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ON CERTAIN RANDOM WALKS ON Sol($\mathbb{Q}_p$)

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Abstract: We give central and off-diagonal estimates for the transition kernels corresponding to certain random walks on the p-adic group $\mathbb{Q}_p^*$ semidirect product with $\mathbb{Q}_p^2$, where $\mathbb{Q}_p^*$ acts on $\mathbb{Q}_p^2$ by $(x, y) \longrightarrow (zx, z^{-1}y)$, $z \in \mathbb{Q}_p^*$.

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1. Introduction

Let p be a fixed prime and let $\mathbb{Q}_p$ denote the field of p-adic numbers (cf. [1], [4]). We shall assume that $\mathbb{Q}_p$ is equipped with its standard absolute value $|.|$. We shall denote by $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p^*,\ \ |x| = 1\}$, where $\mathbb{Q}_p^*$ denotes the multiplicative group of $\mathbb{Q}_p$ and by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p,\ \ |x| \leq 1\}$. Let dx denote the Haar measure on $\mathbb{Q}_p$ normalized by $dx(\mathbb{Z}_p) = 1$ and let d^*x denote the Haar measure on $\mathbb{Q}_p^*$ normalized by $d^*x(\mathbb{Z}_p^*) = 1$. Let

$$\mathrm{Sol}\,(\mathbb{Q}_p) = \mathbb{Q}_p^2 \ltimes \mathbb{Q}_p^* \tag{1}$$

denote the semidirect product of $\mathbb{Q}_p^*$ with the vector space $\mathbb{Q}_p^2$, where $\mathbb{Q}_p^*$ acts on $\mathbb{Q}_p^2$ by

$$(x, y) \longrightarrow (zx, z^{-1}y), \quad (x, y) \in \mathbb{Q}_p^2, \quad z \in \mathbb{Q}_p^*. \tag{2}$$

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The group defined by (1) is a p-adic version of the real Lie group Sol which gives one of the eight model geometries used to describe 3-manifolds in Thurston's geometrization conjecture. It is also one of the simplest examples of non-Abelian compactly generated p-adic groups (in fact the simplest if we restrict ourselves to unimodular groups).

Little is known about random walks on p-adic groups. A first class of examples was considered by the author in [7], [8] (cf. also [9], where a general lower bound was obtained for amenable p-adic groups). The examples studied in [7] and [8] satisfy two assumptions: unimodularity of the group G and symmetry of the random walk. Our aim here is to to focus on the symmetry assumption and study the behaviour of a "simple" non-symmetric random random walk in $\mathrm{Sol}(\mathbb{Q}_p)$. The considerations related to the non-unimodularity will be discussed in a forthcoming paper.

Let $\lambda > 0$ and let $d\mu(g) \in \mathbb{P}(\mathrm{Sol}(\mathbb{Q}_p))$ the probability measure on $\mathrm{Sol}(\mathbb{Q}_p)$ defined by

$$d\mu(g) = \left(\frac{p\lambda}{1+\lambda} I_{\mathbb{Z}_p}(x) I_{p\mathbb{Z}_p}(y) I_{p\mathbb{Z}_p^*}(z) + \frac{p}{1+\lambda} I_{p\mathbb{Z}_p}(x) I_{\mathbb{Z}_p}(y) I_{p^{-1}\mathbb{Z}_p^*}(z) \right) dg\,, \quad (3)$$

where $dg = dx dy d^*z$ denotes the Haar measure on $\mathrm{Sol}(\mathbb{Q}_p)$ and where the notation $I_A(.)$ is used to denote the indicator of a subset A. Let $d\mu^{*n}(g) = d(\mu * ... * \mu)(g) = \varphi_n(g)dg$, $n = 1, 2, ...$ denote the n-fold convolution of the probability measure μ. It is easy to verify that $\mathrm{Sol}(\mathbb{Q}_p)$ is generated by the support of μ, i.e. $\mathrm{Sol}(\mathbb{Q}_p) = \bigcup_{n\geq 1} (\mathrm{supp}(\mu))^n$ and that $\mathrm{Sol}(\mathbb{Q}_p)$ is of exponential volume growth (cf. [3], [13]). Let us observe that, in contrast with the real case, all the compactly generated semidirect products $\mathbb{Q}_p^m \ltimes (\mathbb{Q}_p^*)^l$ are of exponential growth (cf. [13]).

Theorem 1. *Let the notations be as above. We have then:*
(i) Upper central estimate:

$$\varphi_n(e) \leq C \left(\frac{2\lambda^{1/2}}{1+\lambda} \right)^n e^{-cn^{1/3}}, \quad n = 1, 2, ..., \quad (4)$$

where C, $c > 0$.
(ii) Lower central estimate:

$$\varphi_{2n}(e) \geq c \left(\frac{2\lambda^{1/2}}{1+\lambda} \right)^n e^{-Cn^{1/3}}, \quad n = 1, 2, ..., \quad (5)$$

where C, $c > 0$.

