International Journal of Pure and Applied Mathematics
Volume 23 No. 3 2005, 399-407

THE BASIS NUMBER OF THE SEMI-COMPOSITION PRODUCT OF SOME GRAPHS II

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Abstract: The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. We investigate the basis number of the semi-composition product of a path with a cycle and two cycles.

AMS Subject Classification: 05C38, 05C75
Key Words: fold, basis number, cycle space

1. Introduction and Definitions

In this work, we study the basis number of the semi-composition product of a path with a cycle and two cycles. In fact, we show that their basis numbers, under some restriction conditions on the order of their factors, equal to 4. This paper is a continuation of the paper of Jaradat et al [10] in which the authors determined the basis number of the semi-composition product of two paths and a cycle with a path.

All graphs under consideration are undirected, finite and simple. Our terminology and notations will be standard except as indicated. For undefined terms, see Bondy and Murty [5]. We use the symbols $V(G)$ and $E(G)$, respec-

Received: July 25, 2005
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tively, to denote the vertex set and the edge set of G. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph G

$$\dim\mathcal{C}(G) = |E(G)| - |V(G)| + 1. \tag{1}$$

Given any spanning tree T of G, every graph $T + e, e \notin T$, contains exactly one cycle C_e, and the collection of cycles $\{C_e : e \notin T\}$ forms a basis of $\mathcal{C}(G)$, called the fundamental basis corresponding to T. One can observe that each edge outside of T occurs in exactly one cycle of this basis, but each edge of T itself may occur in many cycles of the basis. This observation suggests the following definition.

Definition 1.1. A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a $d-$fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The required basis of G is a basis $\mathcal{B}$ of $b(G)$-fold. An edge e is of fold d if e occurs in d cycles of $\mathcal{B}$.

Basis number of graphs has been extensively examined by many authors. We refer the readers to the papers [1], [2], [3], [4], [6], [7], [8] and [9]. The first important result in the basis number was given by MacLane [11], in 1937, who proved that $b(G) \leq 2$ if and only if G is non planar. In 1981, Schmeichel [12] proved that $b(K_n) \leq 4$ and $b(K_{m,n}) \leq 4$.

We now give the definition of the following two graph products.

Definition 1.2. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The cartesian product $G^* = G \times H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$. (2) The semi-composition product $G^* = G \odot H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1v_2 \notin E(H)\}$.

One can notice that the cartesian product is commutative but the semi-composition product is not commutative. Moreover,

$$d_{G\odot H}(u, v) = |V(H)|d_G(u) + d_H(v) - d_G(u)d_H(v),$$

$$|E(G \odot H)| = |E(G)||V(H)|^2 + |V(G)||E(H)| - 2|E(G)||E(H)|, \qquad (2)$$

where $d_G(x)$ is the degree of the vertex x in the graph G.

Throughout this work, for $B \subset \mathcal{C}(G)$, $f_B(e)$ stands for the number of cycles in B containing the edge e, $\mathcal{C}(B)$ stands for the subspace of $\mathcal{C}(G)$ generated by B and $E(B) = \cup_{c \in B} E(c)$.

2. Main Results

In this section, we investigate the basis number of the semi-composition product of a path with a cycle and two cycles. In fact, we show that, under some restrictions on their orders, the basis number is 4. Let $P_2 = ab$ be a path of order 2 and $U = \{u_1, u_2, ..., u_n\}$ be a set of vertices. It was proven in [10] that the following sets of cycles are linearly independent.

