_m$ and prove the digraph $n \cdot \vec{C}_m$ is graceful if $m = 9, 11, 13$ and n is even.

2. Main Results

Let $\vec{C}^1_m, \vec{C}^2_m, \cdots, \vec{C}^n_m$ denote the n directed cycles in $n \cdot \vec{C}_m$. The common vertex of $\vec{C}^i_m$'s is denoted by v_0, and other $m - 1$ vertices of the $\vec{C}^i_m$ are denoted by v^i_j $(j = 1, 2, \ldots, m-1; i = 1, 2, \cdots, n)$, respectively. For convenience, we put $v^1_0 = v^2_0 = \cdots = v^n_0 = v_0$, and take subscripts j's modulo m. Obviously, $|E(n \cdot \vec{C}_m)| = mn$.

Suppose that $n \cdot \vec{C}_m$ is graceful and f and f' are its graceful labeling and the induced edge's graceful labeling, respectively.

Theorem 1. *For every even integer n, the digraph $n \cdot \vec{C}_9$ is graceful.*

Proof. We have had $f(v_0) = 0$ and for other vertices, define:

$$f(v_j^i) = \begin{cases} \frac{j+1}{2}n + 1 - i, & j = 1,3,5;\ 1 \le i \le n, \\ (j-1)n + 2 - i, & j = 7;\ 1 \le i \le n, \end{cases}$$

$$f(v_j^i) = \begin{cases} 7n + 1 + i, & j = 2;\ 1 \le i \le n, \\ 6n + \frac{n}{2} + 1 + i, & j = 4;\ 1 \le i \le \frac{n}{2}, \\ 8n + i, & j = 4;\ \frac{n}{2} + 1 \le i \le n, \\ 6n + 1 + i, & j = 6;\ 1 \le i \le \frac{n}{2}, \\ 3n + +1 + i, & j = 6;\ \frac{n}{2} + 1 \le i \le n, \\ 3n + i, & j = 8;\ 1 \le i \le \frac{n}{2}, \\ 4n + i, & j = 8;\ \frac{n}{2} + 1 \le i \le n. \end{cases}$$

Firstly, we show that f is an injective mapping from $V(n \cdot \vec{C}_9)$ into $\{0, 1, \cdots, 9n\}$.

Put $S_j = \{f(v_j^i) | 1 \le i \le n,\ 0 \le j \le 8\}$. Then:

$$\begin{aligned}
S_0 &= \{f(v^0)\} = \{0\}, \\
S_1 &= \{f(v_1^i) | 1 \le i \le n\} = \{n + 1 - i | 1 \le i \le n\} = \{1, 2, \cdots, n\}, \\
S_2 &= \{f(v_2^i) | 1 \le i \le n\} = \{7n + 1 + i, 1 \le i \le n\} \\
&= \{7n + 2, 7n + 3, \cdots, 8n + 1\}, \\
S_3 &= \{f(v_3^i) | 1 \le i \le n\} = \{2n + 1 - i, 1 \le i \le n\} \\
&= \{n + 1, n + 2, \cdots, 2n\}, \\
S_4 &= \{f(v_4^i) | 1 \le i \le n\} = \{6n + \tfrac{n}{2} + 1 + i, 1 \le i \le \tfrac{n}{2}\} \cup \{8n + i, \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{6n + \tfrac{n}{2} + 2, 6n + \tfrac{n}{2} + 3, \cdots, 7n + 1\} \cup \{8n + \tfrac{n}{2} + 1, 8n + \tfrac{n}{2} + 2, \cdots, 9n\}, \\
S_5 &= \{f(v_5^i) | 1 \le i \le n\} = \{3n + 1 - i, 1 \le i \le n\} \\
&= \{2n + 1, 2n + 2, \cdots, 3n\}, \\
S_6 &= \{f(v_6^i) | 1 \le i \le n\} = \{6n + 1 + i, 1 \le i \le \tfrac{n}{2}\} \cup \{3n + 1 + i, \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{6n + 2, 6n + 3, \cdots, 6n + \tfrac{n}{2} + 1\} \cup \{3n + \tfrac{n}{2} + 2, 3n + \tfrac{n}{2} + 3, \cdots, 4n + 1\}, \\
S_7 &= \{f(v_7^i) | 1 \le i \le n\} = \{6n + 2 - i, 1 \le i \le n\} \\
&= \{5n + 2, 5n + 3, \cdots, 6n + 1\}, \\
S_8 &= \{f(v_8^i) | 1 \le i \le n\} = \{3n + i, 1 \le i \le \tfrac{n}{2}\} \cup \{4n + i, \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{3n + 1, 3n + 2, \cdots, 3n + \tfrac{n}{2} + 1\} \cup \{4n + \tfrac{n}{2} + 1, 4n + \tfrac{n}{2} + 2, \cdots, 5n\}.
\end{aligned}$$

