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FINITE ELEMENT GALERKIN SOLUTIONS FOR THE NONLINEAR FREE SURFACE PROBLEMS

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Abstract: Numerical computations for the water-wave free surface problem have been extensively studied. But the corresponding numerical analysis have been rarely worked out because of nonlinearity of free surface conditions. We discuss the numerical solutions of a semi-discrete finite element Galerkin method and a Crank-Nicolson type fully discrete one to solve the water-wave problem. Their stability and convergence are discussed and the error estimate of the potential function is expressed in terms of the kinematic free boundary and the potential function on the free boundary given by the Bernoulli's condition.

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1. Introduction

Consider the two-dimensional irrotational perfect flow which is incompressible, inviscid and lacking free surface tension. It is well known that the single-valued velocity potential ϕ satisfies the Laplace equation

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$$\triangle\phi(x,y,t)=0,\quad (x,y,t)\in\Omega_\eta\times(0,T] \tag{1.1}$$

with a kinematic free boundary condition

$$\eta_t+\phi_x\eta_x-\phi_y=0\quad \text{on } y=\eta(x,t) \tag{1.2}$$

and the Bernoulli's condition

$$\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)+g\eta=0\quad \text{on } y=\eta(x,t). \tag{1.3}$$

Here g is the gravitational acceleration in the negative y-direction. The fluid is bounded on top by the free surface S_η described by $y=\eta(x,t)$, where $y=\eta(x,t)$ is the free surface elevation from the bottom. Let $I=(0,1)$, $\Omega_\eta=I\times(0,\eta(x,t))$ and $T\in\mathbb{R}$ is such that $0<T<\infty$. We assume the boundary conditions are given as

$$\frac{\partial\phi}{\partial n}=0\quad \text{on } S_w,S_b, \tag{1.4}$$

where n is the outward normal vector and S_w,S_b are the boundaries of the wall and the no-seepage bottom, respectively. The initial conditions are given as

$$\phi(x,0)=\frac{\partial\phi}{\partial t}(x,0)=0 \tag{1.5}$$

and

$$\eta(x,0)=1. \tag{1.6}$$

Existence of solutions for the irrotational water wave problems were discussed in John [8], Keady and Norbury [9], Miles [14] and Temam [20]. Numerical methods for the free surface problems have been extensively studied. See the review papers Floriyan and Rasmussen [5], Yeung [24], Tsai and Yue [22] and references therein.

ADI finite difference methods for the free boundary problem have been studied by Ryskin and Leal [17], where they applied ADI method on an orthogonal curvilinear coordinate system. DeSilva, Guenther and Hudspeth [4] applied a transformation technique in order to change the curved domain into a rectangular domain and then used finite difference methods with SOR to obtain numerical solutions. The method of lines was applied to the two space dimensional nonlinear water wave generation by Ohring [15]. Finite volume methods for free surface problems have been numerically studied by Hirt and Nichols [6], Thé, Raithby and Stubley [21].

To apply boundary integral methods, Skerget, Hribersek and Kuhn [19] used finite difference methods, Quinn, Oxley and Voika [16] used front-tracking method, and Longuet-Higgins and Cokelet [11] applied Adams-Bashford-Moulton method for the moving boundary computation.

A variational formulation for the problem (1.1)–(1.3) has been done by Luke [13]. Since then finite element methods have been extensively used in order to obtain numerical approximate solutions for the problem. Bai [1]- [2] obtained numerical solutions by using localized finite element methods for the 3-dimensional steady problem. Kim and Bai [10] applied finite element method to the nonlinear formulation in the scope of potential theory without any assumptions on the magnitude of disturbances. Wu and Taylor [23] used mixed finite element methods for numerical solutions. They compared computational efficiency of boundary element methods and finite Galerkin element methods. Idelsohn, Onate and Sacco [7] applied a fractional step method for the free surface equation and applied finite element method for the potential function.

But none of the above works has not discussed error estimates of numerical solutions. Zhang and Babuska [25] have studied convergence of numerical solutions for the steady state periodic free boundary value problems. Saavedra and Scott [18] also studied an error analysis of a finite element method for a steady state free surface problem with surface tension. Recently, Bai, Choo, Chung and Kim [3] have discussed numerical stability and error estimates of a finite element method for the problem (1.1)–(1.6) by using an extended Lax-Richtmyer Equivalence Theorem [12].

In this paper, a nonlinear finite element scheme for (1.1)–(1.6) is considered and error estimates of the approximate solutions are discussed, which is different from [3]. In Section 2, a nonlinear semi-discrete finite element scheme for (1.2)–(1.3) is introduced and the corresponding stability and convergence of approximate solutions are studied for the kinematic equation and the Bernoulli's equation on the free surface. In Section 3, a nonlinear fully discrete finite element scheme for (1.2)–(1.3) is introduced and the corresponding stability and convergence of approximate solutions are studied. In Section 4, convergence of approximate solutions for (1.1)–(1.3) is discussed and error estimates are considered in terms of free surface functions η and ϕ using the idea in Zhang and Babuska [25].

2. Semi-Discrete Finite Element Schemes on S_η

In this section, we consider the finite element approximate solution of (1.2)–(1.3). For the notational simplicity, let $u(x,t) = \phi_x(x,\eta(x,t),t)$, $v(x,t) = \phi_y(x,\eta(x,t),t)$, and $\psi(x,t) = \phi(x,\eta(x,t),t)$. Then the system (1.2)–(1.3) is expressed as

$$\eta_t + u\eta_x - v = 0, \tag{2.1}$$

$$\psi_t + u\psi_x - \frac{1}{2}(u^2 + v^2) + g\eta = 0. \tag{2.2}$$

In order to consider the weak solution of (2.1)-(2.2), we define a vector space $V = \{w \in H^1(I) \cap H^2(I) | w'(0) = w'(1) = 0\}$ and the inner product in V is defined as

$$\langle g, h \rangle = \int_0^1 g(x)h(x)dx, \quad \forall g, h \in V.$$

Then the weak solution of (2.1)–(2.2) is a pair of functions $\eta, \psi \in H^1$ such that for each $\chi \in V$,

$$\langle \eta_t + u\eta_x - v, \chi \rangle = 0 \tag{2.3}$$

and

$$\langle \psi_t + u\psi_x - \frac{1}{2}(u^2 + v^2) + g\eta, \chi \rangle = 0. \tag{2.4}$$

We now consider the finite element numerical solution for (2.3)-(2.4). Let $\delta = \frac{1}{N}$ for any given positive integer N and $x_i = i\delta$ for $i = 0, 1, \cdots, N$. For an integer $r \geq 4$, let V_δ^r be the finite dimensional subspace of V consisting of spline polynomials of degree $\leq r$ on a uniform mesh of width size δ on I, which satisfies the following approximation properties: if $v \in W_2^s(I) \cap W_\infty^m(I)$, then there exits a function $\chi \in V_\delta^r$ such that

$$\sum_{j=0}^{s-1} \delta^j \|v - \chi\|_j \leq C\delta^s \|v\|_s, \quad 1 \leq s \leq r \tag{2.5}$$

and

$$\sum_{j=0}^{m-1} \delta^j \|v - \chi\|_{j,\infty,I} \leq C\delta^m \|v\|_{m,\infty,I}, \quad 1 \leq m \leq r. \tag{2.6}$$

Furthermore, we assume that the following inverse inequalities hold: for all $\chi \in V_\delta^r$,

$$\|\chi\|_\beta \leq C\delta^{-(\beta-\alpha)} \|\chi\|_\alpha, \quad 0 \leq \alpha \leq \beta \leq r - 1 \tag{2.7}$$

and

$$\|\chi\|_{s,\infty,I} \leq C\delta^{-(s+1/2)} \|\chi\|, \quad 0 \leq s \leq r - 1. \tag{2.8}$$

Hereafter a positive constant C will denote a generic constant which is independent of mesh sizes.

Then the solution of continuous time Galerkin method is a pair of functions $\Gamma, \Psi \in V_\delta^r$ such that for each $\chi \in V_\delta^r$,

$$\langle \Gamma_t + U\Gamma_x - V, \chi \rangle = 0 \tag{2.9}$$

and

$$\langle \Psi_t + U\Psi_x - \frac{1}{2}(U^2 + V^2) + g\Gamma, \chi \rangle = 0, \tag{2.10}$$

where U and V are projections of u and v on V_δ^r, respectively.

