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NOTES ON THE CONTINUITY IN IDEAL TOPOLOGICAL SPACES

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Abstract: In this paper it is investigated relationships between group action and $w-I-$continuity and w^*-I-continuity in ideal topological spaces. I try to find an answer to the following question "When $w-I-$continuity implies continuity?". Similarly "when w^*-I-continuity implies continuity?". Also it is given some results for specific ideals.

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1. Introduction

Let (X,τ) be a topological space. As it is well known a nonempty family I of subsets of X is called ideal if the two following conditions are fulfilled: (1) If $A\in I$ and $B\subset A$ then $B\in I$. (2) If $A\in I$ and $B\in I$ then $A\cup B\in I$. Observe that a family of sets is a filter if and only if the family of the complements of these sets is an ideal. A topological space (X,τ) with an ideal I is called an ideal topological space which we denote this (X,τ,I). If $A\subset X$, $A^*(I)$=$\{x\in X : U\cap A\notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ, see [4]. We simply write A^* instead of $A^*(I)$. Let (X,τ,I) be an ideal topological space then there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta\left(I,\tau\right)=\{U\backslash A : U\in\tau$ and $A\in I\}$ but in general $\beta(I,\tau)$ is not always a topology, see [2]. In addition $\mathrm{Cl}^*(A)=A^*\cup A$ defines a

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Kuratowski closure operator for $\tau^*(I)$.

A function $f:(X,\tau)\to(Y,\sigma,I)$ is said to be weakly-I-continuous (briefly, w-I-c) if for each $x\in X$ and each open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(U)\subset\mathrm{Cl}^*(V)$. A complementary form of weak-I-continuity is weak*-I-continuity.

A function $f:(X,\tau)\to(Y,\sigma,I)$ is said to be weak*-I-continuous (briefly, w^*-I-c) if for each open set V in Y, $f^{-1}(fr^*(V))$ is closed in X. Here $fr^*(V)$ denotes the $*-$frontier of V.

The notion of w-I-continuity and w^*-I-continuity was introduced and given a decomposition of continuity by Açıkgöz, Noiri and Yüksel [1]. In this paper some relations are given between w-I and w^*-I-continuities and group actions.

Recall that an ideal topological space (X,τ,I) is an $RI-$ space if, for each $x\in X$ and each open neighbourhood V of x, there exists an open neighbourhood U of x such that $x\in U\subset\mathrm{Cl}^*(U)\subset V$. The notion of $RI-$ space was introduced in [1]. It is clear that if Y be an $RI-$ space then $f:(X,\tau)\to(Y,\sigma,I)$ is w-I-c if and only if f is continuous. But this criteria is not adequate for many cases. Suppose there is an nonempty open set V in I such that $V^*\neq\emptyset$. If $y\in V$ then $y\notin V^*$ since $V\in I$. Thus $V\cap V^*=\emptyset$. Assume that Y is $RI-$ space then there is an open neighbourhood U of y such that $y\in U\subset\mathrm{Cl}^*(U)\subset V$. But this is impossible unless $U^*=\emptyset$. It is because $U\cap U^*=\emptyset$ since U is also contained in I.

On the other hand a triple (G,X,θ) is called a topological transformation group, where G is a topological group, X is a Hausdorff topological space, and $\theta:G\times X\to X$ is a continuous map such that: (1) $\theta(g,\theta(h,x))=\theta(gh,x)$ for all $g,h\in G$ and $x\in X$. (2) $\theta(e,x)=x$ for all $x\in X$, where e is the identity of G. The map θ is called an action of G on X. The space X, together with a given action θ of G, is called a G space. If G is an algebraic group and X be a set, where G acts on X then X is called a G set. Suppose that X is a G set and $x\in X$. The set $G_x=\{g\in G:\theta(g,x)=x\}$ is clearly subgroup of G. Thus G_x is called the isotropy subgroup of G at x. A transformation group (G,X,θ) is said to be trivial if $G_x=G$ for all $x\in X$. Let $K\subseteq G$, then the set of points fixed under K, $X^K=\{x\in X:\theta(g,x)=x \text{ for all } g\in K\}$.

2. Isotropy Subgroups and w-I-continuity

We start our discussion with considering an algebraic group G such that whenever $H,K\neq\{e\}$ be subgroups of G then $H\cap K\neq\{e\}$. Let (Y,σ) be a topological space and Y be a G-set with non-trivial action. Now we define $I = \{A :$ there exists a subgroup K of G such that $K \neq \{e\}$ and $A \subseteq Y^K\}$. Clearly if $A \in I$ and $B \subset A$ then $B \in I$ and if $A, B \in I$ then $A \subseteq Y^K$ and $B \subseteq Y^H$ for some subgroups H, K of G which is different from $\{e\}$. Obviously I be an ideal on Y since $A \cup B \subseteq Y^K \cup Y^H \subseteq Y^{K \cap H}$.