Hence, $S_i \cap S_j = \emptyset$ for $i \ne j, i, j \in \{0, 1, 2, \cdots, 8\}$, which yields that f is an injection from $V(n \cdot \vec{C}_9)$ into $\{0, 1, \cdots, 9n\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n \cdot \vec{C}_9)$ onto $\{1, 2, \cdots, 9n\}$.

For $j (0 \le j \le 8)$, set $B_{j,1} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{9n + 1} | 1 \le i \le \frac{n}{2}\}$, $B_{j,2} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{9n + 1} | \frac{n}{2} + 1 \le i \le n\}$, and $B_j = B_{j,1} \cup B_{j,2}$

and $B=\bigcup_{j=0}^{8} B_j$. Then:

$$
\begin{aligned}
B_0 &= \{[\,f(v_1^i)-f(v_0)|] \pmod{(\,9n+1\,)}|1\le i\le n\}=\{n+1-i|1\le i\le n\}\\
&=\{1,2,\cdots,n\},\\
B_1 &= \{[\,f(v_2^i)-f(v_1^i)] \pmod{(\,9n+1\,)}|1\le i\le n\}\\
&=\{7n+1+i-(n+1-i)=6n+2i|1\le i\le n\}=\{6n+2,6n+4,\cdots,\ 8n\},\\
B_2 &= \{[\,f(v_3^i)-f(v_2^i)] \pmod{(\,9n+1\,)}|1\le i\le n\}\\
&=\{2n+1-i-(7n+1+i)=4n+1-2i|1\le i\le n\}\\
&=\{2n+1,2n+3,\cdots,4n-1\},\\
B_{3,1} &= \{[\,f(x_4^i)-f(x_3^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\
&=\{6n+\tfrac{n}{2}+1+i-(2n+1-i)=4n+\tfrac{n}{2}+2i|1\le i\le \tfrac{n}{2}\}\\
&=\{4n+\tfrac{n}{2}+2,4n+\tfrac{n}{2}+4,\cdots,5n+\tfrac{n}{2}\},\\
B_{3,2} &= \{[\,f(x_4^i)-f(x_3^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\
&=\{8n+i-(2n+1-i)=6n-1+2i|\tfrac{n}{2}+1\le i\le n\}\\
&=\{7n+1,7n+3,\cdots,8n-1\},\\
B_{4,1} &= \{[\,f(x_5^i)-f(x_4^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\
&=\{3n+1-i-(6n+\tfrac{n}{2}+1+i)=6n-\tfrac{n}{2}+1-2i|1\le i\le \tfrac{n}{2}\}\\
&=\{6n-\tfrac{n}{2}-1,6n-\tfrac{n}{2}-3,\cdots,5n-\tfrac{n}{2}+1\},\\
B_{4,2} &= \{[\,f(x_5^i)-f(x_4^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\
&=\{3n+1-i-(8n+i)=4n+2-2i|\tfrac{n}{2}+1\le i\le n\}\\
&=\{3n,3n-2,\ \cdots,2n+2\},\\
B_{5,1} &= \{[\,f(x_6^i)-f(x_5^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\
&=\{6n+1+i-(3n+1-i)=3n+2i|1\le i\le \tfrac{n}{2}\}\\
&=\{3n+2,3n+4,\cdots,4n\},\\
B_{5,2} &= \{[\,f(x_6^i)-f(x_5^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\
&=\{3n+1+i-(3n+1-i)=2i|\tfrac{n}{2}+1\le i\le n\}=\{n+2,n+4,\ \cdots,2n\},\\
B_{6,1} &= \{[\,f(x_7^i)-f(x_6^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\
&=\{6n+2-i-(6n+1+i)=9n+2-2i|1\le i\le \tfrac{n}{2}\}\\
&=\{9n,9n-2,\cdots,8n+2\},\\
B_{6,2} &= \{[\,f(x_7^i)-f(x_6^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\
&=\{6n+2-i-(3n+1+i)=3n+1-2i|\tfrac{n}{2}+1\le i\le n\}\\
&=\{2n-1,2n-3,\ \cdots,n+1\},\\
B_{7,1} &= \{[\,f(x_8^i)-f(x_7^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\
&=\{3n+i-(6n+2-i)=6n-1+2i|1\le i\le \tfrac{n}{2}\}\\
&=\{6n+1,6n+3,\cdots,7n-1\},\\
B_{7,2} &= \{[\,f(x_8^i)-f(x_7^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\
&=\{4n+i-(6n+2-i)=7n-1+2i|\tfrac{n}{2}+1\le i\le n\}\\
&=\{8n+1,8n+3,\ \cdots,9n-1\},
\end{aligned}
$$