(iii) Upper off-diagonal estimate:

$$\varphi_n(g) \leq C_\epsilon n^{-1/2+\epsilon} \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n |z|^{-\frac{\log(\lambda)}{2\log(p)}} \min\left(1, |z^{-1}x|^{-1}\right) \min\left(1, |zy|^{-1}\right) \quad (6)$$

$$\times \exp\left(-\frac{(\log^+ |z^{-1}x|)^2 + (\log^+ |zy|)^2 + (\log |z|)^2}{C_\epsilon n}\right)$$

$$g = (x, y, z) \in \mathrm{Sol}\,(\mathbb{Q}_p), \quad n = 1, 2, \ldots,$$

where $\epsilon > 0$ *can be taken arbitrarily small and where* $C_\epsilon > 0$ *depends on* ϵ.

Let us observe that in the symmetric case (which corresponds to the case $\lambda = 1$), the upper estimate (4) is an immediate consequence of Hebisch and Saloff-Coste work (cf. [6]). Hebisch and Saloff-Coste established a similar estimate for symmetric Markov chains with bounded increments on compactly generated groups with exponential volume growth. The symmetry of the transition kernel plays a crucial rôle in the approach developped in [6] and it seems difficult to incorporate the contribution of the drift (i.e. the factor $\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n$) in this approach.

Concerning the central lower estimates one can distinguish several approachs. The first consists in deducing off-diagonal Gaussian estimates from the central upper estimate and integrate them on balls with appropriate radius. This approach works very well in the polynomial growth case (cf. [6]). The second approach (cf. [5], [10], [11], [12]) is geometric and consists in building Følner subsets adapted to some functionnal inequalities of the type *anti*-Faber-Krahn (cf. [5]) which imply the good central lower bounds for $\varphi_n(e)$. The geometrical approach, which applies in many settings (Riemannian manifolds, graphs, Lie groups, discrete groups, etc.), uses also in a crucial manner the symmetry properties of the kernel and seems difficult to apply in our case. The third approach is probabilistic (cf. [9], [16]) and consists in establishing lower estimates for the probabilities of certain events and using these probabilities in estimating $\varphi_n(e)$. This approach was used in [9] to prove that the $e^{-cn^{1/3}}$ lower bound holds for symmetric random walk on general amenable p-adic groups. This can be applied here in the case $\lambda = 1$.

Let us observe finally that the off-diagonal estimate (6) follows in the symmetric case from the results obtained in [7].

Our estimates are based on an explicit expression for $\varphi_n(g)$ which is established in Section 2. We first express the convolution powers of φ as iterated integrals of appropriate characteristic functions over $\mathbb{Q}_p$ and $\mathbb{Q}_p^*$. We then derive a representation of $\varphi_n(g)$ using only integrals over the multiplicative group $\mathbb{Q}_p^*$ (cf. formula (17) below). The fact that $\mathbb{Q}_p^*$ can be identified to a direct product of $\mathbb{Z}$ and a compact group motivates the translation of (17) into a probabilistic formula. It is this probabilistic interpretation which allows us to relate our estimates to well known potential theoretic properties of simple random walks on the integers. In Section 3 we prove the upper estimate (4) and the lower estimate (5). Section 4 is devoted to the proof of the off-diagonal estimate (6).

2. A Formula for the Convolution Powers

Let φ_n $(n = 1, 2, ...)$ be as in Theorem 1 and let $g = (\xi, \eta, \zeta) \in \mathrm{Sol}\,(\mathbb{Q}_p)$. We have:

$$\begin{aligned}
&\varphi_{n+1}(g) = \int \varphi_n(h)\varphi_1(h^{-1}g)dh = \\
&\int \varphi_1(h^{-1}g)d\mu^{*n}(h) = \int_{\mathrm{Sol}\,(\mathbb{Q}_p)} \cdots \int_{\mathrm{Sol}\,(\mathbb{Q}_p)} \varphi_1\Big((g_1 ... g_n)^{-1}\, g\Big) d\mu(g_1) ... d\mu(g_n) \\
&\qquad = \int_{\mathrm{Sol}\,(\mathbb{Q}_p)} \cdots \int_{\mathrm{Sol}\,(\mathbb{Q}_p)} \varphi_1\Big(- (x_1 + z_1x_2 + \cdots + z_1 ... z_{n-1}x_n) \\
&\qquad\quad + (z_1 ... z_n)^{-1}\xi, - \big(y_1 + z_1^{-1}y_2 + \cdots + (z_1 ... z_{n-1})^{-1}y_n\big) \\
&\qquad\quad + (z_1 ... z_n)\eta; (z_1 \ldots z_n)^{-1}\zeta \Big) \varphi_1(x_1, y_1; z_1) ... \varphi(x_n, y_n; z_n) \\
&\qquad\qquad\qquad\qquad\qquad\qquad dx_1dy_1d^*z_1 ... dx_ndy_nd^*z_n.
\end{aligned}$$