The existence and uniqueness of solution for (2.9)-(2.10) are obtained by the existence theorem of usual ordinary differential equations. We obtain the following stability results for (2.9)-(2.10).

Theorem 2.1. *Suppose $|U_x|_\infty < \infty$ and $\|U^2 + V^2\| < \infty$. Then there is a constant C such that*

$$\|\Gamma(t)\| \le C(\|\Gamma(0)\| + \int_0^t \|V(s)\| ds)$$

and

$$\|\Psi(t)\| \le C(\|\Psi(0)\| + \|\Gamma(0)\| + \int_0^t \|U^2(s) + V^2(s)\| ds).$$

Proof. Setting $\chi = \Gamma$ and integrating by parts, the equation (2.9) becomes

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|^2 = \frac{1}{2}\langle U_x, \Gamma^2 \rangle + \langle V, \Gamma \rangle.$$

Since $|U_x|_\infty < \infty$, we have

$$\frac{d}{dt}\|\Gamma(t)\|^2 \le \|V\|^2 + C\|\Gamma\|^2.$$

Integrating from 0 to t, we obtain

$$\|\Gamma(t)\|^2 \le \|\Gamma(0)\|^2 + \int_0^t \|V(s)\|^2 ds + C\int_0^t \|\Gamma(s)\|^2 ds.$$

Applying Gronwall's inequality, we obtain

$$\|\Gamma(t)\|^2 \le C(\|\Gamma(0)\|^2 + \int_0^t \|V(s)\|^2 ds).$$

Similarly, we obtain the second inequality. Hence the proof is completed. □

If we differentiate (2.1) twice in the temporal direction, then for any $\chi \in V_\delta^r$

$$\big\langle \eta_{txx} + (u\eta_x)_{xx} - v_{xx}, \chi \big\rangle = 0.$$

Integrating by parts, we have

$$\big\langle \eta_{tx}, \chi_x \big\rangle - \big\langle u\eta_x, \chi_{xx} \big\rangle - \big\langle v_x, \chi \big\rangle = 0.$$

It follows from the projection of η that

$$\big\langle \Gamma_{tx}, \chi_x \big\rangle - \big\langle U\Gamma_x, \chi_{xx} \big\rangle - \big\langle V_x, \chi_x \big\rangle = 0. \tag{2.11}$$

Setting $\chi = \Gamma$ in (2.11), we obtain

$$\big\langle \Gamma_{tx}, \Gamma_x \big\rangle - \big\langle U\Gamma_x, \Gamma_{xx} \big\rangle - \big\langle V_x, \Gamma_x \big\rangle = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}\|\Gamma_x\|^2 = -\frac{1}{2}\big\langle U_x, \Gamma_x^2 \big\rangle + \big\langle V_x, \Gamma_x \big\rangle.$$

Since $|U_x|_\infty$ is bounded, applications of Cauchy-Schwarz inequality and Young's inequality give

$$\frac{d}{dt}\|\Gamma_x\|^2 \le C\|\Gamma_x\|^2 + \|V_x\|^2.$$

Integrating from 0 to t and applying Gronwall's inequality, we have

$$\|\Gamma_x(t)\|^2 \le C\big\{\|\Gamma_x(0)\|^2 + \int_0^t \|V_x(s)\|^2 ds\big\}. \tag{2.12}$$

Similarly, it follows from (2.2) that

$$\big\langle \Psi_{tx}, \chi_x \big\rangle - \big\langle U\Psi_x, \chi_{xx} \big\rangle - \big\langle UU_x + VV_x, \chi_x \big\rangle + \big\langle \Gamma_x, \chi_x \big\rangle = 0. \tag{2.13}$$

Taking $\chi = \Psi$ in (2.13), we get

$$\frac{1}{2}\frac{d}{dt}\|\Psi_x\|^2 = -\frac{1}{2}\big\langle U_x, \Psi_x^2 \big\rangle + \big\langle UU_x + VV_x, \Psi_x \big\rangle - \big\langle \Gamma_x, \Psi_x \big\rangle.$$

Applying Young's inequality, we have

$$\frac{d}{dt}\|\Psi_x\|^2 \le C\|\Psi_x\|^2 + \|UU_x + VV_x\|^2 + \|\Gamma_x\|^2.$$

Integrating and applying Gronwall's inequality, we obtain

$$\|\Psi_x(t)\|^2 \le C\big\{\|\Psi_x(0)\|^2 + \int_0^t (\|UU_x + VV_x\|^2 + \|\Gamma_x\|^2) ds\big\}. \tag{2.14}$$

Let $F \in V_\delta^r$ be the projection of a function $f \in V$. Then for each $\chi \in V_\delta^r$,

$$\|F_x\|_\infty \le \|f_x\|_\infty + \|f_x - \chi_x\|_\infty + \|\chi_x - F_x\|_\infty.$$

It follows from (2.5)–(2.8) that

$$\begin{aligned}\|F_x\|_\infty &\le \|f_x\|_\infty + C\delta^{r-1}\|f\|_{r,\infty} + C\delta^{-3/2}\|\chi_x - F_x\| \\ &\le \|f_x\|_\infty + C\delta^{r-1}\|f\|_{r,\infty} + C\delta^{-3/2}\|\chi - f\| + C\delta^{-3/2}\|f - F\| \\ &\le \|f_x\|_\infty + C\delta^{r-1}\|f\|_{r,\infty} + C\delta^{r-3/2}\|f\|_r + C\delta^{r-3/2}\|f\|_r \,.\end{aligned}$$

Since $r \ge 4$, we have

$$\|F_x\|_\infty \le C(f, \delta).$$

Similarly, we can show that $\|F_{xx}\|_\infty$ is bounded.

For any function $w \in V_\delta^r$, let $\theta = \eta - w$ and $\rho = \Gamma - w$. Since the error $\eta - \Gamma = \theta - \rho$ and the estimates of θ is known from the approximation properties (2.5)–(2.5), it is enough to estimate the values on ρ.

Lemma 2.1. *Suppose $|U_x|_\infty < \infty$. Then there is a constant C such that*

$$\|\rho(t)\| \le C\big\{\|\rho(0)\| + \int_0^t (\|\theta_t\| + \|\theta_x\| + \|u - U\| + \|v - V\|) ds\big\}.$$

Proof. It follows from (2.9) that for $\chi \in V_\delta^r$

$$\begin{aligned}\langle \rho_t, \chi\rangle =& \langle \Gamma_t, \chi\rangle - \langle w_t, \chi\rangle \\ =& -\langle U\Gamma_x - V, \chi\rangle - \langle w_t, \chi\rangle + \langle \eta_t + u\eta_x - v, \chi\rangle \\ =& \langle \theta_t, \chi\rangle + \langle u\theta_x, \chi\rangle - \langle U\rho_x, \chi\rangle + \langle (u - U)w_x, \chi\rangle - \langle v - V, \chi\rangle.\end{aligned}$$

Taking $\chi = \rho$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\rho(t)\|^2 = \langle \theta_t, \rho\rangle + \langle u\theta_x, \rho\rangle + \frac{1}{2}\langle U_x, \rho^2\rangle + \langle (u - U)w_x, \rho\rangle - \langle v - V, \rho\rangle.$$

Applying Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\frac{d}{dt}\|\rho(t)\|^2 \le \|\theta_t\|^2 + \|\theta_x\|^2 + \|u - U\|^2 + \|v - V\|^2 + C_1\|\rho\|^2 \,,$$

where $C_1 = 2 + |u|_\infty + |U_x|_\infty + |w_x|_\infty$. It follows from the fact $|w_x|_\infty < \infty$ and integration from 0 to t that

$$\|\rho(t)\|^2 \le \|\rho(0)\|^2 + \int_0^t (\|\theta_t\|^2 + \|\theta_x\|^2 + \|u-U\|^2 + \|v-V\|^2)ds + C_1 \int_0^t \|\rho\|^2 ds.$$

An application of Gronwall's inequality implies

$$\|\rho(t)\|^2 \le C\big\{\|\rho(0)\|^2 + \int_0^t (\|\theta_t\|^2 + \|\theta_x\|^2 + \|u-U\|^2 + \|v-V\|^2)ds\big\}.$$

This completes the proof. □

We now consider an estimate on $\rho_x(t)$.

