International Journal of Pure and Applied Mathematics
Volume 23 No. 3 2005, 335-342

A SUBALGORITHM FOR A QUASIDIFFERENTIABLE EQUATIONS

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Abstract: In this paper, we study a subalgorithm of the generalized Newton method for solving quasidifferentiabl equations. This method can be taken as a subalgorithm for a class of quasidifferentiable equations for certain systems of the nonsmooth optimization.

AMS Subject Classification: 49J52, 65H10, 90C30
Key Words: quasidifferential, subalgorithm, quasidifferentiable function, nonsmooth equations, Newton method

1. Introduction

The methods for solving various kinds of nonsmooth equations

$$H(x) = 0\,,$$

where $H : R^n \to R^n$ is a nonsmooth mapping have been the topic of intense research in recent years. Bobinson [5], [6] developed Newton methods for generalized equations. Pang [3] studied Newton methods for B-differentiable equations in the sense of Robinson [5] and established convergence theorems. Qi and Sun [7] proposed a Newton method for solving semismooth equations.

Received: June 24, 2005

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Other results on Newton and quasi-Newton methods for solving nonsmooth equations, nonlinear complementarity problems have appeared in [4], [2].

As we know, there exist various problems in mechanics, engineering and economics which mathematical models are turned to be the following optimization problem:

$$\min_{x\in R^n} f(x)\,, \tag{1.1}$$

where $f : R^n \to R$ is a function of class $C^{1,1}$ and ∇f is quasidifferentiable. The resolving of problem (1.1) is equivalent to figuring out the solution of the quasidifferentiable equations

$$\nabla f(x) = 0. \tag{1.2}$$

The generalized Newton methods for solving (1.2) are given as follows:

$$x^{k+1} = x^k - V_k^{-1}\nabla f(x)\,,$$

where V_k was an element of generalized Jacobian of ∇f at x. We can use this procedure to approximate the solutions of the quasidifferentiable equations (1.2).

In order to resolve the quasidifferentiable equations (1.2), we consider the following generality quasidifferentiable equations:

$$H(x) = 0\,, \tag{1.3}$$

where $H : R^n \to R^n$ is quasidifferentiable. Newton methods for solving nonsmooth equationsare (1.3) are given as follows:

$$x^{k+1} = x^k - V_k^{-1}H(x^k)\,, \tag{1.4}$$

where V_k was assumed to be an element of b-differential in [8] of H at x^k. In [9], the Newton method for solving equations (1.3) via the quasidifferential was given by

$$x^{k+1} = x^k - (U_k + V_k)^{-1}H(x^k), \quad [U_k, V_k] \in [\underline{\partial} H(x^k), \overline{\partial} H(x^k)]$$

and was proved to be convergence. However, the assumption for the convergence seems to be strong. Newton method, given in (1.4), can be used to approximate the solutions of the quasidifferentiable equations (1.3). In this procedure, it is important for us to resolve the quasidifferentiable equations

$$H(x^k) + H^{'}(x^k; y) = 0, \quad \text{for each fixed } x^k \in R^n. \tag{1.5}$$

And we know that the equations (1.5) themselves are nonsmooth equations and also semismooth.

In this paper, we will study the generalized Newton methods for solving quasidifferentiable equations (1.5). In fact, we will resolve the quasidifferentiable equations $H^{'}(x^k; y) = 0$ for each fixed $x^k \in R^n$. Furthermore, we will introduce a subalgorithm of a class of quasidifferentiable equations under a milder assumption which plays an important role in optimization problems.

2. Preliminaries

We start with the definition of a quasidifferentiable function.