$$\mathcal{A}_{ab}^{(1)} = \{(a, u_j)(b, u_l)(a, u_{j+1})(b, u_{l+1})(a, u_j) : 1 \leq j, l \leq n-1 \text{ and } |j - l| > 2\},$$

$$\mathcal{A}_{ab}^{(2)} = \{\mathcal{A}_2^{(j)} = (a, u_j)(b, u_j)(a, u_{j+2})(b, u_{j+2})(a, u_j) : 1 \leq j \leq n-2\},$$

$$\mathcal{A}_{ab}^{(3)} = \{\mathcal{A}_3^{(j)} = (b, u_n)(a, u_{n-j})(b, u_{n-j})(a, u_{n-j-2})(b, u_n) : 2 \leq j \leq n-3\},$$

$$\mathcal{A}_{ab}^{(4)} = \{\mathcal{A}_4^{(j)} = (a, u_j)(a, u_{j+1})(b, u_{j+3})(a, u_j) : 1 \leq j \leq n-3\} \cup$$
$$\{\mathcal{A}_4^{(n-2)} = (a, u_{n-2})(a, u_{n-1})(b, u_{n-1})(b, u_n)(a, u_{n-2})\},$$

$$\mathcal{A}_{ab}^{(5)} = \{\mathcal{A}_5^{(j)} = (a, u_{n-j})(a, u_{n-j-1})(b, u_{n-j-1})(b, u_{n-j-2})(a, u_{n-j}) :$$
$$0 \leq j \leq n-3\},$$

$$\mathcal{A}_{ab}^{(6)} = \{(b, u_{n-1})(b, u_n)(a, u_2)(a, u_1)(b, u_{n-1})\},$$

$$\mathcal{A}_{ab}^{(7)} = \{(b, u_{n-2})(b, u_{n-1})(a, u_1)(b, u_{n-2})\},$$

$$\mathcal{A}_{ab}^{(8)} = \{\mathcal{A}_8^{(j)} = (b, u_j)(b, u_{j+1})(a, u_{j+3})(b, u_j) : 1 \leq j \leq n-3\}.$$

Moreover, $\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)}$ is a $4-$ fold basis for $\mathcal{C}(P_m \odot P_n)$, where $P_m = a_1 a_2 \ldots a_m$ and $P_n = u_1 u_2 \ldots u_n$ are two paths with m, n vertices, also it was proven that for any $n, m \geq 2$, $b(P_m \odot P_n) \leq 4$ and the equality holds if $n \geq 14$ and $m \geq 2$. Set

$$S_{ab}^{(1)} = (a, u_1)(b, u_{n-1})(a, u_2)(b, u_n)(a, u_1),$$

$$
\begin{aligned}
S_{ab}^{(2)} &= (a,u_{n-1})(b,u_1)(a,u_n)(b,u_2)(a,u_{n-1}),\\
S_{ab}^{(3)} &= \mathcal{A}_3^{(n-3)} = (b,u_n)(a,u_3)(b,u_3)(a,u_1)(b,u_n).
\end{aligned}
$$

Note that $S_{ab}^{(1)}, S_{ab}^{(2)} \in \mathcal{A}_{ab}^{(1)}$ and $S_{ab}^{(3)} \in \mathcal{A}_{ab}^{(3)}$. Moreover, each of which contains an edge of $\{(a,u_1)(b,u_n),(a,u_n)(b,u_1)\}$. Now for $n \geq 6$, we define

$$
\begin{aligned}
T_{ab}^{(1)} &= (a,u_1)(a,u_n)(b,u_4)(a,u_1),\\
T_{ab}^{(2)} &= (a,u_1)(b,u_1)(a,u_{n-1})(b,u_{n-1})(a,u_1),\\
T_{ab}^{(3)} &= (b,u_1)(b,u_n)(a,u_4)(b,u_1).
\end{aligned}
$$

Let $C_n = u_1u_2\dots u_nu_1$ be a cycle of order n. Then, $P_m \odot C_n$ is obtained from $P_m \odot P_n$ by replacing the edge set $\{(a_i,u_1)(a_{i+1},u_n),(a_i,u_n)(a_{i+1},u_1) : 1 \leq i \leq m-1\}$ by the edge set $\{(a_i,u_1)(a_i,u_n) : 1 \leq i \leq m\}$ and hence $|E(P_m \odot C_n)| = |E(P_m \odot P_n)| - 2(m-1) + m = (m-1)n^2 - mn + 2n$. Now, from equation (1) and being $|V(P_m \odot C_n)| = mn$, we have the following lemma.