Using Fubini and the changes of variable $(z_1 ... z_{i-1})x_i \leftrightarrow x_i$, $(z_1 ... z_{i-1})^{-1}y_i \leftrightarrow y_i$, $i = 2, ..., n$, we deduce

$$\begin{aligned}
\varphi_{n+1}(g) = &\int_{\mathbb{Q}_p^*} \cdots \int_{\mathbb{Q}_p^*} \Bigg[\int_{\mathbb{Q}_p^2} \cdots \int_{\mathbb{Q}_p^2} \Bigg[\varphi_1\Big(- (x_1 + x_2 + ... + x_n) \\
&+ (z_1 ... z_n)^{-1}\xi, - (y_1 + y_2 + ... + y_n) + (z_1 ... z_n)\eta; (z_1 ... z_n)^{-1}\zeta \Big) \\
&\varphi_1(x_1, y_1; z_1)\varphi_1(z_1^{-1}x_2, z_1y_2; z_2) \ldots \varphi_1((z_1 ... z_{n-1})^{-1}x_n, \\
&(z_1 ... z_{n-1})y_n; z_n) \Bigg] dx_1dy_1 ... dx_ndy_n \Bigg] d^*z_1 ... d^*z_n, \quad (7)
\end{aligned}$$

where we used the unimodularity of the action (2). Let us now observe that $\varphi_1(g)$ can be rewritten as follows

$$\varphi_1(g) = \max\left(|z|, |z|^{-1}\right) \frac{\lambda}{1+\lambda} I_{\max(1,|z|)\mathbb{Z}_p}(x) I_{\max(1,|z|^{-1})\mathbb{Z}_p}(y) I_{p\mathbb{Z}_p^*}(z)$$
$$+ \max\left(|z|, |z|^{-1}\right) \frac{1}{1+\lambda} I_{\max(1,|z|)\mathbb{Z}_p}(x) I_{\max(1,|z|^{-1})\mathbb{Z}_p}(y) I_{p^{-1}\mathbb{Z}_p^*}(z),$$
$$g = (x, y; z) \in \mathrm{Sol}\,(\mathbb{Q}_p),$$

which also can be expressed as

$$\varphi_1(g) = \max\left(|z|, |z|^{-1}\right) I_{\max(1,|z|)\mathbb{Z}_p}(x) I_{\max(1,|z|^{-1})\mathbb{Z}_p}(y)$$
$$\times \left(\frac{\lambda}{1+\lambda} I_{p\mathbb{Z}_p^*}(z) + \frac{1}{1+\lambda} I_{p^{-1}\mathbb{Z}_p^*}(z)\right), \quad g = (x, y, z) \in \mathrm{Sol}\,(\mathbb{Q}_p).$$

Let

$$d\tilde{\mu}(z) = m(z) d^*z \in \mathbb{P}(\mathbb{Q}_p^*) \tag{8}$$

denote the probability measure on $\mathbb{Q}_p^*$ defined by

$$m(z) = \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{-1} |z|^{\frac{\log(\lambda)}{2\log(p)}} \left(\frac{\lambda}{1+\lambda} I_{p\mathbb{Z}_p^*}(z) + \frac{1}{1+\lambda} I_{p^{-1}\mathbb{Z}_p^*}(z)\right), \qquad z \in \mathbb{Q}_p^*.$$

With these notations we have

$$\varphi_1(g) = \left(\frac{2\lambda^{1/2}}{1+\lambda}\right) |z|^{-\frac{\log(\lambda)}{2\log(p)}} \max\left(|z|, |z|^{-1}\right)$$
$$\times I_{\max(1,|z|)\mathbb{Z}_p}(x) I_{\max(1,|z|^{-1})\mathbb{Z}_p}(y) m(z), \tag{9}$$
$$g = (x, y, z) \in \mathrm{Sol}\,(\mathbb{Q}_p).$$

Let us consider the product

$$\varphi_1(x_1, y_1; z_1)\varphi_1(z_1^{-1}x_2, z_1y_2; z_2) \ldots \varphi_1((z_1 \ldots z_{n-1})^{-1}x_n, (z_1 \ldots z_{n-1})y_n; z_n)$$

that appears in the formula (7). By (9), this product is equal to

$$\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \prod_{i=1}^{n} |z_i|^{-\frac{\log(\lambda)}{2\log(p)}} \max\left(|z_i|, |z_i|^{-1}\right) m(z_i)$$

$$\times I_{\max(1,|z_1|)\mathbb{Z}_p}(x_1)\prod_{i=2}^{n} I_{|z_1...z_{i-1}|^{-1}\max(1,|z_i|)\mathbb{Z}_p}(x_i)$$
$$\times I_{\max(1,|z_1|^{-1})\mathbb{Z}_p}(y_1)\prod_{i=2}^{n} I_{|z_1...z_{i-1}|\max(1,|z_i|^{-1})\mathbb{Z}_p}(y_i). \quad (10)$$