As in [1], $F : R^n \to R^m$ is called quasidifferentiable at $x \in R^n$ in the sense of Demyanov and Rubinov, if it is directionally differentiable at x and there exists a pair of convex compact sets $\underline{\partial} F(x), \overline{\partial} F(x) \subset R^{m\times n}$ such that

$$F^{'}(x; y) = \max_{v\in\underline{\partial} F(x)} v^T y + \min_{w\in\overline{\partial} F(x)} w^T y, \qquad \forall y \in R^n.$$

The pair of sets $DF(x) = [\underline{\partial} F(x), \overline{\partial} F(x)]$ is called the quasidifferential of F at x; $\underline{\partial} F(x)$ and $\overline{\partial} F(x)$ are called the subdifferential and superdifferential, respectively. If $\overline{\partial} F(x) = \{0\}$, F is said to be subdifferentiable and if $\underline{\partial} F(x) = \{0\}$, F is said to be superdifferentiable.

Evidently, $F : R^n \to R^m$ is quasidifferentiable if and only if each component of F is quasidifferentiable. Let f_i be the i-th component of F with a quasidifferential $[\underline{\partial} f_i(x), \overline{\partial} f_i(x)$ and let

$$[\underline{\partial} F(x), \overline{\partial} F(x)] = [\underline{\partial} f_1(x), \overline{\partial} f_1(x)] \times \cdots \times [\underline{\partial} f_n(x), \overline{\partial} f_n(x)].$$

Then, $[\underline{\partial} F(x), \overline{\partial} F(x)]$ is the quasidifferential of F at x.

As we know, quasidifferentiable functions in the sense of Demyanov and Rubinov contain a large class of functions, for instance convex, concave, differentiable, non-Lipschitzian function [2] and other functions.

Let $S \subset R^n$ be convex compact set; the support function of the set S, denoted by $P_S(x)$, is defined by

$$P_S(x) = \max_{s\in S} s^T x.$$

From [1], it follows that P_S is a convex function on R^n with

$$\partial P_S(x) = \{s \in S | s^T x = P_S(x)\},$$

where ∂ denotes subdifferential in the sense of convex analysis.

3. An Algorithm and Related Properties

Let us consider the following quasidifferentiable equations

$$H(x^k) + H^{'}(x^k; y) = 0, \quad \text{for each fixed } \ x^k \in R^n,$$

where $H : R^n \to R^n$ is quasidifferentiable with a quasidifferential $[\underline{\partial} H(x^k), \overline{\partial} H(x^k)]$. As for each fixed $x^k \in R^n$, $H(x^k)$ is a constant value. So the above equations are equivalent to resolving the following quasidifferentiable equations

$$H^{'}(x^k; y) = 0, \quad \text{for each fixed } \ x^k \in R^n. \tag{3.1}$$

For each fixed $x^k \in R^n$, the quasidifferentiable equations (3.1) are also equivalent to

$$\max_{v \in \underline{\partial} H_i(x^k)} vy - \max_{w \in -\overline{\partial} H_i(x^k)} wy = 0, \quad i = 1, \cdots, n. \tag{3.2}$$

Quoting an assistant variant $z_i = (z_{i_1}, \cdots, z_{i_n})^T$, $i = 1, \cdots, n$, then equations (3.2) equate to

$$\begin{aligned} &\max_{v \in \underline{\partial} H_i(x^k)} vy + z_i = 0, \\ -&\max_{w \in -\overline{\partial} H_i(x^k)} wy - z_i = 0, \quad i = 1, \cdots, n. \end{aligned} \tag{3.3}$$

Equations (3.3) are still quasidifferentiable equations with variable $(y, z_i) \in R^{2n}$, where $y, z_i \in R^n$. Evidently:

$$\partial(\max_{v \in \underline{\partial} H_i(x^k)} vy + z_i) = (\partial P_{\underline{\partial} H_i(x^k)}(y) \quad I), \tag{3.4}$$

$$\partial(-\max_{w \in -\overline{\partial} H_i(x^k)} wy - z_i) = (-\partial P_{-\overline{\partial} H_i(x^k)}(y) \quad - I), \quad i = 1, \cdots, n. \tag{3.5}$$

Thus

$$\partial \begin{pmatrix} \max_{v \in \underline{\partial} H_i(x^k)} vy + z_i \\ -\max_{w \in -\overline{\partial} H_i(x^k)} wy - z_i \end{pmatrix} = \begin{pmatrix} \partial P_{\underline{\partial} H_i(x^k)}(y) & I \\ -\partial P_{-\overline{\partial} H_i(x^k)}(y) & - I \end{pmatrix}, \quad i = 1, \cdots, n. \tag{3.6}$$