$$\begin{aligned} B_{8,1} &= \{[\,f(x_0)-f(x_8^i)] \pmod{(\,9n+1\,)}|1\le i\le \tfrac{n}{2}\}\\ &= \{0-(3n+i)=6n+1-i|1\le i\le \tfrac{n}{2}\}=\{6n,6n-1,\cdots,5n+\tfrac{n}{2}+1\},\\ B_{8,2} &= \{[\,f(x_0)-f(x_8^i)] \pmod{(\,9n+1\,)}|\tfrac{n}{2}+1\le i\le n\}\\ &= \{0-(4n+i)=5n+1-i|\tfrac{n}{2}+1\le i\le n\}\\ &= \{4n+\tfrac{n}{2},4n+\tfrac{n}{2}-1,\ \cdots,4n+1\}. \end{aligned}$$

Hence,

$$\begin{aligned} B &= \textstyle\bigcup_{j=0}^{8} B_j\\ &= B_0\cup B_{6,2}\cup B_{5,2}\cup B_2\cup B_{4,2}\cup B_{5,1}\cup B_{8,2}\cup B_{4,1}\cup B_{3,1}\\ &\cup B_{8,1}\cup B_{7,1}\cup B_1\cup B_{3,2}\cup B_{7,2}\cup B_{6,1}\\ &= \{1,2,\cdots,n\}\cup\{2n-1,2n-3,\ \cdots,n+1\}\cup\{n+2,n+4,\ \cdots,2n\}\\ &\quad\cup\{2n+1,2n+3,\cdots,4n-1\}\cup\{3n,3n-2,\ \cdots,2n+2\}\\ &\quad\cup\{3n+2,3n+4,\cdots,4n\}\cup\{4n+\tfrac{n}{2},4n+\tfrac{n}{2}-1,\ \cdots,4n+1\}\\ &\quad\cup\{6n-\tfrac{n}{2}-1,6n-\tfrac{n}{2}-3,5n-\tfrac{n}{2}+1\}\\ &\quad\cup\{4n+\tfrac{n}{2}+2,4n+\tfrac{n}{2}+4,\cdots,5n+\tfrac{n}{2}\}\\ &\quad\cup\{6n,6n-1,\cdots,5n+\tfrac{n}{2}+1\}\cup\{6n+1,6n+3,\cdots,7n-1\}\\ &\quad\cup\{6n+2,6n+4,\cdots,\ 8n\}\cup\{7n+1,7n+3,\cdots,8n-1\,\}\\ &\quad\cup\{8n+1,8n+3,\ \cdots,9n-1\}\cup\{9n,9n-2,\cdots,8n+2\}\\ &= \{1,2,\cdots,9n\}\,, \end{aligned}$$