We have on the other hand (also by (9))

$$\varphi_1\bigg(-(x_1+x_2+....+x_n)+(z_1...z_n)^{-1}\xi, -(y_1+y_2+....+y_n)$$
$$+(z_1...z_n)\eta; (z_1...z_n)^{-1}\zeta\bigg) = \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)|z_1...z_n|^{\frac{\log(\lambda)}{2\log(p)}}|\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}$$
$$\times \max\left(|z_1...z_n||\zeta|^{-1}, |z_1...z_n|^{-1}|\zeta|\right) m\left((z_1...z_n)^{-1}\zeta\right)$$
$$\times I_{\max(1,|z_1...z_n|^{-1}|\zeta|)\mathbb{Z}_p}\left(x_1+x_2+....+x_n-(z_1...z_n)^{-1}\xi\right)$$
$$\times I_{\max(1,|z_1...z_n||\zeta|^{-1})\mathbb{Z}_p}\left(y_1+y_2+....+y_n-(z_1...z_n)\eta\right). \quad (11)$$

Putting together (10) and (11) we deduce that the integrant of (7) can be written as the product of three factors that we shall denote as follows

$$A(z_1,z_2,...,z_n) = \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n+1}|\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}\max\left(|z_1...z_n||\zeta|^{-1}, |z_1...z_n|^{-1}|\zeta|\right)$$
$$\times\prod_{i=1}^{n}\max\left(|z_i|,|z_i|^{-1}\right) m\left((z_1...z_n)^{-1}\zeta\right)\prod_{i=1}^{n} m(z_i), \quad (12)$$

$$B(x_1,x_2,...,x_n) = I_{\max(1,|z_1...z_n|^{-1}|\zeta|)\mathbb{Z}_p}\left(x_1+x_2+....+x_n-(z_1...z_n)^{-1}\xi\right)$$
$$\times I_{\max(1,|z_1|)\mathbb{Z}_p}(x_1)\prod_{i=2}^{n} I_{|z_1...z_{i-1}|^{-1}\max(1,|z_i|)\mathbb{Z}_p}(x_i), \quad (13)$$

$$C(y_1,y_2,...,y_n) = I_{\max(1,|z_1...z_n||\zeta|^{-1})|)\mathbb{Z}_p}\left(y_1+y_2+....+y_n-(z_1...z_n)\eta\right)$$
$$I_{\max(1,|z_1|^{-1})\mathbb{Z}_p}(y_1)\prod_{i=2}^{n} I_{|z_1...z_{i-1}|\max(1,|z_i|^{-1})\mathbb{Z}_p}(y_i). \quad (14)$$

The next step is to compute

$$\int_{\mathbb{Q}_p}...\int_{\mathbb{Q}_p} B(x_1,x_2,...,x_n)dx_1dx_2...dx_n.$$

From (13) it follows that this integral is equal to

$$\int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} I_{\max(1,|z_1 \dots z_n \zeta^{-1}|^{-1})\mathbb{Z}_p}(x_1 + \dots + x_n - (z_1 \dots z_n)^{-1}\xi)$$

$$\times I_{\max(1,|z_1|)\mathbb{Z}_p}(x_1) \prod_{i=2}^{n} I_{|z_1 \dots z_{i-1}|^{-1}\max(1,|z_i|)\mathbb{Z}_p}(x_i) dx_1 dx_2 \dots dx_n$$

$$= \int_{\mathbb{Q}_p} \Big(I_{\max(1,|z_1|)\mathbb{Z}_p} * I_{|z_1|^{-1}\max(1,|z_2|)\mathbb{Z}_p} * \cdots * I_{|z_1 \dots z_{n-1}|^{-1}\max(1,|z_n|)\mathbb{Z}_p} \Big)(x)$$

$$\times I_{\max(1,|z_1 \dots z_n \zeta^{-1}|^{-1})\mathbb{Z}_p}(x - (z_1 \dots z_n)^{-1}\xi) dx,$$

where $*$ denotes the usual convolution in $\mathbb{Q}_p$ (cf. [14]). Hence (cf. [14])

$$\int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} B(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n = \min \Bigg(\max(1, |z_1|),$$

$$\min_{2 \le i \le n} |z_1 \dots z_{i-1}|^{-1} \max(1, |z_i|) \Bigg) \min(1, |z_1|^{-1}) \prod_{i=2}^{n} |z_1 \dots z_{i-1}| \min\big(1, |z_i|^{-1}\big)$$

$$\times \Big[I_{\min\left(\max(1,|z_1|), \min_{2 \le i \le n} |z_1 \dots z_{i-1}|^{-1}\max(1,|z_i|)\right)\mathbb{Z}_p}$$

$$* I_{\max(1,|z_1 \dots z_n \zeta^{-1}|^{-1})\mathbb{Z}_p} \Big] ((z_1 \dots z_n)^{-1}\xi).$$