Lemma 2.2. *There is a constant C such that*

$$\|\rho_x(t)\| \le C\left\{\|\rho_x(0)\| + \int_0^t (\|\theta_{tx}(s)\| + \|\theta_{xx}(s)\| + \|u_x - U_x\| + \|v_x - V_x\|)ds\right\}.$$

Proof. It follows from (2.11) that for $\chi \in V_\delta^r$

$$\begin{aligned}\big\langle \rho_{tx}, \chi_x\big\rangle =&\big\langle \Gamma_{tx}, \chi_x\big\rangle - \big\langle w_{tx}, \chi_x\big\rangle \\ =&\big\langle U\Gamma_x, \chi_{xx}\big\rangle + \big\langle V_x, \chi_x\big\rangle - \big\langle w_{tx}, \chi_x\big\rangle + \big\langle \eta_{tx} + (u\eta_x)_x - v_x, \chi_x\big\rangle \\ =&\big\langle \theta_{tx}, \chi_x\big\rangle - \big\langle u\theta_x, \chi_{xx}\big\rangle + \big\langle U\rho_x, \chi_{xx}\big\rangle \\ &- \big\langle (u-U)w_x, \chi_{xx}\big\rangle - \big\langle v_x - V_x, \chi_x\big\rangle.\end{aligned}$$

Setting $\chi = \rho$ and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_x(t)\|^2 =&\big\langle \theta_{tx}, \rho_x\big\rangle + \big\langle u_x\theta_x + u\theta_{xx}, \rho_x\big\rangle - \frac{1}{2}\big\langle U_x, \rho_x^2\big\rangle \\ &+ \big\langle (u_x - U_x)w_x + (u-U)w_{xx}, \rho_x\big\rangle - \big\langle v_x - V_x, \rho_x\big\rangle.\end{aligned} \tag{2.15}$$

Applying Cauchy-Schwarz inequality and Young's inequality in (2.15), we obtain

$$\frac{d}{dt}\|\rho(t)\|^2 \le \|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \|u_x - U_x\|^2 + \|v_x - V_x\|^2 + C\|\rho_x\|^2.$$

It follows from integration from 0 to t and an application of Gronwall's inequality that

$$\begin{aligned}&\|\rho_x(t)\|^2\\ &\le C\big\{\|\rho_x(0)\|^2+\int_0^t(\|\theta_{tx}(s)\|^2+\|\theta_{xx}(s)\|^2+\|u_x-U_x\|^2+\|v_x-V_x\|^2)ds\big\}.\end{aligned}$$

This completes the proof. □

For any function $w\in V_\delta^r$, let $\xi=\psi-w$ and $\zeta=\Psi-w$. Since the error $\phi-\Psi=\xi-\zeta$ and the approximation of ξ are known from (2.5)–(2.6), it is enough to estimate the values on ζ.

Lemma 2.3. *Suppose $|U_x|_\infty<\infty$. Then there is a constant C such that*

$$\|\zeta(t)\|\le C\big\{\|\zeta(0)\|+\int_0^t(\|\xi_t\|+\|\xi_x\|+\|\eta-\Gamma\|+\|u-U\|+\|v-V\|)ds\big\}.$$

Proof. It follows from (2.10) that for $\chi\in V_\delta^r$

$$\begin{aligned}\langle\zeta_t,\chi\rangle=&\langle\Psi_t,\chi\rangle-\langle w_t,\chi\rangle\\ =&-\langle U\Psi_x,\chi\rangle-\langle\Gamma,\chi\rangle+\frac{1}{2}\langle U^2+V^2,\chi\rangle-\langle w_t,\chi\rangle\\ &+\langle\psi_t+u\psi_x+\eta-\frac{1}{2}(u^2+v^2),\chi\rangle\\ =&\langle\xi_t,\chi\rangle+\langle u\xi_x,\chi\rangle-\langle U\zeta_x,\chi\rangle+\langle\eta-\Gamma,\chi\rangle\\ &+\langle(u-U)w_x,\chi\rangle-\langle\frac{u+U}{2}(u-U),\chi\rangle-\langle\frac{v+V}{2}(v-V),\chi\rangle.\end{aligned}$$

Taking $\chi=\zeta$ and applying Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\frac{d}{dt}\|\zeta\|^2\le\|\xi_t\|^2+\|\xi_x\|^2+\|\eta-\Gamma\|^2+\|u-U\|^2+\|v-V\|^2+C_2\|\zeta\|^2,$$

where $C_2=1+\frac{3}{2}|u|_\infty^2+|U|_\infty+|w_x|_\infty^2+\frac{1}{2}|U|_\infty^2+\frac{1}{2}|v|_\infty^2+\frac{1}{2}|V|_\infty^2$. Integrating from 0 to t and applying Gronwall's inequality, we have

$$\begin{aligned}\|\zeta(t)\|^2\le C\big\{&\|\zeta(0)\|^2\\ &+\int_0^t(\|\xi_t\|^2+\|\xi_x\|^2+\|\eta-\Gamma\|^2+\|u-U\|^2+\|v-V\|^2)ds\big\}.\end{aligned}$$

This completes the proof. □

Lemma 2.4. *Suppose $|U_x|_\infty < \infty$. Then there is a constant C such that*

$$\|\zeta_x(t)\| \le C\Big\{\|\zeta_x(0)\| + \int_0^t (\|\xi_{xt}\| + \|\xi_{xx}\| + \|\eta_x - \Gamma_x\| + \|u_x - U_x\| + \|v_x - V_x\|)ds\Big\}.$$

Proof. It follows from (2.13) that for $\chi \in V_\delta^r$

$$\begin{aligned}\langle \zeta_{tx}, \chi_x\rangle =& \langle \Psi_{tx}, \chi_x\rangle - \langle w_{tx}, \chi_x\rangle \\ =& \langle U\Psi_x, \chi_{xx}\rangle + \langle UU_x + VV_x, \chi_x\rangle - \langle \Gamma_x, \chi_x\rangle - \langle w_{tx}, \chi_x\rangle \\ &+ \langle \psi_{tx}, \chi_x\rangle - \langle u\psi_x, \chi_{xx}\rangle + \langle \eta_x, \chi_x\rangle - \langle uu_x + vv_x), \chi_x\rangle \\ =& \langle \xi_{xt}, \chi_x\rangle + \langle u\xi_x, \chi_{xx}\rangle - \langle U\zeta_x, \chi_{xx}\rangle + \langle \eta_x - \Gamma_x, \chi_x\rangle \\ &+ \langle (u-U)w_x, \chi_{xx}\rangle - \langle uu_x - UU_x, \chi_x\rangle - \langle vv_x - VV_x, \chi_x\rangle.\end{aligned}$$

Taking $\chi = \zeta$, integrating by parts, and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\zeta_x\|^2 =& \langle \xi_{xt}, \zeta_x\rangle - \langle (u\xi_x)_x, \zeta_x\rangle + \frac{1}{2}\langle U_x, \zeta_x^2\rangle + \langle \eta_x - \Gamma_x, \zeta_x\rangle \\ &- \langle ((u-U)w_x)_x, \zeta_x\rangle - \langle uu_x - UU_x, \zeta_x\rangle - \langle vv_x - VV_x, \zeta_x\rangle \\ \le& \|\xi_{xt}\|\|\zeta_x\| + (|u|_\infty\|\xi_{xx}\| + |u_x|_\infty\|\xi_x\|)\|\zeta_x\| + \frac{1}{2}\|U_x\|_\infty\|\zeta_x\|^2 \\ &+ \|\eta_x - \Gamma_x\|\|\zeta_x\| + (|w_x|_\infty\|u-U\| + |w_{xx}|_\infty\|u_x - U_x\|)\|\zeta_x\| \\ &+ \|uu_x - UU_x\|\|\zeta_x\| + \|vv_x - VV_x\|\|\zeta_x\|.\end{aligned}$$

