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THE POLYGON CASE OF THE THETA-CURVE THEOREM

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Abstract: The well known game of Hex uses the following corollary of the Jordan Curve Theorem, so-called *theta-Curve Theorem*: *An open Jordan curve with its endpoints on a closed Jordan curve $\mathcal{K}$, but otherwise located in the bounded part, divides the closure of the bounded part into two parts.*

Some students in a honor class, who have not been previously exposed to homology theory, after being motivated by the game of Hex, may be given direct and elementary proofs of the polygon case of both the Jordan Curve Theorem and the Theta-Curve Theorem.

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1. Introduction

The Jordan Curve Theorem (JCT) claims that a simple closed curve in a plane divides the plane (excluding the points of the curve $\mathcal{K}$ itself) into two regions in the sense that any broken line (curve consisting of connected line segments) connecting two points from different regions intersects the curve, and for any

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two points from the same region there exists a broken line connecting them which does not intersect the curve. Exactly one of these regions is bounded and called the *interior*; the other one is called the *exterior* of the curve. A (bounded) figure Φ determined by a simple closed curve $\mathcal{K}$ is usually defined as the union of the curve $\mathcal{K}$ and its interior. If we denote by A and B two points on the curve $\mathcal{K}$ and by $\mathcal{K}^*$ a simple open curve connecting points A and B which is, excluding its end-points, entirely situated within the interior of $\mathcal{K}$, then we obtain two new simple closed curves, each of them formed by $\mathcal{K}^*$ and one of the two arcs of $\mathcal{K}$ with end-points in A and B (Figure 1). Therefore, we obtain two new figures, say Φ_1 and Φ_2.

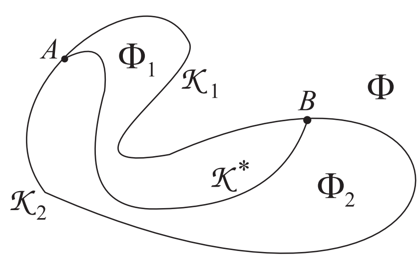

Figure 1: The simple open curve $\mathcal{K}^*$ divides the figure Φ into the figures Φ_1 and Φ_2

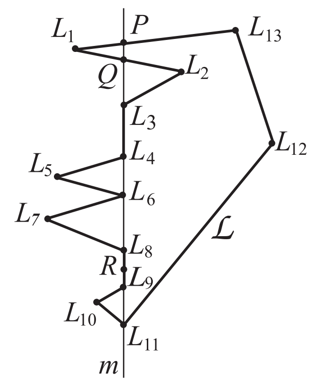

Figure 2: The proper points and proper sides of the intersection of $\mathcal{L}$ and m are P, Q, L_{11} and $[L_3L_4]$

The Jordan Curve Theorem itself (and its generalizations) can be proved using homology theory [5], [4], but the proof can also be carried out in a reasonably elementary fashion [5], [2], [7].

The fact that the figure Φ is divided into figures Φ_1 and Φ_2, so that $\Phi_1 \cup \Phi_2 = \Phi$ and $\Phi_1 \cap \Phi_2 = \mathcal{K}^*$ (Theta-Curve Theorem), is intuitively obvious, but it is not a trivial corollary of the JCT. This theorem can be derived both using homology theory [4] and in an elementary fashion [1], too. The aim of this work is to present a direct and elementary proof of these statements in the polygon case, i.e., when the considered curves are broken lines.

2. Preliminaries

Before we proceed any further, it would be well to adopt some definitions and notations.

Definition 1. Let m be a ray (or a line) and $\mathcal{L}$ a broken line (open or closed) in the plane intersecting m. A point or a side of the broken line $\mathcal{L}$ lying on m is said to be a *proper* point (side) of the intersection in each of the following cases:

1) it is a point belonging to a segment of $\mathcal{L}$ whose end-points lie on opposite sides of m, or

2) it is a vertex of $\mathcal{L}$ whose neighboring vertices on $\mathcal{L}$ lie on opposite sides of m, or

3) it is a side of $\mathcal{L}$ whose adjacent sides (excluding the common points) lie on opposite sides of m.