Let us denote by $S_\zeta(z_1, \dots, z_n) \subset \mathbb{Q}_p$, $z_1, \dots, z_n \in \mathbb{Q}_p^*$, the subset defined by

$$S_\zeta(z_1, \dots, z_n) = \max(1, |z_1|) \wedge \min_{2 \le i \le n} |z_1 \dots z_{i-1}|^{-1} \max(1, |z_i|)$$

$$\wedge \max(1, |z_1 \dots z_n \zeta^{-1}|^{-1}) \mathbb{Z}_p.$$

With this notation we have

$$\int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} B(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

$$= \min \Bigg[1; \min \Big(\max(1, |z_1|), \min_{2 \le i \le n} |z_1 \dots z_{i-1}|^{-1} \max(1, |z_i|) \Big)$$

$$\times \min(1, |z_1 \dots z_n \zeta^{-1}|) \Bigg] \min(1, |z_1|^{-1}) \prod_{i=2}^{n} |z_1 \dots z_{i-1}|$$

$$\times \prod_{i=2}^{n} \min\left(1, |z_i|^{-1}\right) I_{S_\zeta(z_1,\dots,z_n)}((z_1...z_n)^{-1}\xi). \quad (15)$$

Similarly we obtain for (14) the following formula

$$\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} C(y_1, y_2, ..., y_n) dy_1 dy_2 ... dy_n$$
$$\min\left[1; \min\left(\max(1, |z_1|^{-1}), \min_{2\le i\le n} |z_1...z_{i-1}| \max(1, |z_i|^{-1})\right)\right.$$
$$\left.\min(1, |(z_1...z_n)^{-1}\zeta|)\right] \min(1, |z_1|) \prod_{i=2}^{n} |z_1...z_{i-1}|^{-1}$$
$$\prod_{i=2}^{n} \min\left(1, |z_i|\right) I_{S'_\zeta(z_1,\dots,z_n)}((z_1...z_n)\eta), \quad (16)$$

where $S'_\zeta(z_1, ..., z_n)$, $z_1, ..., z_n \in \mathbb{Q}_p^*$, denote the subset of $\mathbb{Q}_p$ defined by

$$S'_\zeta(z_1, ..., z_n) = \max(1, |z_1|^{-1}) \wedge \min_{2\le i\le n} |z_1...z_{i-1}| \max(1, |z_i|^{-1})$$
$$\wedge \max(1, |(z_1...z_n)^{-1}\zeta|))\mathbb{Z}_p.$$

Putting together (7), (12), (15), (16) we deduce the formula

$$\varphi_{n+1}(g)$$
$$= \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n+1} |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}} \int_{\mathbb{Q}_p^*} \cdots \int_{\mathbb{Q}_p^*} \left[\max\left(|z_1...z_n||\zeta|^{-1}, |z_1...z_n|^{-1}|\zeta|\right)\right.$$
$$\min\left[1; \min\left(\max(1, |z_1|), \min_{2\le i\le n} |z_1...z_{i-1}|^{-1} \max(1, |z_i|)\right)\right.$$
$$\left.\min(1, |z_1...z_n\zeta^{-1}|)\right] \min\left[1; \min\left(\max(1, |z_1|^{-1}),\right.\right.$$
$$\left.\left.\min_{2\le i\le n} |z_1...z_{i-1}| \max(1, |z_i|^{-1})\right) \min(1, |(z_1...z_n)^{-1}\zeta|)\right]$$
$$I_{S_\zeta(z_1,\dots,z_n)}((z_1...z_n)^{-1}\xi) I_{S'_\zeta(z_1,\dots,z_n)}((z_1...z_n)\eta)$$
$$\left. m\left((z_1...z_n)^{-1}\zeta\right) \prod_{i=1}^{n} m(z_i)\right] d^*z_1...d^*z_n. \quad (17)$$

Let now $Z_1, Z_2, ... \in \mathbb{Q}_p^*$ be a sequence of independent identically $\mathbb{Q}_p^*$-valued random variables with distribution on $\mathbb{Q}_p^*$ given by

$$\mathbf{P}[Z_j \in dz] = d\tilde{\mu}(z), \tag{18}$$

where $d\tilde{\mu}(z)$ is defined by (8). We can rewrite (17) as follows

$$\begin{aligned}
&\varphi_{n+1}(g) \\
&\quad = \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n+1} |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}} \mathbf{E}\Bigg[\max\left(|Z_1...Z_n||\zeta|^{-1}, |Z_1...Z_n|^{-1}|\zeta|\right) \\
&\qquad \min\left[1; \min\left(\max(1,|Z_1|), \min_{2\le i\le n} |Z_1...Z_{i-1}|^{-1}\max(1,|Z_i|)\right)\right. \\
&\qquad\quad \left.\min(1,|Z_1...Z_n\zeta^{-1}|)\right] \min\left[1; \min\left(\max(1,|Z_1|^{-1}),\right.\right. \\
&\qquad \left.\left.\min_{2\le i\le n} |Z_1...Z_{i-1}|\max(1,|Z_i|^{-1})\right)\min(1,|(Z_1...Z_n)^{-1}\zeta|)\right] \\
&\quad I_{S_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)^{-1}\xi) I_{S'_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)\eta) m\left((Z_1...Z_n)^{-1}\zeta\right)\Bigg].
\end{aligned} \tag{19}$$