Lemma 2.1. $\dim C(P_m \odot C_n) = \dim C(P_m \odot P_n) - 2(m-1) + m = (n-1)^2(m-1) - (m-2)$.

To find a basis for $C(P_m \odot C_n)$, we delete all cycles in $\mathcal{B}(P_m \odot P_n) = \cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_ia_{i+1}}^{(k)}$ that contain edges of

$$\{(a_i,u_1)(a_{i+1},u_n),\{(a_i,u_n)(a_{i+1},u_1) : 1 \leq i \leq m-1\}$$

and replace them by cycles that contain edges of $\{(a_i,u_1)(a_i,u_n) : 1 \leq i \leq m\}$ and some other cycles. Set

$$S = \cup_{i=1}^{m-1} \cup_{k=1}^{3} S_{a_ia_{i+1}}^{(k)},$$

and

$$T = (\cup_{i=1}^{m-1} \cup_{k=1}^{2} T_{a_ia_{i+1}}^{(k)}) \cup T_{a_{m-1}a_m}^{(3)}.$$

Note that any cycle of S contains an edge of

$$\{(a_i,u_1)(a_{i+1},u_n),(a_{i+1},u_1)(a_i,u_n) : 1 \leq i \leq m-1\}$$

and any cycle of $(\cup_{i=1}^{m-1} T_{a_ia_{i+1}}^{(1)}) \cup T_{a_{m-1}a_m}^{(3)}$ contains an edge of $\{(a_i,u_1)(a_i,u_n) : 1 \leq i \leq m\}$.

Lemma 2.2. *$(\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_ia_{i+1}}^{(k)}) \cup T - S$ is linearly independent.*

Proof. By [10], $(\cup_{i=1}^{m-1}\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}})$ is a linearly independent set of cycles. Since $S\subseteq(\cup_{i=1}^{m-1}\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}), (\cup_{i=1}^{m-1}\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}})-S$ is linearly independent. Now, $T^{(1)}_{a_ia_{i+1}}$ contains the edge $(a_i,u_1)(a_i,u_n)$ which is not in any other cycle of $(\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))-S$. Thus, $(\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))-S$ is linearly independent. The cycle $T^{(3)}_{a_{m-1}a_m}$contains the edge $(a_m,u_1)(a_m,u_n)$ which is not in any cycle of $(\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))-S$. Hence, $((\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))\cup T^{(3)}_{a_{m-1}a_m})-S$ is linearly independent. Since $E(T^{(2)}_{a_ia_{i+1}})\cap E(T^{(2)}_{a_ka_{k+1}})=\phi$ for each $i\neq k$, $\cup_{i=1}^{m-1}T^{(2)}_{a_ia_{i+1}}$ is linearly independent. Therefore, it suffices to show that any linear combination C of $\cup_{i=1}^{m-1}T^{(2)}_{a_ia_{i+1}}$ cannot be written as a linear combination of cycles of $((\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))\cup T^{(3)}_{a_{m-1}a_m})-S$. Let C be a linear combination of cycles of $\cup_{i=1}^{m-1}T^{(2)}_{a_ia_{i+1}}$. Then C is either a cycle or an edge disjoint union of cycles each of which is a cycle of $\cup_{i=1}^{m-1}T^{(2)}_{a_ia_{i+1}}$. Therefore, C contains an edge of the form $(a_i,u_1)(a_{i+1},u_1)$ for some $1\le i\le m-1$. Assume that C is a linear combination of cycles from $((\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))\cup T^{(3)}_{a_{m-1}a_m})-S$, say $R=\{R_1,R_2,\ldots,R_y\}$. Since C contains the edge $(a_i,u_1)(a_{i+1},u_1)$ and the only cycle of $((\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))\cup T^{(3)}_{a_{m-1}a_m})-S$ contains such an edge is $\mathcal{A}^{(1)}_2=(a_i,u_1)(a_{i+1},u_1)(a_i,u_3)(a_{i+1},u_3)(a_i,u_1)\in\mathcal{A}^{(2)}_{a_ia_{i+1}}$, as a result $\mathcal{A}^{(1)}_2$ must belong to R , say $R_1=\mathcal{A}^{(1)}_2$. Since $(a_i,u_1)(a_{i+1},u_3)\in E(\mathcal{A}^{(1)}_2)$ and no other cycles of $((\cup_{i=1}^{m-1}(\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}\cup T^{(1)}_{a_ia_{i+1}}))\cup T^{(3)}_{a_{m-1}a_m})-S$ contain such an edge, we have that $(a_i,u_1)(a_{i+1},u_3)$ is an edge of $R_1+R_2+\ldots+R_y$ (mod2) but $(a_i,u_1)(a_{i+1},u_3)\notin E(T^{(2)}_{a_ia_{i+1}})$ for each $i=1,2\ldots,m-1$ and so $(a_i,u_1)(a_{i+1},u_3)\notin E(C)$. This is a contradiction. The proof is complete. □