The right-hand sides of (3.4) and (3.5) are both $n \times 2n$ order identity matrixs. It is $2n \times 2n$ order matrix in the right-hand side of (3.6), i.e.

$$(\partial P_{\underline{\partial} H_i(x^k)}(y) \quad I) = \{(V^{(i_1)}_{n\times n} \quad I_{n\times n}) \mid V^{(i_1)}_{n\times n} \in \partial P_{\underline{\partial} H_i(x^k)}(y)\},$$
$$(-\partial P_{-\overline{\partial} H_i(x^k)}(y) \quad -I) = \{(-V^{(i_2)}_{n\times n} \quad -I_{n\times n}) \mid V^{(i_2)}_{n\times n} \in \partial P_{-\overline{\partial} H_i(x^k)}(y)\},$$
$$i = 1, \cdots, n.$$

Then

$$\begin{pmatrix} \partial P_{\underline{\partial} H_i(x^k)}(y) & I \\ -\partial P_{-\overline{\partial} H_i(x^k)}(y) & -I \end{pmatrix} = \{\begin{pmatrix} V^{(i_1)}_{n\times n} & I_{n\times n} \\ -V^{(i_2)}_{n\times n} & -I_{n\times n} \end{pmatrix}$$
$$\mid V^{(i_1)}_{n\times n} \in \partial P_{\underline{\partial} H_i(x^k)}(y), V^{(i_2)}_{n\times n} \in \partial P_{-\overline{\partial} H_i(x^k)}(y)\},$$

$i = 1, \cdots, n$, where $I_{n\times n}$ denotes $n \times n$ order identity matrix. We denote by

$$V^i = \begin{pmatrix} V^{(i_1)}_{n\times n} & I_{n\times n} \\ -V^{(i_2)}_{n\times n} & -I_{n\times n} \end{pmatrix} \in \begin{pmatrix} \partial P_{\underline{\partial} H_i(x^k)}(y) & I \\ -\partial P_{-\overline{\partial} H_i(x^k)}(y) & -I \end{pmatrix}, \; i = 1, \cdots, n.$$

Applying iterative method (1.4) to equations (3.2), we have

$$\begin{pmatrix} x_i^{k+1} \\ z_i^{k+1} \end{pmatrix} = \begin{pmatrix} x_i^k \\ z_i^k \end{pmatrix} - (V_k^i)^{-1} \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i^k \\ -\max\limits_{w \in -\overline{\partial} H_i(x^k)} wy - z_i^k \end{pmatrix},$$

$$V_k^i \in \partial \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i^k \\ -\max\limits_{w \in -\overline{\partial} H_i(x^k)} wy - z_i^k \end{pmatrix}, \; i = 1, \cdots, n.$$

It is obtained that

$$V_k^i \begin{pmatrix} x_i^{k+1} \\ z_i^{k+1} \end{pmatrix} = V_k^i \begin{pmatrix} x_i^k \\ z_i^k \end{pmatrix} - \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i^k \\ -\max\limits_{w \in -\overline{\partial} H_i(x^k)} wy - z_i^k \end{pmatrix},$$
$$i = 1, \cdots, n. \quad (3.7)$$

Noting that

$$V_k^i = \begin{pmatrix} V_k^{(i_1)} & I \\ -V_k^{(i_2)} & -I \end{pmatrix}, \quad V_k^{(i_1)} \in \partial P_{\underline{\partial} H_i(x^k)}(y),$$

$$V_k^{(i_2)} \in \partial P_{-\overline{\partial} H_i(x^k)}(y), \quad i = 1, \cdots, n. \quad (3.8)$$