3. Proof of the Central Estimates

We now face the task of estimating the above expectation for the random variables Z_1, $Z_2, \ldots$. The fact that the Z_j's are supported in $p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*$ in (19) allows us to deduce that

$$\varphi_n(e) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\left(\min\left[1, \min_{1\le i\le n} |Z_1...Z_i|^{-1}\right] \min\left[1, \min_{1\le i\le n} |Z_1...Z_i|\right]\right).$$

We shall use the projection

$$\alpha : \mathbb{Q}_p^* \equiv p^{\mathbb{Z}} \times \mathbb{Z}_p \longrightarrow \mathbb{Z} \tag{20}$$

and project on $\mathbb{Z}$ the random walk controlled by (18). The random walk obtained via (20) is the standard random walk $S_n = X_1 + X_2 + ... + X_n$ $(n \ge 1)$ defined by $\mathbf{P}[X_j = 1] = \mathbf{P}[X_j = -1] = 1/2$. With these notations we have:

$$\varphi_n(e) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\left(\min\left(1, \min_{1\le i\le n} p^{-S_i}\right) \min\left(1, \min_{1\le i\le n} p^{S_i}\right)\right),$$

and therefore

$$\varphi_n(e) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\left(\min\left(1, \min_{1\le i\le n} p^{-S_i^+}\right) \min\left(1, \min_{1\le i\le n} p^{S_i^-}\right)\right),$$

where we denote $n^+ = \max(n,0)$, $n \in \mathbb{Z}$, (resp. $n^- = \min(n,0)$, $n \in \mathbb{Z}$). We get then

$$\varphi_n(e) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\left(p^{-\left(\max_{1\le i\le n} S_i^+ + \max_{1\le i\le n} -S_i^-\right)}\right),$$

and this gives

$$\varphi_n(e) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\left(p^{-\max_{1\le i\le n} |S_i|}\right), \quad n = 1, 2, \dots .$$

On the other hand we have (for an elementary proof cf. [15])

$$\mathbf{P}\left[\max_{1\le i\le n} |S_i| \le \lambda\right] \le C \exp\left(-c\frac{n}{\lambda^2}\right), \quad \lambda \ge 1, \quad n = 1, 2, ...$$

from which it follows that

$$\begin{aligned}\mathbf{E}\left(p^{-\max_{1\le i\le n} |S_i|}\right) &= \sum_{\lambda=0}^{\infty} p^{-\lambda} \mathbf{P}\left[\max_{1\le i\le n} |S_i| = \lambda\right] \\ &\le \sum_{\lambda=0}^{\infty} p^{-\lambda} \mathbf{P}\left[\max_{1\le i\le n} |S_i| \le \lambda + 1\right] \le C \sum_{\lambda=1}^{\infty} p^{-c\frac{n}{\lambda^2} - \lambda} \le C e^{-cn^{1/3}}.\end{aligned}$$

The estimate (4) follows at once.

To prove the lower estimate (5) we first deduce from (19), as before, the lower bound

$$\begin{aligned}\varphi_n(e) \ge \frac{1}{C}\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\Bigg(&\min\left[1, \min_{1\le i\le n} |Z_1 ... Z_i|^{-1}\right] \\ &\times \min\left[1, \min_{1\le i\le n} |Z_1 ... Z_i|\right] m(Z_1 ... Z_n)\Bigg).\end{aligned}$$

From this it follows that

$$\varphi_n(e) \geq \frac{1}{C}\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\Bigg(\min\left[1, \min_{1\leq i\leq n} |Z_1...Z_i|^{-1}\right] \min\left[1, \min_{1\leq i\leq n} |Z_1...Z_i|\right]$$
$$\times\, m(Z_1...Z_n) I\left[p^{-n^{1/3}} \leq |Z_1...Z_i| \leq p^{n^{1/3}}, \quad i=1,...n\right]\Bigg),$$

and then

$$\varphi_n(e) \geq c e^{-Cn^{1/3}} \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{E}\Bigg(m(Z_1...Z_n)$$
$$\times\, I\left[p^{-n^{1/3}} \leq |Z_1...Z_i| \leq p^{n^{1/3}}, \quad i=1,...,n\right]\Bigg).$$

Using the projection (20) we deduce that

$$\varphi_{2n}(e) \geq c e^{-Cn^{1/3}} \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n \mathbf{P}\left(|S_j| \leq n^{1/3},\ j=0,...,2n;\quad S_{2n}=0\right),$$
$$n=1,2,.... \quad (21)$$

The lower estimate (5) is an immediate consequence of (21) and the following estimate

$$\mathbf{P}\left(|S_j| \leq cn^{1/3},\ j=0,...,n;\quad S_{2n}=0\right) \geq c e^{-Cn^{1/3}}, \quad n=1,2,.... \quad (22)$$

This estimate is an automatic consequence of the well known estimate

$$\mathbf{P}\left[\max_{1\leq i\leq n} |S_i| \leq \lambda\right] \geq C \exp\left(-c\frac{n}{\lambda^2}\right), \quad \lambda=1,2,...;\quad n=1,2,.... \quad (23)$$