In Figure 2 the points P, Q and L_{11} are the proper points of intersection of $\mathcal{L} = L_1L_2 \ldots L_{13}L_1$ and the line m, but the points L_3, L_4, L_6, L_8, R and L_9 are not. Also, the segment $[L_3L_4]$ is a proper side of this intersection, but the segment $[L_8L_9]$ is not.

From now on, we denote the considered plane by $\mathcal{E}^2$ (Euclidean 2-space) and $d(X,Y)$ denotes the Euclidean distance between two points X and Y.

Definition 2. The set $\mathcal{N}(A,\delta) \stackrel{\text{def}}{=} \{X \mid \ d(A,X) < \delta\}$ is the *neighborhood* of A with centeer A and radius δ.

A point P is an *interior point* of a set Φ ($\Phi \subseteq \mathcal{E}^2$) if there exists a neighborhood of P that is contained in Φ. A point P is an *exterior point* to set Φ if there exists a neighborhood of P that is contained in the complement of Φ ($Cp(\Phi)$). A point P is a *boundary point* to both Φ and $Cp(\Phi)$ if every neighborhood of P intersects both Φ and $Cp(\Phi)$.

The *interior* of Φ, denoted by $\mathrm{In}(\Phi)$ is the set of all points interior to Φ. The *exterior* of Φ, denoted by $\mathrm{Ex}(\Phi)$ is the set of all points exterior to Φ. The *boundary* of Φ and $Cp(\Phi)$, denoted by $Bd(\Phi)$, is the set of all boundary points to Φ and $Cp(\Phi)$.

Obviously, the interior of Φ is contained in Φ, the exterior of Φ is contained in the complement of Φ and $\mathcal{E}^2 = \mathrm{In}(\Phi) \cup \mathrm{Bd}(\Phi) \cup \mathrm{Ex}(\Phi)$, and the sets in this union are pairwise disjoint.

The *distance* from a point X to a set φ (or between two sets φ_1 and φ_2) is defined by $d(X,\varphi) \stackrel{\text{def}}{=} \inf\{d(X,Y) \ \mid \ Y \in \varphi\}$ ($d(\varphi_1,\varphi_2) \stackrel{\text{def}}{=} \inf\{d(X,Y) \ \mid \ X \in \varphi_1 \wedge Y \in \varphi_2\}$).

For a given coordinate system Oxy, we denote by $r^+(Z)$ and $r^-(Z)$ the half-lines extending from Z in the positive and negative directions of the y-axis, respectively. Let $\mathcal{L}$ be a broken line and Z a point which does not belong to $\mathcal{L}$. Then, $n(Z,\mathcal{L})$ denotes the number of all proper points and proper sides of the intersection of $\mathcal{L}$ and $r^+(Z)$. In the case of self-crossing of the broken line $\mathcal{L}$, i.e., if the broken line is not simple, we are counting each proper point (side) of intersection according to its multiplicity.

Now we can define a function, with domain in the set of all points of the plane excluding the points of $\mathcal{L}$, by

$$N(Z,\mathcal{L}) = \begin{cases} 0 \quad , \quad \text{for } \; n(Z,\mathcal{L}) \;\; \text{even} \; , \\ 1 \quad , \quad \text{for } \; n(Z,\mathcal{L}) \;\; \text{odd}. \end{cases} \tag{1}$$

Lemma 1. *Let $\mathcal{L} : L_1L_2 \ldots L_n$ and $\mathcal{M} : M_1M_2 \ldots M_m$ be two disjoint broken lines (not necessarily simple) in the plane Oxy. If the broken line $\mathcal{L}$ is closed or $\mathcal{M}$ lies entirely between the vertical lines (lines parallel to the y-axis) through the points L_1 and L_n, respectively, then $N(Z,\mathcal{L})$ is constant, for all $Z \in \mathcal{M}$.*