In some cases w-I-c is not different from continuity. For example suppose (Y, σ) admits such kind of G action with finitely many distinct isotropy subgroups and all of them are different from $\{e\}$. Remember that $I = \{A :$ there exists a subgroup K of G such that $K \neq \{e\}$ and $A \subseteq Y^K\}$. Then $V^* = \emptyset$ for all open set V in Y. Because for any subspace A of Y, it is clear that $A \subseteq \bigcup_{a \in A} Y^{G_a}$ since $y \in Y^{G_y}$ for any $y \in Y$. On the other hand there are $a_1, a_2, ...a_n \in A$ such that $A \subseteq \bigcup_{a \in A} Y^{G_a} = \bigcup_{i=1}^{n} Y^{G_{a_i}} \subseteq Y^{\bigcap_{i=1}^{n} G_{a_i}}$, since there are finitely many distinct isotropy subgroups. Thus there is no difference between w-I-c and continuity for any $f : (X, \tau) \to (Y, \sigma, I)$. In this case, observe that each $f : (X, \tau) \to (Y, \sigma, I)$ is $w^* - I - c$.

Theorem 2.1. *Let Y be a compact, topological manifold. Suppose for any ideal J on Y, there is a continuous Abelian $p-$group action (p, prime) on Y such that all isotropy subgroups are different from $\{e\}$ and all open sets of I contained in J. Then there is no difference between $w - J - c$, continuity for any $f : (X, \tau) \to (Y, \sigma, J)$.*

Proof. Recall that any (continuous) compact Lie group action on any compact topological manifold, There are at most a finite number of non-conjugate isotropy subgroups, see [5]. Observe that an Abelian $p-$ group G is a compact Lie group. Thus there are finitely many distinct isotropy subgroups of G. On the other hand if H, K are subgroups of G different from $\{e\}$ then $H \cap K \neq \{e\}$. Let V be an open set in Y and suppose that $V^* \neq \emptyset$. Let $y \in V^*$ then for all open set U containing y, $V \cap U \notin J$. This implies that $V \cap U \notin I$ since for all open sets in I contained in J. Hence $V \cap U \notin Y^K$ for all subgroup K of G different from $\{e\}$. But this is impossible because of finiteness of distinct isotropy subgroups. □

Example 2.2. Let p be a prime. Consider linear circle, S^1, action φ on $2k-1$ sphere, $Y = S^{2k-1}$, such that $\varphi(\theta, (z_1, z_2, ...z_k)) = (\theta^p z_1, \theta^{p^2} z_2, ..., \theta^{p^k} z_k)$. It is easy to see that isotropy subgroups of this action are $S^1, \mathbb{Z}_p, \mathbb{Z}_{p^2}, ..., \mathbb{Z}_{p^k}$. Let $S(k)$ be the subgroup of elements θ of S^1 such that $p^k\theta = 1$, the identity element of S^1. Clearly $S(k) \cong \mathbb{Z}_{p^k}$. Consider the restricted $S(k)$ action on Y then isotropy subgroups of this restricted action are $\mathbb{Z}_p, \mathbb{Z}_{p^2}, ..., \mathbb{Z}_{p^k}$. Since $\mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset .... \subset \mathbb{Z}_{p^k}$, $Y = Y^{\mathbb{Z}_p} \supseteq Y^{\mathbb{Z}_{p^2}} \supseteq ... \supseteq Y^{\mathbb{Z}_{p^k}}$. This implies that all open sets in Y contained in I. Thus if J be an ideal on Y such that all open sets in Y contained in J then there is no difference between $w-J-c$ and continuity for any $f:(X,\tau)\to(S^{2k-1},\sigma,J)$ and each $f:(X,\tau)\to(S^{2k-1},\sigma,J)$ is w^*-J-c.

Theorem 2.3. *Suppose that all isotropy subgroups of G are different from $\{e\}$ (may not be finitely many number of distinct isotropy subgroups). Then a function $f:(X,\tau)\to(Y,\sigma,I)$ is continuous if and only if it is w-I-c.*

Proof. The necessity is clear. Sufficiency part of theorem: Assume that f is not continuous at $x_0\in X$. Then there exists an open set V in Y such that $f(x_0)\in V$ and $f(U)\nsubseteq V$ for all open set U containing x_0. On the other hand since f is w-I-c, there exists an open set W containing x_0 such that $f(W)\subset \mathrm{Cl}^*V=V^*\cup V$. Since f is not continuous at x_0, $f(W)\nsubseteq V$. Hence there exists $z\in W$ such that $f(z)\in V^*$ and $f(z)\notin V$. Thus we obtain $V\cap U\notin I$ for all open set U containing $f(z)$. This implies there exists an element z' of V such that $z'\notin Y^K$ for all subgroup K of G different from $\{e\}$. But this contradicts the assumption since $z'\in Y^{G_{z'}}$. □