It follows from applications of Young's inequality that

$$\frac{d}{dt}\|\zeta_x\|^2 \le \|\xi_{xt}\|^2 + \|\xi_{xx}\|^2 + \|\eta_x - \Gamma_x\|^2 + \|u_x - U_x\|^2 + \|v_x - V_x\|^2 + C\|\zeta_x\|^2.$$

Integrating from 0 to t and applying Gronwall's inequality, we have

$$\|\zeta_x(t)\|^2 \le C\big\{\|\zeta_x(0)\|^2 + \int_0^t (\|\xi_{xt}\|^2 + \|\xi_{xx}\|^2 + \|\eta_x - \Gamma_x\|^2 + \|u_x - U_x\|^2 + \|v_x - V_x\|^2)ds\big\}.$$

This completes the proof. □

3. A Fully Discrete Finite Element Scheme on S_η

We now consider the fully discrete Crank-Nicolson type finite element Galerkin method. For a positive integer J, let k be a temporal step size such that $k = T/J$. Define $w^n = w(x, nk)$ for a continuous function $w : [0,T] \to L^2(I)$ and

$$\bar{\partial} w^n = \frac{w^{n+1} - w^n}{k}, \quad w^{n+1/2} = \frac{w^{n+1} + w^n}{2}, \quad n = 0, 1, \cdots, J-1.$$

Then the problem of a fully discrete approximation scheme for (1.2)-(1.3) is to find a pair of functions $\Gamma^n, \Psi^n \in V_\delta^r$ such that for $n = 1, 2, \cdots, J$,

$$\langle \bar{\partial}\Gamma^n + U^n \Gamma_x^{n+1/2} - V^n, \chi \rangle = 0 \tag{3.1}$$

and

$$\langle \bar{\partial}\Psi^n + U^n \Psi_x^{n+1/2} + \Gamma^{n+1/2} - \frac{1}{2}((U^n)^2 + (V^n)^2), \chi \rangle = 0. \tag{3.2}$$

Then we obtain the stability of solutions for (3.1)-(3.2).

Theorem 3.1. *Suppose $\|U_x\|_{L^\infty(L^\infty)} < \infty$. Then the solutions of (3.1)–(3.2) are stable, i.e., there is a constant C such that*

$$\|\Gamma^n\| \le C(\|\Gamma^0\| + k \sum_{m=0}^{n-1} \|V^m\|)$$

and

$$\|\Psi^n\| \le C\{\|\Psi^0\| + k \sum_{m=0}^{n-1} (\|(U^m)^2\| + \|(V^m)^2\| + \|\Gamma^m\|)\}.$$

Proof. If we take $\chi = \Gamma^{m+1/2}$, then the equation (3.1) becomes

$$\begin{aligned}\frac{1}{2}\bar{\partial}\|\Gamma^m\|^2 &= -\langle U^m \Gamma_x^{m+1/2}, \Gamma^{m+1/2} \rangle + \langle V^m, \Gamma^{m+1/2} \rangle \\ &= \frac{1}{2}\langle U_x^m, (\Gamma^{m+1/2})^2 \rangle + \langle V^m, \Gamma^{m+1/2} \rangle.\end{aligned}$$

It follows from Cauchy-Schwarz inequality and Young's inequality that

$$\begin{aligned}\|\Gamma^{m+1}\|^2 - \|\Gamma^m\|^2 &\le k\|V^m\|^2 + Ck\|\Gamma^{m+1/2}\|^2 \\ &\le k\|V^m\|^2 + \frac{1}{2}Ck(\|\Gamma^{m+1}\|^2 + \|\Gamma^m\|^2).\end{aligned}$$

Summing from $m = 0$ to $m = n-1$, we obtain

$$(1 - \frac{Ck}{2})\|\Gamma^n\|^2 \le (1 - \frac{Ck}{2})\|\Gamma^0\|^2 + k\sum_{m=0}^{n-1}\|V^m\|^2 + Ck\sum_{m=0}^{n-1}\|\Gamma^m\|^2.$$

If we now apply the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$, then we obtain the required inequality

$$\|\Gamma^n\|^2 \le C(\|\Gamma^0\|^2 + k\sum_{m=0}^{n-1}\|V^m\|^2).$$

Similarly, we obtain the second inequality in the theorem. □

Following the proof of Theorem 3.1, we may obtain estimates of $\|\Gamma_x^n\|^2$ and $\|\Psi_x^n\|^2$. If we take $\chi = \Gamma_{xx}^{m+1/2}$ in (3.1), then it becomes

$$\begin{aligned}\frac{1}{2}\bar{\partial}\|\Gamma_x^m\| &= \langle U^m\Gamma_x^{m+1/2}, \Gamma_{xx}^{m+1/2}\rangle + \langle V_x^m, \Gamma_x^{m+1/2}\rangle \\ &= -\frac{1}{2}\langle U_x^m, (\Gamma_x^{m+1/2})^2\rangle + \langle V_x^m, \Gamma_x^{m+1/2}\rangle.\end{aligned}$$

Applying Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned}\|\Gamma^{m+1}\|^2 - \|\Gamma^m\|^2 &\le k\|V_x^m\|^2 + Ck\|\Gamma_x^{m+1/2}\|^2 \\ &\le k\|V_x^m\|^2 + \frac{1}{2}Ck(\|\Gamma_x^{m+1}\|^2 + \|\Gamma_x^m\|^2).\end{aligned}$$

Summing the above inequality from $m = 0$ to $m = n-1$, we obtain

$$(1 - \frac{Ck}{2})\|\Gamma_x^n\|^2 \le (1 - \frac{Ck}{2})\|\Gamma_x^0\|^2 + k\sum_{m=0}^{n-1}\|V_x^m\|^2 + Ck\sum_{m=0}^{n-1}\|\Gamma_x^m\|^2.$$

It follows from the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$ that

$$\|\Gamma_x^n\|^2 \le C(\|\Gamma_x^0\|^2 + k\sum_{m=0}^{n-1}\|V_x^m\|^2). \tag{3.3}$$

Similarly, we may obtain

$$\|\Psi_x^n\| \le C\{\|\Psi_x^0\| + k\sum_{m=0}^{n-1}(\|(U_x^m)^2\| + \|(V_x^m)^2\| + \|\Gamma_x^m\|)\}. \tag{3.4}$$

For any function $w \in V_\delta^r$, let $\theta^n = \eta^n - w^n$ and $\rho^n = \Gamma^n - w^n$. Since the error $\eta^n - \Gamma^n = \theta^n - \rho^n$ and the approximation on θ^n are known from (2.5)–(2.6), it is enough to estimate the values on ρ^n.

Lemma 3.1. *Suppose $\|U_x\|_{L^\infty(L^\infty)} < \infty$. Then there is a constant C such that*

$$\|\rho^n\| \le C\big\{\|\rho^0\| + k^2 \max_t \|\eta_{ttt}\| + k\sum_{m=0}^{n-1}(\|\bar\partial\theta^m\| + \|\theta_x^m\| + \|u^m - U^m\| + \|v^m - V^m\|)\big\}.$$

Proof. It follows from (3.1) that for $\chi \in V_\delta^r$

$$\begin{aligned}\big\langle\bar\partial\rho^m,\chi\big\rangle =&\big\langle\bar\partial\Gamma^m,\chi\big\rangle - \big\langle\bar\partial w^m,\chi\big\rangle\\ =&-\big\langle U^m\Gamma_x^{m+1/2} - V^m,\chi\big\rangle - \big\langle\bar\partial w^n,\chi\big\rangle\\ &+\big\langle\eta_t^m + u^m\eta_x^{m+1/2} - v^m,\chi\big\rangle - \big\langle\bar\partial\eta^m,\chi\big\rangle + \big\langle\bar\partial\eta^m,\chi\big\rangle\\ =&\big\langle\eta_t^m - \bar\partial\eta^m,\chi\big\rangle + \big\langle\bar\partial\theta^m,\chi\big\rangle + \big\langle u^m\theta_x^{m+1/2},\chi\big\rangle - \big\langle U^m\rho_x^{m+1/2},\chi\big\rangle\\ &+\big\langle(u^m - U^m)w_x^{m+1/2},\chi\big\rangle - \big\langle v^m - V^m,\chi\big\rangle.\end{aligned}$$