Proof. Denote by $p_1, p_2, \ldots, p_k$ ($k \leq n+m$) all possible vertical lines passing through the vertices of $\mathcal{L} \bigcup \mathcal{M}$ in such a way that p_i lies between p_{i-1} and p_{i+1} ($2 \leq i \leq k-1$). On every segment of the broken line $\mathcal{M}$ between two adjacent vertical lines, including the end points, the function $N(Z,\mathcal{L})$ is constant. To prove it, consider one such segment with end-points Z_0 and Z_1 ($Z_0 \in p_i$, $Z_1 \in p_{i+1}$) and an arbitrary point Z between Z_0 and Z_1. For every proper point (or proper side) of the intersection of $\mathcal{L}$ and the ray $r^+(Z_0)$ there is exactly one segment of $\mathcal{L}$ entirely or in part situated within the strip between the lines p_i and p_{i+1} which contains (adjoins) this proper point (proper side) of the intersection of $\mathcal{L}$ and $r^+(Z_0)$. The intersection of this segment and the ray $r^+(Z)$ is a proper point of intersection (first three pictures in Figure 3), too. Similarly, for every point (or side) of $\mathcal{L}$ on the ray $r^+(Z_0)$ which is not a proper point (side) of the intersection, there is no segment or there are exactly two segments entirely or in part situated within the strip between the lines p_i and p_{i+1} and incident on this point (or this segment) (the last two pictures in Figure 3). In the second case, the ray $r^+(Z)$ crosses these two segments of $\mathcal{L}$ in two proper points of intersection. From above, we see that the numbers $n(Z_0,\mathcal{L})$ and $n(Z,\mathcal{L})$ are of the same parity, i.e., $N(Z_0,\mathcal{L}) = N(Z,\mathcal{L})$. Similarly, we can obtain that $N(Z_1,\mathcal{L}) = N(Z,\mathcal{L})$ for every $Z \in [Z_0Z_1]$. Since the function $N(Z,\mathcal{L})$ is constant on any two adjacent segments of $\mathcal{M}$, it is constant for all $Z \in \mathcal{M}$. □

3. Polygon Case of the Jordan Curve Theorem

Theorem 1. (Polygon Case of the JCT) *Any simple closed broken line*

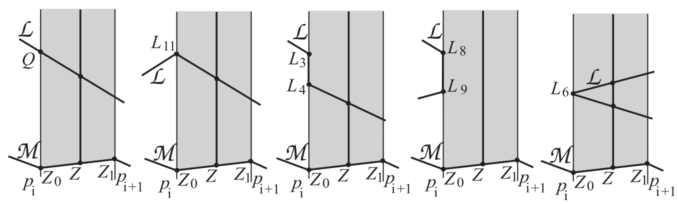

Figure 3: The numbers $n(Z_0, \mathcal{L})$ and $n(Z, \mathcal{L})$ are of the same parity

$\mathcal{L} : L_1 L_2 \dots L_n L_1$ in a plane divides the plane (excluding the points of $\mathcal{L}$) into two regions.

Proof. We begin by choosing a coordinate system with the y-axis not being parallel to any of sides of $\mathcal{L}$. Without loss of generality, we can assume that no three successive points of $\mathcal{L}$ lie in a same line. Let we define two relations τ and σ in the set of all points of the plane ($\mathcal{E}^2$) excluding the set of points of $\mathcal{L}$:

— $X \tau Y$ iff there exists a broken line connecting the points X and Y which does not intersect $\mathcal{L}$;

— $X \sigma Y$ iff $N(X, \mathcal{L}) = N(Y, \mathcal{L})$.

Note that both relations are equivalence (RST) relations. Since $N(Z, \mathcal{L}) \in \{0, 1\}$, where $Z \in \mathcal{E}^2 \backslash \mathcal{L}$, we conclude that the relation σ has at most two equivalence classes.

Exercise 1. Prove that the sets $\{Z \in \mathcal{E}^2 \backslash \mathcal{L} \mid \quad N(Z, \mathcal{L}) = 0\}$ and $\{Z \in \mathcal{E}^2 \backslash \mathcal{L} \mid \quad N(Z, \mathcal{L}) = 1\}$ are not empty, i.e., the relation σ has exactly two equivalence classes.

(*Hint.* Look for two points, one from each of these sets, in a small enough neighborhood of a point lying on $\mathcal{L}$ which is not a vertex of $\mathcal{L}$.)

Therefore, for the purpose of proving that the relation τ has exactly two equivalence classes we will prove the coincidence of the relations τ and σ, i.e., $X \sigma Y$ iff $X \tau Y$, for all $X, Y \in \mathcal{E}^2 \backslash \mathcal{L}$.