Theorem 2.1. *For any integers $n\ge 3, m\ge 2$, $b(P_m\odot C_n)\le 4$. Moreover, the equality holds if $n\ge 14$ and $m\ge 2$.*

Proof. To prove the first part of the theorem, it suffices to exhibit a 4-fold basis. The theorem is clear for $n=3$, 4 and 5. For $n\ge 6$, define $\mathcal{B}(P_m\odot C_n)=(\cup_{i=1}^{m-1}\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}})\cup T-S$. By Lemma 2.2, $\mathcal{B}(P_m\odot C_n)$ is linearly independent. Since $T\subseteq\mathcal{C}(P_m\odot C_n)$ for $n\ge 6$,

$$|T|=2(m-1)+1,\quad\text{and}\quad |S|=3(m-1),$$

we have,

$$\begin{aligned}|\mathcal{B}(P_m\odot C_n)|&=|\cup_{i=1}^{m-1}\cup_{k=1}^{8}\mathcal{A}^{(k)}_{a_ia_{i+1}}|+|T|-|S|=(n-1)^2(m-1)+2(m-1)\\&\quad+1-3(m-1)=(n-1)^2(m-1)-(m-2)=\dim\ \mathcal{C}(P_m\odot C_n).\end{aligned}$$

Therefore, $\mathcal{B}(P_m \odot C_n)$ is a basis of $\mathcal{C}(P_m \odot C_n)$. We now show that $\mathcal{B}(P_m \odot C_n)$ is a 4-fold basis. Let $e \in E(P_m \odot C_n)$. If $e \in E((\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)}) - (S \cup T))$, then from [10] $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 4$. Now suppose $e \in E(T)$. Then:

(1) if $e = (a_i, u_1)(a_i, u_n)$, then $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 1$.

(2) If $e = (a_i, u_1)(a_{i+1}, u_1)$, then $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 2$.

(3) If $e = (a_i, u_1)(a_{i+1}, u_{n-1})$ or $e = (a_i, u_{n-1})(a_{i+1}, u_1)$, then $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 2$.

(4) If $e = (a_i, u_1)(a_{i+1}, u_4)$ or $e = (a_{i+1}, u_1)(a_i, u_4)$, then $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 4$.

(5) If $e = (a_i, u_n)(a_{i+1}, u_4)$ or $e = (a_{i+1}, u_n)(a_i, u_4)$, then $f_{\mathcal{B}(P_m \odot C_n)}(e) \leq 4$.