Taking $\chi = \rho^{m+1/2}$ and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{2}\bar\partial\|\rho^m\|^2 \le&\|\bar\partial\theta^m\|\|\rho^{m+1/2}\| + |u^m|_\infty\|\theta_x^{m+1/2}\|\|\rho^{m+1/2}\| + \frac{1}{2}|U_x^m|_\infty\|\rho^{m+1/2}\|^2\\ &+\|\eta_t^m - \bar\partial\eta^m\|\|\rho^{m+1/2}\| + |w_x^{m+1/2}|_\infty\|u^m - U^m\|\|\rho^{m+1/2}\|\\ &+\|v^m - V^m\|\|\rho^{m+1/2}\|.\end{aligned}$$

Applications of Young's inequality imply

$$\begin{aligned}\|\rho^{m+1}\| - \|\rho^m\| \le& k\|\bar\partial\theta^m\|^2 + k\|\theta_x^{m+1/2}\|^2 + k\|\eta_t^m - \bar\partial\eta^m\|^2\\ &+ k\|u^m - U^m\|^2 + k\|v^m - V^m\|^2 + Ck\|\rho^{m+1/2}\|^2.\end{aligned}$$

Summing the above inequality from $m = 0$ to $m = n-1$, we obtain

$$\begin{aligned}(1 - \frac{Ck}{2})\|\rho^n\|^2 \le (1 - \frac{Ck}{2})\|\rho^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar\partial\theta^m\|^2 + \|\theta_x^{m+1/2}\|^2 + \|\eta_t^m - \bar\partial\eta^m\|^2\\ + \|u^m - U^m\|^2 + \|v^m - V^m\|^2) + Ck\sum_{m=0}^{n-1}\|\rho^m\|^2.\end{aligned}$$

If we apply the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$, then

$$\begin{aligned}\|\rho^n\|^2 \le C\Big\{&\|\rho^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar\partial\theta^m\|^2 + \|\theta_x^{m+1/2}\|^2\\ &+ \|\eta_t^m - \bar\partial\eta^m\|^2 + \|u^m - U^m\|^2 + \|v^m - V^m\|^2)\Big\}.\end{aligned}$$

This completes the proof. □

For an estimation on $\|\rho_x^n\|$, we obtain the following lemma.

Lemma 3.2. *Suppose $\|U_x\|_{L^\infty(L^\infty)} < \infty$. Then there is a constant C such that*

$$\|\rho_x^n\| \le C\big\{\|\rho_x^0\|+k^2\max_t\|\eta_{xttt}\|+k\sum_{m=0}^{n-1}(\|\bar\partial\theta_x^m\|+\|\theta_{xx}^m\|+\|u_x^m-U_x^m\|+\|v_x^m-V_x^m\|)\big\}.$$

Proof. It follows from (3.1) that for $\chi \in V_\delta^r$

$$\begin{aligned}\big\langle\bar\partial\rho^m,\chi\big\rangle =&\big\langle\eta_t^m-\bar\partial\eta^m,\chi\big\rangle+\big\langle\bar\partial\theta^m,\chi\big\rangle+\big\langle u^m\theta_x^{m+1/2},\chi\big\rangle-\big\langle U^m\rho_x^{m+1/2},\chi\big\rangle\\&+\big\langle(u^m-U^m)w_x^{m+1/2},\chi\big\rangle-\big\langle v^m-V^m,\chi\big\rangle.\end{aligned}$$

Taking $\chi = \rho_{xx}^{m+1/2}$ and applying Cauchy-Schwarz inequality after integration by parts, we obtain an inequality

$$\begin{aligned}\frac{1}{2}\bar\partial\|\rho_x^m\|^2 \le&\|\bar\partial\theta_x^m\|\|\rho_x^{m+1/2}\|+|u_x^m|_\infty\|\theta_x^{m+1/2}\|\|\rho_x^{m+1/2}\|+\frac{1}{2}|U_x^m|_\infty\|\rho_x^{m+1/2}\|^2\\&+|u^m|_\infty\|\theta_{xx}^{m+1/2}\|\|\rho_x^{m+1/2}\|+|u_x^m|_\infty\|\theta_x^{m+1/2}\|\|\rho_x^{m+1/2}\|\\&+\|\eta_{xt}^m-\bar\partial\eta_x^m\|\|\rho_x^{m+1/2}\|+|w_x^{m+1/2}|_\infty\|u_x^m-U_x^m\|\|\rho_x^{m+1/2}\|\\&+|w_{xx}^{m+1/2}|_\infty\|u^m-U^m\|\|\rho_x^{m+1/2}\|+\|v_x^m-V_x^m\|\|\rho_x^{m+1/2}\|.\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}\|\rho_x^{m+1}\|-\|\rho_x^m\| \le& k\|\bar\partial\theta_x^m\|^2+k\|\theta_{xx}^{m+1/2}\|^2+k\|\eta_{xt}^m-\bar\partial\eta_x^m\|^2+k\|u^m-U^m\|^2\\&+k\|u_x^m-U_x^m\|^2+k\|v_x^m-V_x^m\|^2+Ck\|\rho_x^{m+1/2}\|^2,\end{aligned}$$

where $C = 3+|u^m|_\infty+|u_x^m|_\infty+|U_x^m|_\infty+|w_x^m|_\infty+|w_{xx}^m|_\infty$. It follows from Poincare's inequality and summation from $m=0$ to $m=n-1$ that

$$\begin{aligned}(1-\frac{Ck}{2})\|\rho_x^n\|^2 \le (1-\frac{Ck}{2})\|\rho_x^0\|^2+k\sum_{m=0}^{n-1}(\|\bar\partial\theta_x^m\|^2+\|\theta_{xx}^{m+1/2}\|^2+\|\eta_{xt}^m-\bar\partial\eta_x^m\|^2\\+\|u_x^m-U_x^m\|^2+\|v_x^m-V_x^m\|^2)+Ck\sum_{m=0}^{n-1}\|\rho_x^m\|^2.\end{aligned}$$

If we now apply the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$, then we have

$$\|\rho_x^n\|^2 \le C\Big\{\|\rho_x^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar{\partial}\theta_x^m\|^2 + \|\theta_{xx}^{m+1/2}\|^2 + \|\eta_{xt}^m - \bar{\partial}\eta_x^m\|^2 + \|u_x^m - U_x^m\|^2 + \|v_x^m - V_x^m\|^2)\Big\}.$$

This completes the proof. □

Now for $w \in V_\delta^r$, let $\xi^n = \psi^n - w^n$ and $\zeta^n = \Psi^n - w^n$. Since the error $\psi^n - \Psi^n = \xi^n - \zeta^n$ and the approximation properties (2.5)–(2.6) hold for ζ^n, it is enough to estimate the values on ζ^n in order to obtain the error estimates on $\psi^n - \Psi^n$.

Lemma 3.3. *Suppose $\|U_x\|_{L^\infty(L^\infty)} < \infty$. Then there is a constant C such that*

$$\|\zeta^n\| \le C\Big\{\|\zeta^0\| + k^2\max_t\|\psi_{ttt}\| + k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi^m\| + \|\xi_x^m\| + \|\eta - \Gamma\| + \|u^m - U^m\| + \|v^m - V^m\|)\Big\}.$$

Proof. It follows from (3.2) that for $\chi \in V_\delta^r$

$$\begin{aligned}\langle\bar{\partial}\zeta^m, \chi\rangle =& \langle\bar{\partial}\xi^m, \chi\rangle + \langle u^m\xi_x^{m+1/2}, \chi\rangle - \langle U^m\zeta_x^{m+1/2}, \chi\rangle \\ &+ \langle\eta^{m+1/2} - \Gamma^{m+1/2}, \chi\rangle \\ &+ \langle\psi_t^m - \bar{\partial}\psi^m, \chi\rangle + \langle(u^m - U^m)w_x^m, \chi\rangle \\ &- \frac{1}{2}\langle(u^m + U^m)(u^m - U^m), \chi\rangle - \frac{1}{2}\langle(v^m + V^m)(v^m - V^m), \chi\rangle.\end{aligned}$$