Corollary 2.4. *Let (Y,σ,J) be an ideal topological space. If Y admits such kind of action such that all open sets of I contained in J then $f:(X,\tau)\to(Y,\sigma,J)$ is continuous if and only if it is $w-J-c$.*

Proof. Let J be such an ideal on Y. Since open sets in I also contained in J, if $f:(X,\tau)\to(Y,\sigma,J)$ is $w-J-c$ then it is w-I-c. Hence the proof is quite similar to that of Theorem 2.3. □

Theorem 2.5. *Let (Y,σ) is a Hausdorff space. A function $f:(X,\tau)\to(Y,\sigma,J)$ is continuous if and only if it is $w-J-c$.*

Proof. The necessity is clear. The sufficiency part: Suppose f is $w-J-c$. Let V be a non-empty open set in Y and $y\notin V$. Since (Y,σ) is Hausdorff topological space there exists an open set U containing y such that $U\cap V=\emptyset$. This implies that $y\notin V^*$ since $\emptyset\in J$. Hence $(Y-V)\cap V^*=\emptyset$. So $V^*\subseteq V$ and $f^{-1}(fr^*(V))=f^{-1}(V^*-V)=f^{-1}(\emptyset)=\emptyset$ for all $f:(X,\tau)\to(Y,\sigma,J)$. This means that f is w^*-J-c. Since Decomposition Theorem (see [1]), we obtain that f is continuous. □

From now on (Y,σ) be a topological space such that whenever $U,V\neq\emptyset$ be open sets of Y then $U\cap V\neq\emptyset$ (for example, left ray or right ray topology on R) and $W\subset Y$ unless otherwise stated. Now we define $I=\{A:A\subseteq W\}$. Clearly I be an ideal on Y.

Theorem 2.6. *Suppose that $Y-W$ dense in Y then any function $f:(X,\tau)\to(Y,\sigma,I)$ is w-I-c.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Let $y \in Y$ and U be any open set of Y containing y then $V \cap U \neq \emptyset$ since $U, V \neq \emptyset$. Thus there exists an element $z' \in U \cap V$ such that $z' \in Y - W$ since $Y - W$ is dense in Y. This implies that $U \cap V \notin I$ for all open set U containing y. Hence we obtain that $V^* = Y$. Thus, f is w-I-c. □

Corollary 2.7. *Let J be an ideal on Y such that $J \subseteq I$ then a function $f : (X, \tau) \to (Y, \sigma, J)$ is continuous if and only if it is $w^* - I - c$.*

Proof. The necessity is clear. Sufficiency part of theorem: Suppose f is $w^* - J - c$. Since f is w-I-c and $J \subseteq I$, it is easy to see that f is $w - J - c$. Thus we obtain that f is continuous by Decomposition Theorem (see [1]). □

There are many examples of group action such that complement of fixed point set is dense.

Example 2.8. Let $Y = R$, real numbers, with left ray (respectively right ray) topology. Let G be compact Lie group and it acts differentiably on Y (i.e action of G on Y is differentiable). If we consider the usual topology on Y then Y^G is a compact submanifold of R [3]. Thus Y^G must be zero dimensional. Otherwise Y^G is a circle but this is impossible since Y^G is a submanifold of R. This implies that Y^G must be a finite set. Y^G may not be submanifold of Y if we consider left ray topology (respectively right-ray topology) on R but $Y - Y^G$ is dense in Y.

Theorem 2.9. *Let J be any ideal on Y such that $(Y - V) \subseteq V^*$ for all open set V in Y. Then a function $f : (X, \tau) \to (Y, \sigma, J)$ is continuous if and only if it is $w^* - J - c$.*

Proof. The necessity is clear. The sufficiency part of theorem: Suppose that f is not continuous then there exists an open set V in Y such that $f^{-1}(V)$ is not open in X. Hence there exists an element x_0 of $f^{-1}(V)$ such that $f(U) \nsubseteq f^{-1}(V)$ for all open set U containing x_0. On the other hand $X - f^{-1}(fr^*(V)) = X - f^{-1}(V^* - V)$ open in X since f is $w^* - J - c$. Observe that $x_0 \in X - f^{-1}(V^* - V)$ since $f(x_0) \in V$. Therefore there exists an element z of $X - f^{-1}(V^* - V)$ such that $f^{-1}(V)$ does not contain z. This implies that $f(z) \in Y - V^*$ and $f(z) \notin V$. Furthermore $f(z) \in V$ because of $(Y - V) \subseteq V^*$, a contradiction. □

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