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RATIONAL MAPS BETWEEN DOMAINS OF REAL BANACH SPACES

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Abstract: Fix an even integer $p \geq 2$. Fix $P \in \ell_p$ and $r > 0$. Let $\bar{B}(P,r) \subset \ell_p$ be the closed ball with center P and ratius r. Here we prove the existence of bijections $\phi_p : \ell_p \backslash \{0\} \to \ell_p$ and $\psi_p : \ell_p \backslash \bar{B}(P,r) \to \ell_p$ which are no-pole componentwise rational maps and whose inverse are real analytic.

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Let V, W be locally convex and Hausdorff real topological spaces and $\Omega \subseteq V$ an open subset. As in the complex case done in [2] (or see [3], pp. 60–61, for the real case) one can define the real vector space of all degree d continuous homogeneous polynomials defined on V and with values in W. A continuous W-valued polynomial on V (or a continuous polynomial map $V \to W$) is a finite sum of continuous W-valued homogeneous polynomials. A rational map from V into W is just a fraction P/Q with P, Q continuous W-valued polynomials on V and $Q \neq 0$, up to the obvious identification of P/Q with P_1/Q_1 if and only if $Q_1 P = Q P_1$. A rational map f from V into W is defined on Ω if for every $x \in \Omega$ there is a representation $f = P/Q$ with $Q(x) \neq 0$. In this case f defines a real analytic map $\Omega \to W$. Notice that with these definition for

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a rational map with a fixed representation P/Q both $\deg(P)$ and $\deg(Q)$ are finite. We want to weaken this condition when $\dim(W)$ is not finite, in such a way that (if for instance W is a separable Hilbert space with orthonormal basis e_i, $i \geq 1$, we may write $f(x) = (p_1(x)/q_1(x)e_1 + \cdots)$ with each p_i/q_i a rational $\mathbb{R}$-valued rational function, but $\deg(q_i)$ and/or $\deg(p_i)$, $i \geq 1$, not bounded. Let $g : \Omega \to W$ be a real analytic map. We fix a linear subspace Λ of the dual W'. We say that g is a componentwise rational map with respect to Λ and with no pole in Ω (or defined in Ω) if $\phi \circ f$ is rational for every $\phi \in \Lambda$.

Just notational modifications of the proof of [1], Theorem 2.1, will allow us to get the following results.

Theorem 1. *Let H be a separable real Hilbert space. Then there is bijection $\phi : H\backslash\{0\} \to H$ defined by a no-pole componentwise (with respect to any fixed orthonormal basis of H) rational map whose inverse $\phi^{-1} : H \to H\backslash\{0\}$ is real analytic.*

Theorem 2. *Let H be a separable real Hilbert space. Fix $P \in H$ and $r > 0$. Let $\bar{B}(P,r) \subset H$ be the closed ball with center P and ratius r. Then there is a bijection $\phi : H\backslash\bar{B}(P,r) \to H$ defined by a no-pole componentwise (with respect to the duals of any fixed orthonormal basis of H) rational map whose inverse $\phi^{-1} : H \to H\backslash\bar{B}(P,r)$ is real analytic.*

Theorem 3. *Fix an even integer $p \geq 2$. Fix $P \in \ell_p$ and $r > 0$. Let $\bar{B}(P,r) \subset \ell_p$ be the closed ball with center P and ratius r. There are bijections $\phi_p : \ell_p \to \ell_p\backslash\{0\}$ and $\psi_p : \ell_p \to \ell_p\backslash\bar{B}(P,r)$ which are no-pole componentwise rational maps with respect to the duals of the standard basis and whose inverse are real analytic.*

Question 1. Fix $1 \leq p < +\infty$. Is there a continuous polynomial bijection $\phi_p : \ell_p\backslash\{0\} \to \ell_p$ and $\psi_p : \ell_p\backslash\bar{B}(P,r) \to \ell_p$ whose inverse is real analytic

Guess. NO! Concerning the other direction $\ell_p\backslash\{0\} \to \ell_p$ a negative answer is easy for the following reason.

Proposition 1. *Let V, W be metrizable topological vector spaces and $\Omega \subsetneq V$ a proper open subset. Assume that for every sequence $\{x_k\}_{k\geq 1} \subset W$ without any accumulation point there is a continuous polynomial $u : W \to \mathbb{R}$ such that $\sup_{k\geq 1} |u(x_k)| = +\infty$. Then there is no surjective polynomial map $\psi : \Omega \to W$ which is a homeomorphism.*

Proof. Assume the existence of such a map ψ. Fix any $P \in \partial(\Omega)$ and any sequence $\{P_k\}_{k\geq 1} \subset \Omega$ converging to P and take the corresponding polynomial u. Set $x_k := \psi(P_k)$. Since $\psi^*(u)$ is a continuous polynomial, it extends continuously to P, contradiction. $\square$

Proof of Theorem 1. Fix an orthonormal basis of H and use it to identify H with the set of all sequences $x = (x_1, \dots) : x_i \in \mathbb{R}$ and $|x|^2 = \sum_{i=1}^{\infty} x_i^2 < +\infty$. Set $H_0 := \{x = (x_1, \dots) : |x|_0^2 := \sum_{i=1}^{\infty} (x_i/i \cdot 2^i)^2 < \infty$. Set $\alpha(t) = (1/4)(1 + t^2)^{-1}$ for all $0 \le t \le 1$. Hence $|\alpha'(t)| \le 1/2$ for $0 \le t \le 1$. For $0 \le t \le 1$ set $\tilde{p}(t) := (t, t^2, t^3, \dots)$. Hence $\tilde{p}(t) \in H$ for $0 \le t < 1$ and $\tilde{p}(1) \in H_0 \backslash H$. Set $\phi(y) := y + \tilde{p}(\alpha(|y|_0)$. The only difference with respect to the proof of [1], Theorem 2.1, is that we use t^2 instead of t in the definition of α to use the polynomial function $|y|_0^2$ instead of its square-root. The property $|\alpha'(t)| \le 1/2$ for $0 \le t \le 1$, is (as in [1]) sufficient to check the surjectivity of ϕ solving an easy fixed-point problem. $\square$

Proof of Theorem 2. Up to a translation and a dilatation we reduce to the case $P = 0$ and $r = 1$. Set $\tilde{\alpha}(t) := t^{-2}/4$ for all $t \ge 1$ and copy the proof of Theorem 1 using the function $\tilde{\alpha}$ instead of the function α. $\square$

Proof of Theorem 3. Set $H_{p,0} := \{x = (x_1, \dots) : |x|_p^p : \sum_{i=1}^{\infty} (x_i/i \cdot 2^i)^2 < \infty$. Use the functions $\alpha_p(t) := (1/4)(1 + t^2)^{-1}$ and $\tilde{\alpha}_p := t^{-p}/4$. $\square$

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