which implies that f' is surjective, hence, bijective. So we prove that $n\cdot\vec{C}_9$ is a graceful digraph for even n. □

Theorem 2. *For every even integer n, the digraph $n\cdot\vec{C}_{11}$ is graceful.*

Proof. Define

$$f(v_0)=0,$$

and

$$f(v_j^i)=\begin{cases} \frac{j+9}{2}n+1-i, & j=1,3,5;\ 1\le i\le n,\\ 3n+2-i, & j=7;\ 1\le i\le n,\\ 4n+1-i, & j=9;\ 1\le i\le \frac{n}{2},\\ 11n+2-i, & j=9;\ \frac{n}{2}\le i\le n. \end{cases}$$

$$f(v_j^i)=\begin{cases} (6+\frac{j}{2})n+\frac{j-2}{2}+i, & j=2,4;\ 1\le i\le n,\\ n+i, & j=6;\ 1\le i\le n,\\ 9n+1+i, & j=8;\ 1\le i\le n,\\ i, & j=10,\ 1\le i\le n. \end{cases}$$

Firstly, we show that f is an injective mapping from $V(n\cdot\vec{C}_{11})$ into $\{0,1,$

$\cdots, 11n\}$. Set $S_j = \{f(v_j^i)|1 \le i \le n,\ 0 \le j \le 10\}$ and $S = \bigcup_{i=0}^{10} S_i$. Then:

$$\begin{aligned}
S_0 = &\ \{f(v_0)\} = \{0\}\\
S_1 = &\ \{f(v_1^i)|1 \le i \le n\} = \{5n+1-i|1 \le i \le n\} = \{5n, 5n-1, \cdots, 4n+1\},\\
S_2 = &\ \{f(v_2^i)|1 \le i \le n\} = \{7n+i|1 \le i \le n\} = \{7n+1, 7n+2, \cdots, 8n\},\\
S_3 = &\ \{f(v_3^i)|1 \le i \le n\} = \{6n+1-i|1 \le i \le n\} = \{6n, 6n+1, \cdots, 5n+1\},\\
S_4 = &\ \{f(v_4^i)|1 \le i \le n\} = \{8n+1+i|1 \le i \le n\} = \{8n+2, 8n+3, \cdots, 9n+1\},\\
S_5 = &\ \{f(v_5^i)|1 \le i \le n\} = \{7n+1-i|1 \le i \le n\} = \{7n, 7n-1, \cdots, 6n+1\},\\
S_6 = &\ \{f(v_6^i)|1 \le i \le n\} = \{n+i|1 \le i \le n\} = \{n+1, n+2, \cdots, 2n\},\\
S_7 = &\ \{f(v_7^i)|1 \le i \le n\} = \{3n+2-i|1 \le i \le n\} = \{3n+1, 3n, \cdots, 2n+2\},\\
S_8 = &\ \{f(v_8^i)|1 \le i \le n\} = \{9n+1+i|1 \le i \le n\} = \{9n+2, 9n+3, \cdots, 10n+1\},\\
S_9 = &\ \{f(v_9^i)|1 \le i \le n\} = \{4n+1-i|1 \le i \le \tfrac{n}{2}\} \cup \{11n+2-i|\tfrac{n}{2} \le i \le n\}\\
= &\ \{4n, 4n-1, \cdots, 3n+\tfrac{n}{2}+1\} \cup \{11n-\tfrac{n}{2}+1, 11n-\tfrac{n}{2}, \cdots, 10n+2\},\\
S_{10} = &\ \{f(v_{10}^i)|1 \le i \le n\} = \{i,\ 1 \le i \le n\} = \{1, 2, \cdots, n\}.
\end{aligned}$$