Taking $\chi = \zeta^{m+1/2}$ and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{2}\bar{\partial}\|\zeta^m\|^2 \le& \|\bar{\partial}\xi^m\|\|\zeta^{m+1/2}\| + |u^m|_\infty\|\xi_x^{m+1/2}\|\|\zeta^{m+1/2}\| + \frac{1}{2}|U_x^m|_\infty\|\zeta^{m+1/2}\|^2 \\ &+ \|\eta^{m+1/2} - \Gamma^{m+1/2}\|\|\zeta^{m+1/2}\| + \|\psi_t^m - \bar{\partial}\psi^m\|\|\zeta^{m+1/2}\| \\ &+ |w_x^m|_\infty\|u^m - U^m\|\|\zeta^{m+1/2}\| \\ &+ \frac{1}{2}|u^m + U^m|_\infty\|u^m - U^m\|\|\zeta^{m+1/2}\| \\ &+ \frac{1}{2}|v^m + V^m|_\infty\|v^m - V^m\|\|\zeta^{m+1/2}\|.\end{aligned}$$

It follows from Young's inequality that

$$\begin{aligned}\bar{\partial}\|\zeta^m\|^2 \leq& \|\bar{\partial}\xi^m\|^2 + \|\xi_x^{m+1/2}\|^2 + \|\eta^{m+1/2} - \Gamma^{m+1/2}\|^2 + \|\psi_t^m - \bar{\partial}\psi^m\|^2 \\ &+ \|u^m - U^m\|^2 + \|v^m - V^m\|^2 + C\|\zeta^{m+1/2}\|^2.\end{aligned}$$

Summing from $m = 0$ to $m = n-1$, we obtain

$$\begin{aligned}(1 - \frac{Ck}{2})\|\zeta^n\|^2 \leq& (1 - \frac{Ck}{2})\|\zeta^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi^m\|^2 + \|\xi_x^{m+1/2}\|^2 \\ &+ \|\eta^{m+1/2} - \Gamma^{m+1/2}\|^2 + \|\psi_t^m - \bar{\partial}\psi^m\|^2 \\ &+ \|u^m - U^m\|^2 + \|v^m - V^m\|^2) + Ck\sum_{m=0}^{n-1}\|\zeta^m\|^2.\end{aligned}$$

If we apply the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$, then

$$\begin{aligned}\|\zeta^n\|^2 \leq& C\Big\{\|\zeta^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi^m\|^2 + \|\xi_x^{m+1/2}\|^2 + \|\eta^{m+1/2} - \Gamma^{m+1/2}\|^2 \\ &+ \|\psi_t^m - \bar{\partial}\psi^m\|^2 + \|u^m - U^m\|^2 + \|v^m - V^m\|^2)\Big\}.\end{aligned}$$

This completes the proof. □

Similarly, we obtain an estimate on $\|\xi_x^n\|$ as in the following lemma.

Lemma 3.4. *Suppose $\|U_x\|_{L^\infty(L^\infty)} < \infty$. Then there is a constant C such that*

$$\begin{aligned}\|\zeta_x^n\| \leq C\Big\{&\|\zeta_x^0\| + k^2\max_t\|\psi_{xttt}\| \\ &+ k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi_x^m\| + \|\xi_{xx}^m\| + \|\eta_x - \Gamma_x\| + \|u_x^m - U_x^m\| + \|v_x^m - V_x^m\|)\Big\}.\end{aligned}$$

Proof. It follows from (3.2) that for $\chi \in V_\delta^r$

$$\begin{aligned}\langle\bar{\partial}\zeta^m, \chi\rangle =& \langle\bar{\partial}\xi^m, \chi\rangle + \langle u^m\xi_x^{m+1/2}, \chi\rangle - \langle U^m\zeta_x^{m+1/2}, \chi\rangle \\ &+ \langle\eta^{m+1/2} - \Gamma^{m+1/2}, \chi\rangle \\ &+ \langle\psi_t^m - \bar{\partial}\psi^m, \chi\rangle + \langle(u^m - U^m)w_x^m, \chi\rangle \\ &- \frac{1}{2}\langle(u^m + U^m)(u^m - U^m), \chi\rangle - \frac{1}{2}\langle(v^m + V^m)(v^m - V^m), \chi\rangle.\end{aligned}$$

Taking $\chi = \zeta_{xx}^{m+1/2}$, integrating by parts and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\zeta_x^m\|^2 \leq \|\bar{\partial}\xi_x^m\|\|\zeta_x^{m+1/2}\| + |u_x^m|_\infty\|\xi_x^{m+1/2}\|\|\zeta_x^{m+1/2}\| \\
+ |u^m|_\infty\|\xi_{xx}^{m+1/2}\|\|\zeta_x^{m+1/2}\| + \frac{1}{2}|U_x^m|_\infty\|\zeta_x^{m+1/2}\|^2 \\
+ \|\eta_x^{m+1/2} - \Gamma_x^{m+1/2}\|\|\zeta_x^{m+1/2}\| + \|\psi_{xt}^m - \bar{\partial}\psi_x^m\|\|\zeta_x^{m+1/2}\| \\
+ |w_{xx}^m|_\infty\|u^m - U^m\|\|\zeta^{m+1/2}\| + |w_x^m|_\infty\|u_x^m - U_x^m\|\|\zeta_x^{m+1/2}\| \\
+ \frac{1}{2}(|u_x^m + U_x^m|_\infty\|u^m - U^m\| + |u^m + U^m|_\infty\|u_x^m - U_x^m\|)\|\zeta_x^{m+1/2}\| \\
+ \frac{1}{2}(|v_x^m + V_x^m|_\infty\|v^m - V^m\| + |v^m + V^m|_\infty\|v_x^m - V_x^m\|)\|\zeta_x^{m+1/2}\|.
\end{aligned}$$

It follows from Young's inequality and Poincare's inequality that

$$\begin{aligned}
\bar{\partial}\|\zeta_x^m\|^2 \leq &\|\bar{\partial}\xi_x^m\|^2 + \|\xi_{xx}^{m+1/2}\|^2 + \|\eta_x^{m+1/2} - \Gamma_x^{m+1/2}\|^2 + \|\psi_{xt}^m - \bar{\partial}\psi_x^m\|^2 \\
&+ \|u_x^m - U_x^m\|^2 + \|v_x^m - V_x^m\|^2 + C\|\zeta_x^{m+1/2}\|^2\,,
\end{aligned}$$

where $C = 3 + 2|u_x^m|_\infty + 2|U_x^m|_\infty + |w_{xx}^m|_\infty + |v_x^m|_\infty + |V_x^m|_\infty$. Summing from $m = 0$ to $m = n-1$, we obtain

$$\begin{aligned}
(1 - \frac{Ck}{2})\|\zeta_x^n\|^2 \leq &(1 - \frac{Ck}{2})\|\zeta_x^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi_x^m\|^2 + \|\xi_{xx}^{m+1/2}\|^2 \\
&+ \|\eta_x^{m+1/2} - \Gamma_x^{m+1/2}\|^2 + \|\psi_{xt}^m - \bar{\partial}\psi_x^m\|^2 \\
&+ \|u_x^m - U_x^m\|^2 + \|v_x^m - V_x^m\|^2) + Ck\sum_{m=0}^{n-1}\|\zeta_x^m\|^2.
\end{aligned}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $1 - \frac{Ck}{2} > 0$, we have

$$\begin{aligned}
\|\zeta_x^n\|^2 \leq &C\Big\{\|\zeta_x^0\|^2 + k\sum_{m=0}^{n-1}(\|\bar{\partial}\xi_x^m\|^2 + \|\xi_{xx}^{m+1/2}\|^2 + \|\eta_x^{m+1/2} - \Gamma_x^{m+1/2}\|^2 \\
&+ \|\psi_{xt}^m - \bar{\partial}\psi_x^m\|^2 + \|u_x^m - U_x^m\|^2 + \|v_x^m - V_x^m\|^2)\Big\}.
\end{aligned}$$

This completes the proof. $\square$

4. Finite Element Schemes on Ω_Γ

Consider the following steady state free boundary problem

$$\triangle\phi = 0 \quad \text{in } \Omega_\eta = I \times (0, \eta(x)), \tag{4.1}$$

$$\phi = \psi(x) \quad \text{on } S_\eta, \tag{4.2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } S_w, S_b. \tag{4.3}$$