Exercise 2. Prove that $\tau \subseteq \sigma$, i.e., if $X \tau Y$ then $X \sigma Y$, for all $X, Y \in \mathcal{E}^2 \backslash \mathcal{L}$.

(*Hint.* Use Lemma 1.)

The proof in reverse direction is semi-constructive. Let $X \sigma Y$, i.e., $N(X, \mathcal{L}) = N(Y, \mathcal{L})$. If the segment $[XY]$ does not intersect $\mathcal{L}$, then $X \tau Y$. Therefore, we assume that $[XY] \cap \mathcal{L}$ is not empty set. By P and Q we denote the first and the last points of the intersection on the segment $[XY]$, respectively. We assume that these two points (not necessary two) do not belong to the set of vertices of $\mathcal{L}$.

Exercise 3. Why is it sufficient to prove the case when we assume that no vertex of $\mathcal{L}$ lies on $[XY]$?

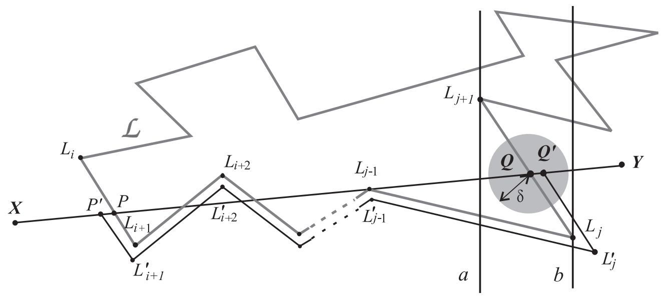

Figure 4: The broken line $P'L'_{i+1}\dots L'_jQ'$ is "close" to the broken line $PL_{i+1}\dots L_jQ$

(*Hint.* Consider the point Y' close to the point Y with property that no vertex of $\mathcal{L}$ lies on $[XY']$.)

Let P and Q belong to open segments $]L_iL_{i+1}[$ and $]L_jL_{j+1}[$ $(1 \le i, j \le n$, $L_{n+1} \stackrel{\text{def}}{=} L_1)$.

Let a and b be the lines through L_j and L_{j+1}, respectively, which are parallel to y-axis. We can choose a positive number δ so that

(1) the neighborhood $\mathcal{N}(Q,\delta)$ is settled into the strip between lines a and b,

(2) the neighborhood $\mathcal{N}(Q,\delta)$ and the open broken line $L_{j+1}L_{j+2}\dots L_iL_{i+1}\dots L_{j-1}L_j$ (i.e., $\mathcal{L}\setminus]L_jL_{j+1}[$) are disjoint, and

(3) the point Y does not belong to $\mathcal{N}(Q,\delta)$.

We introduce now a point P' between the points X and P $(X - P' - P)$ close to P and a broken line $P'L'_{i+1}L'_{i+2}\dots L'_jQ'$ "close" to the broken line $PL_{i+1}L_{i+2}\dots L_jQ$ (Figure 4) in the following meaning: these two broken lines are disjoint (corresponding segments can be parallel) and the point Q' lies on the segment $[XY]$ inside $\mathcal{N}(Q,\delta)$.

Exercise 4. Explain the construction of points L'_k in dependence on the fact that the point L'_{k-1} lies in the interior or in the exterior of the angle $\angle L_{k-1}L_kL_{k+1}$.

If we are sure that Q' lies between the point Q and Y, then we get the broken line $XP'L'_{i+1}L'_{i+2}\dots L'_jQ'Y$ connecting X and Y which does not intersect $\mathcal{L}$, i.e., $X\tau Y$. But, are we sure in it?

Exercise 5. Prove that the point Q' lies between Q and Y.

(*Hint.* Suppose that the point Q lies between Q' and Y and introduce a point Q'' between the points Q and Y inside $\mathcal{N}(Q,\delta)$ (Figure 5). By applying Lemma 1 prove that the values $n(Q',\mathcal{L}\setminus]L_jL_{j+1}[)$ and $n(Q'',\mathcal{L}\setminus]L_jL_{j+1}[)$ are of the same parity, i.e., the values $n(Q', \mathcal{L})$ and $n(Q'', \mathcal{L})$ are of opposite parity. By using Lemma 1 twice again obtain $N(X, \mathcal{L}) = N(Q', \mathcal{L})$ and $N(Q'', \mathcal{L}) = N(Y, \mathcal{L})$. From $N(Q', \mathcal{L}) \neq N(Q'', \mathcal{L})$ obtain $N(X, \mathcal{L}) \neq N(Y, \mathcal{L})$, which is in contradiction with our assumption $X \sigma Y$.)