Hence, $S_i \cap S_j = \emptyset$ for $i \ne j, i, j \in \{0, 1, 2, \cdots, 10\}$, which yields that f is an injection from $V(n \cdot \vec{C_{11}})$ into $\{0, 1, \cdots, 11n\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n \cdot \vec{C}_{11})$ onto $\{1, 2, \cdots, 11n\}$.

For $j(0 \le j \le 10)$, set $B_{j,1} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{11n+1}|1 \le i \le \frac{n}{2}\}$, $B_{j,2} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{11n+1}|\frac{n}{2}+1 \le i \le n\}$, and $B_j = B_{j,1} \cup B_{j,2}$ and $B = \bigcup_{j=0}^{10} B_j$. Then

$$\begin{aligned}
B_0 &= \{[f(v_1^i) - f(v_0)] \pmod{11n+1}|1 \le i \le n\} = \{5n+1-i|1 \le i \le n\}\\
&= \{5n, 5n-1, \cdots, 4n+1\},\\
B_1 &= \{[f(v_2^i) - f(v_1^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{7n+i-(5n+1-i) = 2n-1+2i, |1 \le i \le n\}\\
&= \{2n+1, 2n+3, \cdots, 4n-1\},\\
B_2 &= \{[f(v_3^i) - f(v_2^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{6n+1-i-(7n+i) = -(n-1+2i) = 10n+2-2i, |1 \le i \le n\}\\
&= \{10n, 10n+2, \cdots, 8n+2\},\\
B_3 &= \{[f(v_4^i) - f(v_3^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{8n+1+i-(6n+1-i) = 2n+2i|1 \le i \le n\}\\
&= \{2n+2, 2n+4, \cdots, 4n\},\\
B_4 &= \{[f(v_5^i) - f(v_4^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{7n+1-i-(8n+1+i) = 10n+1-2i|1 \le i \le n\}\\
&= \{10n-1, 10n-3, \cdots, 8n+1\},\\
B_5 &= \{[f(v_6^i) - f(v_5^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{n+i-(7n+1-i) = 5n+2i|1 \le i \le n\}\\
&= \{5n+2, 5n+4 \cdots, 7n\},\\
B_6 &= \{[f(v_7^i) - f(v_6^i)] \pmod{11n+1}|1 \le i \le n\}\\
&= \{3n+2-i-(n+i) = 2n+2-2i|1 \le i \le n\}\\
&= \{2n, 2n-2, \cdots, 2\},
\end{aligned}$$

$$
\begin{aligned}
B_7 &= \{[f(v_8^i) - f(v_7^i)] \pmod{11n+1} | 1 \le i \le n\} \\
&= \{9n + 1 + i - (3n + 2 - i) = 6n - 1 + 2i = |1 \le i \le n\} \\
&= \{6n + 1, 6n + 3, \cdots, 8n - 1\}, \\
B_{8,1} &= \{[f(v_9^i) - f(v_8^i)] \pmod{11n+1} | 1 \le i \le \tfrac{n}{2}\} \\
&= \{4n + 1 - i - (9n + 1 + i) = 6n + 1 - 2i | 1 \le i \le \tfrac{n}{2}\} \\
&= \{6n - 1, 6n - 3, \cdots, 5n + 1\}, \\
B_{8,2} &= \{[f(v_9^i) - f(v_8^i)] \pmod{11n+1} | \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{11n + 2 - i - (9n + 1 + i)) = 2n + 1 - 2i | \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{n - 1, n - 3, \ \cdots, 1\}, \\
B_{9,1} &= \{[f(v_{10}^i) - f(v_9^i)] \pmod{11n+1} | 1 \le i \le \tfrac{n}{2}\} \\
&= \{i - (4n + 1 - i) = 7n + 2i | 1 \le i \le \tfrac{n}{2}\} \\
&= \{7n + 2, 7n + 4, \cdots, 8n\}, \\
B_{9,2} &= \{[f(v_{10}^i) - f(v_9^i)] \pmod{11n+1} | \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{i - (11n + 2 - i)) = 2i - 1 | \tfrac{n}{2} + 1 \le i \le n\} \\
&= \{n + 1, n + 3, \ \cdots, 2n - 1\}, \\
B_{10} &= \{[f(v_0) - f(v_{10}^i)] \pmod{11n+1} | 1 \le i \le n\} \\
&= \{-i = 11n + 1 - i \ 1 \le i \le n\} \\
&= \{11n, 11n - 1, \cdots, 10n + 1\}.
\end{aligned}
$$