Indeed, let $I_\lambda \subset \mathbb{Z}$ ($\lambda \geq 1$) denote the interval consisting of the integers $z = -\lambda, -\lambda+1, ..., \lambda-1, \lambda$, and let $p_n^\lambda(x,y)$, $n=1,2,...$, $x, y \in I_\lambda$, denote the transition kernel corresponding to the simple random walk with killing outside of I_λ. We have

$$\mathbf{P}\left[|S_j| \leq \lambda,\ j=0,...,n\right] = \sum_{y\in I_\lambda} p_n^\lambda(0,y), \quad n=1,2,....$$

Having in mind that

$$p_{2n}^\lambda(0,0) = \sum_{y\in I_\lambda} p_n^\lambda(0,y) p_n^\lambda(y,0) = \sum_{y\in I_\lambda} \left(p_n^\lambda(0,y)\right)^2, \quad n=1,2,...$$

we see that with an appropriate $C > 0$ we have

$$C\lambda^{1/2}\left(p_{2n}^{\lambda}(0,0)\right)^{1/2} \geq \sum_{y\in I_\lambda} p_n^{\lambda}(0,y), \quad n = 1,2,...$$

and therefore

$$C\lambda p_{2n}^{\lambda}(0,0) \geq \left(\mathbf{P}\left[|S_j| \leq \lambda, \;\; j = 0,...,n\right]\right)^2, \quad \lambda = 1,2,...; \quad n = 1,2,... .$$

This combined with (23) gives

$$\lambda\mathbf{P}\left[|S_j| \leq \lambda, \;\; j = 0,...,2n; \quad S_{2n} = 0\right] \geq c\exp\left(-C\frac{n}{\lambda^2}\right), \tag{24}$$

$$\lambda = 1,2,...; \quad n = 1,2,... .$$

Choosing $\lambda \approx n^{1/3}$ in (24) we deduce (22). This completes the proof the lower estimate (5).

4. The Off-Diagonal Estimate

We shall now derive the off-diagonal estimate (6). All our previous notations are preserved. Taking advantage, as in §3, of the fact that $Z_1, Z_2, ..., Z_n \in p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*$, in the expectation that appears in the right hand side of (19) we first deduce

$$\begin{aligned}\varphi_{n+1}(g) \;\; \leq \;\; & C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n+1} |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}\mathbf{E}\Bigg(\max\Big(|Z_1...Z_n\zeta^{-1}|, \\ & \times|\left(Z_1...Z_n\right)^{-1}\zeta|\Big)\min\left[1, \min_{2\leq i\leq n}|Z_1...Z_{i-1}|^{-1}\right. \\ & \left.\times\min(1,|Z_1...Z_n\zeta^{-1}|)\right]\min\left[1, \min_{2\leq i\leq n}|Z_1...Z_{i-1}|\right. \\ & \left.\times\min(1,|(Z_1...Z_n)^{-1}\zeta|)\right] I_{S_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)^{-1}\xi) \\ & \times I_{S'_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)\eta)m\left((Z_1...Z_n)^{-1}\zeta\right)\Bigg).\end{aligned}$$

Using the definition of m we see that the previous estimate reduces to

$$\begin{aligned}\varphi_{n+1}(g) \quad \leq \quad & C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n+1}|\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}\mathbf{E}\Bigg(\min\left[1,\min_{2\leq i\leq n}|Z_1...Z_{i-1}|^{-1}\right] \\ & \times\min\left[1,\min_{2\leq i\leq n}|Z_1...Z_{i-1}|\right]I_{S_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)^{-1}\xi) \\ & \times I_{S'_\zeta(Z_1,...,Z_n)}((Z_1...Z_n)\eta)m\left((Z_1...Z_n)^{-1}\zeta\right)\Bigg).\end{aligned}$$

It follows then that

$$\begin{aligned}&\varphi_n(g) \\ &\quad \leq C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n}|\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}\mathbf{E}\Bigg(\min\left(1,\min_{2\leq i\leq n}p^{-S_i}\right)\min\left(1,\min_{2\leq i\leq n}p^{S_i}\right) \\ &\quad I_{\min\left(1,\min_{2\leq i\leq n}p^{-S_i}\right)\mathbb{Z}_p}((Z_1...Z_n)^{-1}\xi)I_{\min\left(1,\min_{2\leq i\leq n}p^{S_i}\right)\mathbb{Z}_p}((Z_1...Z_n)\eta) \\ &\qquad\qquad m\left((Z_1...Z_{n-1})^{-1}\zeta\right)\Bigg), \quad n\geq 2.\end{aligned}$$

Hence

$$\varphi_n(g) \leq C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^{n}|\zeta|^{-\frac{\log(\lambda)}{2\log(p)}}\mathbf{E}\Bigg(p^{-\max_{1\leq i\leq n}S_i^+}p^{\min_{1\leq i\leq n}S_i^-} \tag{25}$$

$$I_{\frac{1}{C}p^{-\max_{1\leq i\leq n}S_i^+}\mathbb{Z}_p}(\zeta^{-1}\xi)I_{\frac{1}{C}p^{\min_{1\leq i\leq n}S_i^-}\mathbb{Z}_p}(\zeta\eta)m\left((Z_1...Z_n)^{-1}\zeta\right)\Bigg).$$