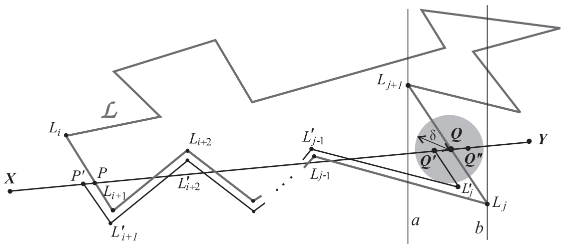

Figure 5: The values $n(Q', \mathcal{L})$ and $n(Q'', \mathcal{L})$ are of opposite parity

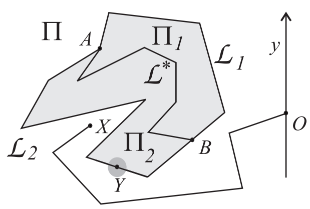

Figure 6: Polygon case of the Theta-Curve Theorem

The last exercise completes the proof of the polygon case JCT. □

We obtained two regions $\mathcal{R}_1$ and $\mathcal{R}_2$. How do we distinguish them?

Exercise 6. Prove that exactly one of these regions is bounded.

(*Hint.* Cover the broken line $\mathcal{L}$ by an enough big disk. Then prove that all points out of the disk are in the same region of $\mathcal{L}$. All points from the another region will be inside the disk.)

The bounded region (say $\mathcal{R}_1$) of $\mathcal{L}$ is called *interior* of $\mathcal{L}$; the other one ($\mathcal{R}_2$) is called the *exterior* of $\mathcal{L}$. In the proof of Theorem 1 we chose the special direction of y-axis. But, we can find the value $N(Z, \mathcal{L})$ for an arbitrary direction of y-axis and for any point $Z \in \mathcal{E}^2 \backslash \mathcal{L}$. Can we establish the position of the point Z with respect to $\mathcal{L}$ if we know the value $N(Z, \mathcal{L})$?

Exercise 7. Prove that for an arbitrary direction of y-axis,

$$N(Z, \mathcal{L}) = \begin{cases} 0 \quad , & \text{for all } \ Z \in \mathcal{O}_2 \\ 1 \quad , & \text{for all } \ Z \in \mathcal{O}_1 \ . \end{cases}$$

(*Hint.* Use the assertion of Theorem 1 and the fact that there exists a point Z (out of the mentioned disk) for which $N(Z, \mathcal{L}) = 0$ for any direction of y-axis.)

A *polygon* $\Pi : \ L_1 L_2 \ldots L_n$ determined by $\mathcal{L}$ is usually defined as the union of the simple closed broken line $\mathcal{L}$ and its interior.

Exercise 8. Prove that the interior, the exterior and the boundary of Π, i.e., $In(\Pi)$, $Ex(\Pi)$ and $Bd(\Pi)$, coincide with the interior of $\mathcal{L}$, exterior of $\mathcal{L}$ and the broken line $\mathcal{L}$ itself, respectively.

(*Hint.* Note that it is sufficient to prove that $\mathcal{R}_1 \subseteq In(\Pi)$, $\mathcal{R}_2 \subseteq Ex(\Pi)$ and $\mathcal{L} \subseteq Bd(\Pi)$.)

4. Polygon Case of the Theta-Curve Theorem

Theorem 2. (Polygon case of the Theta-Curve Theorem) *Every polygon* Π *is divided into two polygons by a simple broken line* $\mathcal{L}^*$ *whose end-points* A *and* B *lie on the boundary* $\mathcal{L}$ *of the polygon* Π *while other its points lie in the interior of* Π.