Hence, $B = \bigcup_{i=0}^{10} B_i$ is the set of labels of all edges, and

$$
\begin{aligned}
B &= B_{8,2} \cup B_{9,2} \cup B_6 \cup B_1 \cup B_3 \cup B_0 \cup B_{8,1} \cup B_7 \cup B_5 \cup B_{9,1} \cup B_4 \cup B_2 \cup B_{10} \\
&= \{1, 3, \cdots, n - 1\} \cup \{n + 1, n + 3, \cdots, 2n - 1\} \cup \{2, 4, \cdots, 2n\} \\
&\quad \cup \{2n + 1, 2n + 3, \cdots, 4n - 1\} \cup \{2n + 2, 2n + 4, \cdots, 4n\} \\
&\quad \cup \{5n, 5n - 1, \cdots, 4n + 1\} \cup \{6n - 1, 6n - 3, \cdots, 5n + 1\} \\
&\quad \cup \{6n + 1, 6n + 3, \cdots, 8n - 1\} \cup \{5n + 2, 5n + 4 \cdots, 7n\} \\
&\quad \cup \{7n + 2, 7n + 4, \cdots, 8n\} \cup \{10n - 1, 10n - 3, \cdots, 8n + 1\} \\
&\quad \cup \{10n, 10n + 2, \cdots, 8n + 2\} \cup \{11n, 11n - 1, \cdots, 10n + 1\} \\
&= \{1, 2, \ldots, 11n\}.
\end{aligned}
$$

It shows that f' is a bijection from $E(n \cdot \vec{C}_{11})$ onto $\{1, 2, \cdots, |E(n \cdot \vec{C}_{11})|\}$. So we conclude that $n \cdot \vec{C}_{11}$ is graceful for even n. □

Theorem 3. *For every even integer n, the digraph $n \cdot \vec{C}_{13}$ is graceful.*

Proof. Define

$$f(v_0) = 0,$$

and

$$
f(v_j^i) = \begin{cases} (13 - \frac{j-1}{2})n + 1 - i, & j = 1, 3, 5, 7;\ 1 \le i \le n, \\ \frac{j-3}{2}n + 2 - i, & j = 9, 11;\ 1 \le i \le n. \end{cases}
$$

$$f(v_j^i) = \begin{cases} i, & j=2;\ 1 \le i \le n, \\ 8n+i, & j=4;\ 1 \le i \le n, \\ 5n+i, & j=6;\ 1 \le i \le n, \\ 4n-1+i, & j=8;\ 1 \le i \le \frac{n}{2}, \\ n-1+i, & j=8;\ \frac{n}{2} \le i \le n, \\ (12-\frac{j}{2})n+i, & j=10,12;\ 1 \le i \le n. \end{cases}$$

Similar to the proof of Theorem 1 and Theorem 2, it can be shown that this assignment provides a graceful labeling of $n \cdot \vec{C}_{13}$ for even n. Hence $n \cdot \vec{C}_{13}$ is graceful for even n. □

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