On the other hand, to show that $b(P_m \odot C_n) \geq 4$ for any $n \geq 14$ and $m \geq 2$, we have to exclude any possibility for the cycle space $\mathcal{C}(P_m \odot C_n)$ to have a 3-fold basis for any $n \geq 14$ and $m \geq 2$. Suppose that $\mathcal{B}$ is a 3-fold basis of the cycle space, then we have the following three cases:

Case 1. Suppose that $\mathcal{B}$ consists only of 3-cycles. Then $|\mathcal{B}| \leq 3mn$ because any 3-cycle must contain an edge of the form $(a_i, u_j)(a_i, u_{j+1})$ or the form $(a_i, u_1)(a_i, u_n)$ for some $1 \leq i \leq m; 1 \leq j \leq n-1$ and each edge is of fold at most 3. That is equivalent to the inequality $(m-1)(n-1)^2 - (m-2) \leq 3mn$ which implies that $m(n^2 - 5n) \leq n^2 - 2n - 1$. But $m \geq 2$, so $n \leq 8$. This is a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4. Then $4|\mathcal{B}| \leq 3|E(P_m \odot P_n)|$ because the length of each cycle of $\mathcal{B}$ greater than or equal to 4 and each edge is of fold at most 3. Thus, $4(m-1)(n-1)^2 - 4(m-2) \leq 3((m-1)n^2 - mn + 2n)$, which is equivalent to $m(n^2 - 5n) \leq n^2 - 2n - 4$. But $m \geq 2$, so $n \leq 7$. This is a contradiction.

Case 3. Suppose that $\mathcal{B}$ consists of s 3-cycles and t cycles of length greater than or equal to 4. Then as in Case 1 $s \leq 3mn$. Since the length of each cycle of s is 3 and each cycle of t is at least 4 and the fold of each edge is at most 3, we have that $4t + 3s \leq 3|E(P_m \odot C_n)|$. But $t = |\mathcal{B}| - s = |E(P_m \odot C_n)| - |V(P_m \odot C_n)| + 1 - s$, so $4t + 3s = 4|E(P_m \odot C_n)| - 4|V(P_m \odot C_n)| + 4 - s \leq 3|E(P_m \odot C_n)|$ which implies that $|E(P_m \odot C_n)| - 4|V(P_m \odot C_n)| + 4 = (m-1)n^2 - mn + 2n - 4mn + 4 \leq s \leq 3mn$. Thus, $(m-1)n^2 - mn + 2n - 4mn + 4 \leq 3mn$ which is equivalent to $m(n^2 - 8n) \leq n^2 - 2n - 4$. But $m \geq 2$, so $n \leq 13$. This is a contradiction. The proof is complete. □

Let $C_m = a_1 a_2 \ldots a_m a_1$ be a cycle of order m. Then, from equations (2), we have $|E(C_m \odot C_n)| = mn^2 - mn$. Now $|V(C_m \odot C_n)| = mn$, so from equation

(1), we have the following lemma.

Lemma 2.3. $\dim C(C_m \odot C_n) = m(n^2 - 2n) + 1$.

Theorem 2.2. *For any $n, m \geq 3$, we have $b(C_m \odot C_n) \leq 4$. Moreover, the equality holds if $n \geq 9$ and $m \geq 4$.*

Proof. To prove that $b(C_m \odot C_n) \leq 4$, it suffices to exhibit a 4-fold basis. It is clear for $n = 3$, 4 and 5. For $n \geq 6$, define $\mathcal{B} = (\mathcal{B}(P_m \odot C_n) \cup (\cup_{k=1}^{8} \mathcal{A}_{a_1 a_m}^{(k)} \cup T_{a_1 a_m}^{(2)})) - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)}$, where $\mathcal{B}(P_m \odot C_n) = (\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)}) \cup T - S$ is defied as in Theorem 2.1. Since $E(\mathcal{B}(P_m \odot C_n)) \cap E(\cup_{k=1}^{8} \mathcal{A}_{a_1 a_m}^{(k)} \cup T_{a_1 a_m}^{(2)} - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)}) = E(a_1 \times P_n) \cup E(a_m \times P_n)$, then any linear combination of cycles of $(\cup_{k=1}^{8} \mathcal{A}_{a_m a_1}^{(k)} \cup T_{a_1 a_m}^{(2)}) - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)}$ must contain an edge of the form $(a_1, u_j)(a_m, u_l)$ for some j, l which is not in any cycle of $\mathcal{B}(P_m \odot C_n)$, Thus $\mathcal{B}$ is linearly independent. Now, consider the following cycle:

$$C = (a_1, u_1)(a_2, u_1) \ldots (a_{m-1}, u_1)(a_m, u_1)(a_1, u_1).$$

We show that C is independent of cycles of $\mathcal{B}$. Suppose that C is a sum modulo 2 of cycles of $\mathcal{B}$. Then

$$C = \sum_{i=1}^{m} \beta_i \pmod 2,$$

where β_i $(1 \leq i \leq m-2)$ is a linear combination of cycles of $(\cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)} \cup_{k=1}^{2} T_{a_i a_{i+1}}^{(k)}) - \cup_{k=1}^{3} S_{a_i a_{i+1}}^{(k)}$, β_{m-1} is a linear combination of cycles of

$$(\cup_{k=1}^{8} \mathcal{A}_{a_{m-1} a_m}^{(k)} \cup_{k=1}^{2} T_{a_{m-1} a_m}^{(k)} \cup T_{a_{m-1} a_m}^{(3)}) - \cup_{k=1}^{3} S_{a_{m-1} a_m}^{(k)},$$

and β_m is a linear combination of cycles of $(\cup_{k=1}^{8} \mathcal{A}_{a_1 a_m}^{(k)} \cup T_{a_1 a_m}^{(2)}) - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)}$. Thus,

$$\beta_m = C + \sum_{i=1}^{m-1} \beta_i \pmod 2.$$

Therefore,

$$\begin{aligned} &E(\beta_m) = E(C \oplus \beta_1 \oplus \cdots \oplus \beta_{m-1}) \\ &\subseteq (E(\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)} \cup T - S) \cup E(C)) \cap E(\cup_{k=1}^{8} \mathcal{A}_{a_1 a_m}^{(k)} \cup T_{a_1 a_m}^{(2)} - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)}), \end{aligned}$$

where $\oplus$ is the ring sum. But,

$$(E(\cup_{i=1}^{m-1} \cup_{k=1}^{8} \mathcal{A}_{a_i a_{i+1}}^{(k)} \cup T - S) \cup E(C)) \cap E(\cup_{k=1}^{8} \mathcal{A}_{a_1 a_m}^{(k)} \cup T_{a_1 a_m}^{(2)} - \cup_{k=1}^{3} S_{a_1 a_m}^{(k)})$$

$$= E(a_1 \times P_n) \cup E(a_m \times P_n) \cup \{(a_1, u_1)(a_m, u_1)\},$$

which is an edge set of a tree. This contradicts the fact that β_m is a cycle or an edge disjoint union of cycles. Therefore, $\mathcal{B}(C_m \odot C_n) = \mathcal{B} \cup \{C\}$ is linearly independent. Since

$$\begin{aligned}|\mathcal{B}(C_m \odot C_n)| = |\mathcal{B}| + |C| &= (n-1)^2(m-1) - (m-2) + (n-1)^2 + 1 - 3 + 1 \\ &= (n^2 - 2n)m + 1 = \dim \mathcal{C}(C_n \odot C_m),\end{aligned}$$

$\mathcal{B}(C_m \odot C_n)$ is a basis of $\mathcal{C}(C_m \odot C_n)$. It is easy to show that $\mathcal{B}(C_m \odot C_n)$ is a 4-fold basis.

On the other hand, to show that $b(C_m \odot C_n) \geq 4$, we follow, more or less, the same arguments as in the three cases of Theorem 2.1. The proof is complete. □

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