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p-BANACH COMMUTATIVE ALGEBRAS WHOSE RADICAL IDEALS ARE FINITELY GENERATED OR CLOSED

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Abstract: Let A be a commutative unitary and complete p-normed algebra, $0 < p \leq 1$. Consider the following conditions:

(i) every prime ideal of A is closed;

(ii) every radical ideal of A is closed;

(iii) every radical ideal of A is finitely generated.

(iv) every chain of prime ideals of A is stationary;

(v) every chain of radical ideals of A is stationary;

(vi) every radical ideal of A is the radical of a finitely generated ideal.

Then: (i) $\Longleftrightarrow$ (ii); (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv); (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) and (iii) $\Longrightarrow$ (ii).

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1. Closed and Finitely Generated Radical Ideals

Let A be a topological complex algebras. In many cases if a certain property Γ is true for all ideals of A, then another property Λ is true and sometimes

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also the reverse is true (see [2], [3], [6] and [7] for the case Γ = "closed " and Λ = "finitely generated"). Here we consider the same type of results, when the properties Γ and Λ are assumed to be true only for a class of ideal (e.g. all prime ideals or all radical ideal). We prove the following result.

Theorem 1. *Let A be a commutative unitary and complete p-normed algebra, $0 < p \leq 1$. Consider the following conditions:*
(i) every prime ideal of A is closed;
(ii) every radical ideal of A is closed;
(iii) every radical ideal of A is finitely generated.
(iv) every chain of prime ideals of A is stationary;
(v) every chain of radical ideals of A is stationary;
(vi) every radical ideal of A is the radical of a finitely generated ideal.
Then: (i) $\Longleftrightarrow$ (ii); (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv); (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) and (iii) $\Longrightarrow$ (ii).

For many properties of p-normes topological vector spaces, see [1], [5], p. 3, or [4], pp. 40–41.

Lemma 1. *Let A be any commutative ring and $I \subset A$ an ideal such that $x^2 \in I$ implies $x \in I$. Then I is radical.*

Proof. Fix $x \in A$ such that $x^k \in I$ for some integer $k > 0$. We need to check that $x \in I$. This is obviously true if $k \leq 2$. Assume $k \geq 3$ and let a be the first integer such that $k \leq 2^a$. Notice that $x^{2^a} = x^k x^{2^a - k} \in I$. Hence we get $x^{2^{a-1}} \in I$. Iterating this trick, we conclude. □

Lemma 2. *Let $(A, \| \|_p)$ be a commutative unitary and complete p-normed algebra, $0 < p \leq 1$. Then the 2-power map $\phi_2 : A^* \to A^*$ defined by $y \mapsto y^2$ is open.*

Proof. A is a Q-algebra (use [4], Lemma I.6.2, and the classical proof for Banach algebras). It is sufficient to prove it in a neighborhood of e. Take $x \in A$ such that $\|x - e\|_p \ll 1$. As in the classical Banach case use the expansion $\sum_{n \geq 0} \binom{\frac{1}{2}}{n}(x - e)$ to get $x^{\frac{1}{2}}$. □

Lemma 3. *Let $(A, \| \|_p)$ be a commutative unitary and complete p-normed algebra, $0 < p \leq 1$. Then $\bar{I}$ is radical.*

Proof. By Lemma 1 it is sufficient to prove that $x^2 \in \bar{I}$ implies $x \in \bar{I}$. Fix $x \in A$ such that $x^2 \in \bar{I}$ for some $k > 0$. By Lemma 2 the map $y \mapsto y^2$ is open at e. Hence there is an open neighborhood V of 0 such that $e + V \subset A^*$ and the set $S := \{x^2(e + t)^2\}_{t \in V}$ contains an open neighborhood U of x^2. Since

$x^2 \in \bar{I}$, there is $z \in I$ such that $z = x^2(e+t)^2$ for some $t \in V$. Since I is a radical ideal, we get $x(e+t) \in I$. Taking instead of V a fundamental system of neighborhoods of 0 contained in V we get $x \in \bar{I}$. □

Remark 1. Let A be any algebra such that all radical ideals are finitely generated. Since the union of a filtered set of radical ideals is radical, it is straightforward to check that the set of all radicals ideals of A satifies the Ascending Chain Condition.

Proof of Theorem 1. Since every prime ideal is radical, (ii) implies (i) and (iv) implies (v). Since every radical ideal of A is an intersection of prime ideals, (i) implies (ii).

(a) Here we assume (ii) and prove (v). Let $\{I_n\}_{n\geq 1}$ be an increasing sequence of radical ideal. The ideal $I := \bigcup_{n\geq 1} I_n$ is radical. Hence I is closed. Since each I_n is closed, we have $I = I_m$ for some $m \geq 1$ by Baire's Theorem.

(b) Here we assume (iii) and prove (ii). Assume that (ii) is false. By Remark 1 the set of all radical ideals of A satisfies the Ascending Chain Condition. Hence by Zorn Lemma there is a maximal non-closed radical ideal I. By Lemma 3 $\bar{I}$ is a radical ideal. Hence it is finitely generated. As in the proof of [6], Lemma 4, or [7], Lemma 7, we get $I = \bar{I}$, contradiction.

(c) The implication "(v) $\Longrightarrow$ (vi)" is elementary and true in any commutative ring. □

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