Zhang and Babuska [25] have studied convergence of numerical solutions for a steady state free boundary value problem. We modify the results of Zhang and Babuska [25] for the problem (4.1)–(4.3) with periodic boundary conditions. Let $\Phi = \Phi(x,y)$ and $y = \Gamma(x)$ be approximate solutions of $\phi(x,y)$ and $\eta(x)$, respectively. That is, $\Phi = \Phi(x,y)$ and $y = \Gamma(x)$ satisfy the following:

$$\triangle\Phi = 0 \quad \text{in } \Omega_\Gamma, \tag{4.4}$$

$$\Phi(x, \Gamma(x)) = \Psi(x) \quad \text{on } S_\Gamma, \tag{4.5}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } S_w, S_b. \tag{4.6}$$

Since $\eta(x)$ cannot be equal to $\Gamma(x)$, the potential function $\phi(x,y)$ cannot be defined on $\Omega_\Gamma - \Omega_\eta$. Hence it is necessary to define an extension $\bar{\phi}(x,y)$ of $\phi(x,y)$. In order to simplify the notation, we will use ϕ instead of $\bar{\phi}$ hereafter. If Γ is a Lipschitz continuous function, then $\sqrt{1+\Gamma'(x)^2}$ is an L_∞-function. That is, there are positive constants L and M such that

$$L = \sup_{0<x<1} \sqrt{1+\Gamma'(x)^2} \quad \text{and} \quad M = \sup_{0<x<1} \Gamma(x).$$

Let $e = \phi - \Phi$ be the error. Then we obtain the following lemma.

Lemma 4.1. *Let e be the error function defined as above. Then*

$$\|e\|_{L^2(\Gamma)} \le \sqrt{2ML}(\|e\|_{L^2(S_b)} + \|\nabla e\|_{L^2(\Omega_\Gamma)}).$$

Proof. Note that

$$e(x, \Gamma(x)) = e(x, 0) + \int_0^{\Gamma(x)} \frac{\partial e}{\partial y}(x, s)\, ds.$$

If we apply Young's inequality and Cauchy-Schwarz inequality to the above relation, then

$$\begin{aligned} e(x,\Gamma(x))^2 &\le 2e(x,0)^2 + 2[\int_0^{\Gamma(x)} \frac{\partial e}{\partial y}(x,s)ds]^2 \\ &\le 2e(x,0)^2 + 2\int_0^{\Gamma(x)} 1^2 ds \int_0^{\Gamma(x)} [\frac{\partial e}{\partial y}(x,s)]^2 ds \\ &\le 2e(x,0)^2 + 2M\int_0^{\Gamma(x)} [\frac{\partial e}{\partial y}(x,s)]^2 ds \\ &\le 2e(x,0)^2 + 2M\int_0^{\Gamma(x)} |\nabla e(x,s)|^2 ds. \end{aligned}$$

Multiplying the above inequality by $\sqrt{1+(\Gamma'(x))^2}$ and integrating from 0 to 1, we obtain

$$\begin{aligned} \|e\|^2_{L^2(\Gamma)} &= \int_0^1 e(x,\Gamma(x))^2\sqrt{1+(\Gamma'(x))^2}dx \\ &\le L\int_0^1 e(x,\Gamma(x))^2 dx \\ &\le 2L\int_0^1 e(x,0)^2dx + 2ML\int_0^1\int_0^{\Gamma(x)} |\nabla e(x,s)|^2 dsdx \\ &\le 2ML[\|e\|^2_{L^2(S_b)} + \|\nabla e\|^2_{L^2(\Omega_\Gamma)}] \\ &\le 2ML[\|e\|_{L^2(S_b)} + \|\nabla e\|_{L^2(\Omega_\Gamma)}]^2. \end{aligned}$$

This completes the proof. □

We express the outward unit normal vector $n(x)$ of Ω_η at a point $(x,\eta(x))$ and the outward unit normal vector $n_\Gamma(x)$ of Ω_Γ at a point $(x,\Gamma(x))$ as

$$n(x) = \frac{1}{\sqrt{1+\eta'(x)^2}}(-\eta'(x),1) \quad \text{and} \quad n_\Gamma(x) = \frac{1}{\sqrt{1+\Gamma'(x)^2}}(-\Gamma'(x),1),$$

respectively. Then the two tangential directions $s(x)$ and $s_\Gamma(x)$ are given as

$$s(x) = \frac{1}{\sqrt{1+\eta'(x)^2}}(1,\eta'(x)), \quad s_\Gamma(x) = \frac{1}{\sqrt{1+\Gamma'(x)^2}}(1,\Gamma'(x)).$$

Let $\alpha(x)$ denote the angle from $n(x)$ to $n_\Gamma(x)$. Then we obtain the following lemma as in Zhang and Babuska [25].

Lemma 4.2. *Let $\alpha(x)$ be the angle defined as above. Then*

$$|\sin\alpha(x)| \le |\eta'(x) - \Gamma'(x)| \quad \textit{and} \quad |1-\cos\alpha(x)| \le |\eta'(x)-\Gamma'(x)|^2.$$

Let $v(x) = \phi_y(x,\eta(x))$ and $V(x) = \Phi_y(x,\Gamma(x))$. Then we obtain an error estimate of $\phi - \Phi$.

Lemma 4.3. *Let $e = \phi - \Phi$ be the error. Then there is a constant C such that*

$$\|\frac{\partial e}{\partial n}\|_{L^2(\Gamma)} \le C(\|\psi' - \Psi'\|_{L^2(I)} + \|v - V\|_{L^2(I)} + \|\eta - \Gamma\|_{L^2(I)} + \|\eta' - \Gamma'\|^2_{L^4(I)}).$$

Proof. Note that

$$\frac{\partial \phi}{\partial n_\Gamma}(x,\Gamma(x)) = \frac{\partial \phi}{\partial n}(x,\Gamma(x))\cos\alpha(x) - \frac{\partial \phi}{\partial s}(x,\Gamma(x))\sin\alpha(x).$$

Since

$$\frac{\partial \phi}{\partial n}(x,\Gamma(x)) = \frac{\partial \phi}{\partial n}(x,\eta(x)) + \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial n\partial y}(x,s)ds$$

and

$$\frac{\partial \phi}{\partial s}(x,\Gamma(x)) = \frac{\partial \phi}{\partial s}(x,\eta(x)) + \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial s\partial y}(x,s)ds,$$

we obtain the relation

$$\begin{aligned}\frac{\partial \phi}{\partial n_\Gamma}(x,\Gamma(x)) = &\frac{\partial \phi}{\partial n}(x,\eta(x))\cos\alpha(x) - \frac{\partial \phi}{\partial s}(x,\eta(x))\sin\alpha(x)\\ &+ \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial n\partial y}(x,s)ds\cos\alpha(x) - \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial s\partial y}(x,s)ds\sin\alpha(x).\end{aligned}$$

Hence it follows from the relation $e = \phi - \Phi$ that

$$\begin{aligned}\frac{\partial e}{\partial n_\Gamma}(x,\Gamma(x)) =& \frac{\partial \phi}{\partial n_\Gamma}(x,\Gamma(x)) - \frac{\partial \Phi}{\partial n_\Gamma}(x,\Gamma(x))\\ =& \frac{\partial \phi}{\partial n}(x,\eta(x))(\cos\alpha(x) - 1)\\ &+ \frac{\partial \phi}{\partial n}(x,\eta(x)) - \frac{\partial \Phi}{\partial n_\Gamma}(x,\Gamma(x))\\ &- \frac{\partial \phi}{\partial s}(x,\eta(x))\sin\alpha(x)\\ &+ \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial n\partial y}(x,s)ds\cos\alpha(x) - \int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial s\partial y}(x,s)ds\sin\alpha(x)\\ =:& I + II - III + IV.\end{aligned}$$