The fact that

$$\zeta^{-1}\xi\in\frac{1}{C}p^{-\max_{1\leq i\leq n}S_i^+}\mathbb{Z}_p$$

in (25) implies that

$$|\zeta^{-1}\xi|\leq Cp^{\max_{1\leq i\leq n}S_i^+}$$

and thus

$$p^{-\max_{1\leq i\leq n}(S_i)^+}\leq C\min\left(1,|\zeta^{-1}\xi|^{-1}\right). \tag{26}$$

Similarly, we have

$$p^{\min_{1\leq i\leq n}S_i^-}\leq C\min\left(1,|\zeta\eta|^{-1}\right). \tag{27}$$

Putting together (25), (26) and (27) we deduce that

$$\varphi_n(g) \le C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}} \min\left(1, |\zeta^{-1}\xi|^{-1}\right) \min\left(1, |\zeta\eta|^{-1}\right)$$

$$\times \mathbf{E}\left(I_{\frac{1}{C}p^{-\max_{1\le i\le n} S_i^+}\mathbb{Z}_p}(\zeta^{-1}\xi) I_{\frac{1}{C}p^{\min_{1\le i\le n} S_i^-}\mathbb{Z}_p}(\zeta\eta) I\left[|S_n - \alpha(\zeta)| \le C\right]\right).$$

Using Hölder we obtain

$$\begin{aligned}\varphi_n(g) \quad \le \quad & C\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}} \min\left(1, |\zeta^{-1}\xi|^{-1}\right) \min\left(1, |\zeta\eta|^{-1}\right) \\ & \times \left(\mathbf{P}\left[|S_n - \alpha(\zeta)| \le C\right]\right)^{1-\frac{1}{k}} \left(\mathbf{P}\left[\max_{1\le i\le n} S_i^+ \ge c\log^+|\zeta^{-1}\xi|\right]\right)^{1/2k} \\ & \times \left(\mathbf{P}\left[\max_{1\le i\le n} (-S_i)^+ \ge c\log^+|\zeta\eta|\right]\right)^{1/2k},\end{aligned}$$

where the k can be made as large as we like. We can replace $\max_{1\le i\le n} S_i^+$ and $\max_{1\le i\le n}(-S_i)^+$ by $\max_{1\le i\le n}|S_i|$ in the above estimate and this gives

$$\begin{aligned}\varphi_n(g) \le C & \left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n |\zeta|^{-\frac{\log(\lambda)}{2\log(p)}} \min\left(1, |\zeta^{-1}\xi|^{-1}\right) \min\left(1, |\zeta\eta|^{-1}\right) \\ & \times \left(\mathbf{P}\left[|S_n - \alpha(\zeta)| \le C\right]\right)^{1-\frac{1}{k}} \left(\mathbf{P}\left[\max_{1\le i\le n} |S_i| \ge c\log^+|\zeta^{-1}\xi|\right]\right)^{1/2k} \\ & \times \left(\mathbf{P}\left[\max_{1\le i\le n} |S_i| \ge c\log^+|\zeta\eta|\right]\right)^{1/2k}. \end{aligned} \tag{28}$$

We have on the other hand the following maximal estimate

$$\mathbf{P}\left[\max_{1\le i\le n} |S_i| \ge c\log^+|\zeta^{-1}\xi|\right] \le C\exp\left(-\frac{c(\log^+|\zeta^{-1}\xi|)^2}{n}\right). \tag{29}$$

The argument to prove (29) is well known. First we observe that by symmetry

$$\mathbf{P}\left[\max_{1\le i\le n} |S_i| \ge c\log^+|\zeta^{-1}\xi|\right] \le 2\mathbf{P}\left[\max_{1\le i\le n} S_i \ge c\log^+|\zeta^{-1}\xi|\right].$$

We then use the reflection principle (cf. [2], Chapter 1.4) and this gives the required estimate. Similarly we have

$$\mathbf{P}\left[\max_{1\le i\le n}|S_i|\ge c\log^+|\zeta\eta|\right]\le C\exp\left(-\frac{c(\log^+|\zeta\eta|)^2}{n}\right). \tag{30}$$

Putting together (28), (29) and (30) and choosing k large enough we see that $\varphi_n(g)$ can be estimated by

$$\varphi_n(g)\le C_\epsilon n^{-1/2+\epsilon}\left(\frac{2\lambda^{1/2}}{1+\lambda}\right)^n|z|^{-\frac{\log(\lambda)}{2\log(p)}}\min\left(1,|\zeta^{-1}\xi|^{-1}\right)\min\left(1,|\zeta\eta|^{-1}\right)$$

$$\times\exp\left(-\frac{(\log^+|\zeta^{-1}\xi|)^2+(\log^+|\zeta\eta|)^2+(\log|\zeta|)^2}{C_\epsilon n}\right).$$

This completes the proof of the off-diagonal estimate (6).

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