Combining (3.7) together with (3.8) leads to

$$\begin{pmatrix} V_k^{(i_1)} x_i^{k+1} + z_i^{k+1} \\ -V_k^{(i_2)} x_i^{k+1} - z_i^{k+1} \end{pmatrix} = \begin{pmatrix} V_k^{(i_1)} x_i^k + z_i^k \\ -V_k^{(i_2)} x_i^k - z_i^k \end{pmatrix} - \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i^k \\ - \max\limits_{w \in -\overline{\partial} H_i(x^k)} wy - z_i^k \end{pmatrix},$$

$i = 1, \cdots, n$. Via elimination theory, one has that

$$(V_k^{(i_1)} + V_k^{(i_2)}) x_i^{k+1} = (V_k^{(i_1)} + V_k^{(i_2)}) x_i^k - (\max_{v \in \underline{\partial} H_i(x^k)} vy - \max_{w \in -\overline{\partial} H_i(x^k)} wy), \quad i = 1, \cdots, n.$$

When $V_k^{(i_1)} - V_k^{(i_2)}$ is nonsingular, we obtain that

$$x_i^{k+1} = x_i^k - (V_k^{(i_1)} - V_k^{(i_2)})^{-1} (\max_{v \in \underline{\partial} H_i(x^k)} vy - \max_{w \in -\overline{\partial} H_i(x^k)} wy),$$
$$V_k^{(i_1)} \in \partial P_{\underline{\partial} H_i(x^k)}(y), V_k^{(i_2)} \in \partial P_{-\overline{\partial} H_i(x^k)}(y), \; i = 1, \cdots, n. \quad (3.9)$$

Iterative method (3.9) can enable us to resolve quasidifferentiable equations (3.1).

Proposition 3.1. *Let* $V^{(i_1)} \in \partial P_{\underline{\partial} H_i(x^k)}(y), V^{(i_2)} \in \partial P_{-\overline{\partial} H_i(x^k)}(y), \; i = 1, \cdots, n$. *Then*

$$V^i = \begin{pmatrix} V^{(i_1)} & I \\ -V^{(i_2)} & -I \end{pmatrix} \in \partial \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i \\ - \max\limits_{w \in -\overline{\partial} H_i(x^k)} wy - z_i \end{pmatrix} \quad (i = 1, \cdots, n)$$

is nonsingular if and only if $V^{(i_1)} - V^{(i_2)}$ $(i = 1, \cdots, n)$ *is nonsingular.*

Proof. By means of elementary transformation in matrix

$$\begin{pmatrix} V^{(i_1)} & I \\ -V^{(i_2)} & -I \end{pmatrix},$$

we obtain that

$$V^i = \begin{pmatrix} V^{(i_1)} & I \\ -V^{(i_2)} & -I \end{pmatrix} \to \begin{pmatrix} V^{(i_1)} - V^{(i_2)} & 0 \\ -V^{(i_2)} & -I \end{pmatrix}, i = 1, \cdots, n. \quad (3.10)$$

From (3.10), one has that V^i $(i = 1, \cdots, n)$ is nonsingular if and only if $V^{(i_1)} - V^{(i_2)}$ $(i = 1, \cdots, n)$ is nonsingular. □

Let y^* be a solution of equations (3.2) and (y^*, z^*) be a solution of the equivalent equations (3.3). Let any

$$V^{(i_1)} \in \partial P_{\underline{\partial} H_i(x^k)}(y^*), V^{(i_2)} \in \partial P_{-\overline{\partial} H_i(x^k)}(y^*) \qquad (i = 1, \cdots, n)$$

satisfy that $V^{(i_1)} - V^{(i_2)}$ $(i = 1, \cdots, n)$ is nonsingular. By virtue of Proposition 3.1, we know that it is nonsingular for any

$$V^i \in \partial \begin{pmatrix} \max\limits_{v \in \underline{\partial} H_i(x^k)} vy + z_i \\ - \max\limits_{w \in -\overline{\partial} H_i(x^k)} w^y - z_i \end{pmatrix}_{|_{y=y^*, z=z^*}}, \quad i = 1, \cdots, n.$$

Thus, iterative method (3.9) is well-posed and the sequence $\{x^k\}$ converges to x^* superlinearly, it follows from that it is obtained from method (1.2) by iterative, that is to say, they have the same convergence.

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