Firstly, it follows from Lemma 4.2 that

$$\begin{aligned}\int_0^1 I^2 dx &= \int_0^1 (\frac{\partial \phi}{\partial n}(x,\eta(x)))^2(\cos\alpha(x)-1)^2 dx \\ &\leq |\phi|^2_{1,\infty,\eta}\int_0^1 |\eta'(x)-\Gamma'(x)|^4 dx.\end{aligned} \tag{4.7}$$

Secondly, we estimate $\int_0^1 II^2 dx$. Since $\psi(x) = \phi(x,\eta(x))$ and $v(x) = \phi_y(x,\eta(x))$, $\phi_x(x,\eta(x)) = \psi'(x) - v(x)\eta'(x)$. Using this relation, we obtain

$$\begin{aligned}\frac{\partial \phi}{\partial n}(x,\eta(x)) &= \frac{1}{\sqrt{1+\eta'(x)^2}}(-\phi_x(x,\eta(x))\eta'(x)+\phi_y(x,\eta(x))) \\ &= -\frac{1}{\sqrt{1+\eta'(x)^2}}\psi'(x)+\sqrt{1+\eta'(x)^2}v(x).\end{aligned}$$

Similarly, we have

$$\frac{\partial \Phi}{\partial n_\Gamma}(x,\Gamma(x)) = -\frac{1}{\sqrt{1+\Gamma'(x)^2}}\Psi'(x)+\sqrt{1+\Gamma'(x)^2}V(x).$$

We now split II into two parts A and B as follows.

$$\begin{aligned}II =& \frac{\partial \phi}{\partial n}(x,\eta(x)) - \frac{\partial \Phi}{\partial n_\Gamma}(x,\Gamma(x)) \\ =& \frac{-\sqrt{1+\Gamma'(x)^2}\psi'(x)+\sqrt{1+\eta'(x)^2}\Psi'(x)}{\sqrt{1+\eta'(x)^2}\sqrt{1+\Gamma'(x)^2}} \\ &+ (\sqrt{1+\eta'(x)^2}v(x) - \sqrt{1+\Gamma'(x)^2}V(x)) \\ =:& A+B.\end{aligned}$$

Then A and B are expressed as, respectively,

$$A = \frac{\psi'(x)(\sqrt{1+\eta'(x)^2}-\sqrt{1+\Gamma'(x)^2})+\sqrt{1+\eta'(x)^2}(\psi'(x)-\Psi'(x))}{\sqrt{1+\eta'(x)^2}\sqrt{1+\Gamma'(x)^2}}$$

and

$$B = v(x)(\sqrt{1+\eta'(x)^2}-\sqrt{1+\Gamma'(x)^2})+\sqrt{1+\Gamma'(x)^2}(v(x)-V(x)).$$

Using the inequality $|\sqrt{1+a^2}-\sqrt{1+b^2}| \leq |a-b|$, we obtain

$$\begin{aligned}\int_0^1 II^2 dx \leq& 2|\psi|^2_{1,\infty,I}\int_0^1 |\eta'(x)-\Gamma'(x)|^2 dx + \int_0^1 |\psi'(x)-\Psi'(x)|^2 dx \\ &+ 2|f|^2_{\infty,I}\int_0^1 |\eta'(x)-\Gamma'(x)|^2 dx + L^2\int_0^1 |v(x)-V(x)|^2 dx.\end{aligned} \tag{4.8}$$

Next, it follows from Lemma 4.2 that

$$\begin{aligned}\int_0^1 III^2 dx &= \int_0^1 (\frac{\partial \phi}{\partial s}(x,\eta(x)))^2 \sin^2\alpha(x)dx \\ &\le |\phi|^2_{1,\infty,\eta}\int_0^1 |\eta'(x)-\Gamma'(x)|^2 dx.\end{aligned} \tag{4.9}$$

Finally, we obtain

$$\begin{aligned}\int_0^1 IV^2 dx \le& 2\int_0^1 (\int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial n\partial y}(x,s)ds)^2\cos^2\alpha(x) \\ &+ (\int_{\eta(x)}^{\Gamma(x)} \frac{\partial^2\phi}{\partial s\partial y}(x,s)ds)^2 \sin^2\alpha(x))dx \\ \le& 2|\phi|^2_{2,\infty,\triangle}\int_0^1 (\int_{\eta(x)}^{\Gamma(x)} 1ds)^2 dx \\ \le& 2|\phi|^2_{2,\infty,\triangle}\int_0^1 |\eta(x)-\Gamma(x)|^2 dx,\end{aligned} \tag{4.10}$$

where $\triangle = (\Omega_\Gamma - \Omega_\eta)\cup(\Omega_\eta - \Omega_\Gamma)$. Using the above estimates (4.7)–(4.10), we have

$$\begin{aligned}\|\frac{\partial w}{\partial n_\Gamma}\|^2_{L^2(\Gamma)} =& \int_0^1 (\frac{\partial w}{\partial n_\Gamma}(x,\Gamma(x)))^2\sqrt{1+\Gamma'(x)^2}dx \\ \le& 2L\int_0^1 (I^2 + II^2 + III^2 + IV^2)dx \\ \le& C(L,\phi,\psi,v)\int_0^1 (|\eta(x)-\Gamma(x)|^2 + |\eta'(x)-\Gamma'(x)|^2 \\ &+ |\eta'(x)-\Gamma'(x)|^4 + |\psi'(x)-\Psi'(x)|^2 + |v(x)-V(x)|^2)dx.\end{aligned}$$

Hence, we complete the proof. □

Theorem 4.1. *Let (ϕ,η) and (Φ,Γ) be the solution of problem (4.1)–(4.3) and (4.4)–(4.6), respectively. Then there exists a constant C such that*

$$\begin{aligned}&\|\nabla(\phi-\Phi)\|_{L^2(\Omega_\Gamma)} \\ &\quad \le C(\|\psi'-\Psi'\|_{L^2(I)} + \|v-V\|_{L^2(I)} + \|\eta-\Gamma\|_{L^2(I)} + \|\eta'-\Gamma'\|^2_{L^4(I)}).\end{aligned}$$

Proof. Let $e = \phi - \Phi$. Then applying Green's formula, Cauchy-Schwarz inequality and Lemma 4.1, we have

$$\begin{aligned}\|\nabla e\|^2_{L^2(\Omega_\Gamma)} &= \int_{\Omega_\Gamma} |\nabla e|^2 dxdy \\ &= \int_{\partial\Omega_\Gamma} e\frac{\partial e}{\partial n_\Gamma} ds_\Gamma \\ &= \int_\Gamma e\frac{\partial e}{\partial n_\Gamma} ds_\Gamma \\ &\le \|e\|_{L^2(\Gamma)} \|\frac{\partial e}{\partial n_\Gamma}\|_{L^2(\Gamma)} \\ &\le \sqrt{2ML}(\|e\|_{L^2(S_b)} + \|\nabla e\|_{L^2(\Omega_\Gamma)})\|\frac{\partial e}{\partial n_\Gamma}\|_{L^2(\Gamma)}.\end{aligned}$$

Using the inequality $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$, we obtain

$$(1 - ML\epsilon)\|\nabla e\|^2_{L^2(\Omega_\Gamma)} \le ML\epsilon\|e\|^2_{L^2(S_b)} + \frac{1}{\epsilon}\|\frac{\partial e}{\partial n_\Gamma}\|^2_{L^2(\Gamma)}.$$

Hence for sufficiently small ϵ such that $1 - ML\epsilon > 0$, we have

$$\|\nabla e\|^2_{L^2(\Omega_\Gamma)} \le \frac{ML\epsilon}{1 - ML\epsilon}\|e\|^2_{L^2(S_b)} + \frac{1}{\epsilon(1 - ML\epsilon)}\|\frac{\partial e}{\partial n_\Gamma}\|^2_{L^2(\Gamma)}.$$

Lemma 4.3 completes the proof. □

5. Concluding Remarks

We have established numerical stability and convergence for the nonlinear water-wave free surface problem. Finite element Galerkin methods were applied to obtain approximate solutions for the potential function of the problem using the idea of Zhang and Babuska [25] and the Crank-Nicolson method was used for the free surface conditions. It has been shown that H^1-error estimate for the potential function is described by the sum of H^1-error of the kinematic free boundary and L^2- and $W^{1,4}$-error of the free surface Bernoulli's equation.

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