Proof. By introducing a simple broken line $\mathcal{L}^*$ we obtain two new simple closed broken lines, each of them formed by $\mathcal{L}^*$ and one of two open simple broken lines ($\mathcal{L}_1$ and $\mathcal{L}_2$) of $\mathcal{L}$ with end-points in A and B. Therefore, we obtain two new polygons, say Π_1 ($Bd(\Pi_1) = \mathcal{L}_1 \cup \mathcal{L}^*$) and Π_2 ($Bd(\Pi_2) = \mathcal{L}_2 \cap \mathcal{L}^*$). Our task is to prove four inclusions: $\Pi = \Pi_1 \cup \Pi_2$ and $\Pi_1 \cap \Pi_2 = \mathcal{L}^*$.

We begin by choosing a coordinate system with the y-axis (and thus the origin O) outside a disk covering the broken line $\mathcal{L}$. This disk also covers the region $In(\Pi)$, and consequently, broken lines $\mathcal{L}' \stackrel{\text{def}}{=} \mathcal{L}_1 \cup \mathcal{L}^*$ and $\mathcal{L}'' \stackrel{\text{def}}{=} \mathcal{L}_2 \cup \mathcal{L}^*$, and their interiors $In(\Pi_1)$ and $In(\Pi_2)$. Thus, $O \in Ex(\Pi) \cap Ex(\Pi_1) \cap Ex(\Pi_2)$.

Step I. Instead of proving that $\Pi_1 \cup \Pi_2 \subseteq \Pi$, we will prove its contrapositive.

Exercise 9. Prove that

$$Ex(\Pi) \subseteq Ex(\Pi_1) \cap Ex(\Pi_2). \tag{2}$$

(*Hint.* Consider an arbitrary point X from the set $Ex(\Pi)$ and the broken line connecting X and O which does not intersect the broken line $\mathcal{L}$. Note that this broken line lies entirely in the set $Ex(\Pi)$. Can it meet the broken line $\mathcal{L}^*$? Prove that X belongs to the same region as O with respect to both the broken line $\mathcal{L}'$ and the broken line $\mathcal{L}''$ (Figure 6).)

To prove the left inclusions, we need the following assertion.

Exercise 10. $N(Z, \mathcal{L}) = N(Z, \mathcal{L}') \oplus N(Z, \mathcal{L}'')$, for all $Z \in \mathcal{E}^2 \backslash (\mathcal{L} \cup \mathcal{L}^*)$, where $\oplus$ denotes addition modulo 2.

(*Hint.* Note that the number $n(Z, \mathcal{L}') + n(Z, \mathcal{L}'')$ is greater than $n(Z, \mathcal{L})$ for the double number of all proper points and proper sides of the intersection of $\mathcal{L}^*$ and $r^+(Z)$.)

Step II. To prove that $\Pi_1 \cap \Pi_2 = \mathcal{L}^*$, it is sufficient to prove that

$$Bd(\Pi_2) \cap In(\Pi_1) = \emptyset \tag{3}$$

(and analogously that $Bd(\Pi_1) \cap In(\Pi_2) = \emptyset$, as well), and that

$$In(\Pi_1) \cap In(\Pi_2) = \emptyset. \tag{4}$$

II a) Suppose that the relation (3) is not true, i.e., that there exists a point Y on the broken line $\mathcal{L}_2$, different from both A and B, which belongs to the set

$In(\Pi_1)$. Since this point does not belong to the broken line $\mathcal{L}' = \mathcal{L}_1 \cup \mathcal{L}^*$, there is a neighborhood of Y, say $\mathcal{N}(Y, \delta)$ ($\delta > 0$), which is disjoint from $\mathcal{L}'$ (δ can be chosen to be less than the distance of the point Y from $\mathcal{L}'$). Observe that all points of $\mathcal{N}(Y, \delta)$ are in the same region with respect to $\mathcal{L}'$, i.e., in $In(\Pi_1)$. Since $\Pi_1 \cup \Pi_2 \subseteq \Pi$ (2), we obtain $\mathcal{N}(Y, \delta) \subseteq In(\Pi_1) \subseteq \Pi_1 \subseteq \Pi$, which is in contradiction with $Y \in \mathcal{L}_2 \subseteq \mathcal{L} = Bd(\Pi)$.

II b)

Exercise 11. Prove that $In(\Pi_1) \cap In(\Pi_2) = \emptyset$.

(*Hint.* Suppose that the relation (4) is not true, i.e., that there exists a point $Z \in In(\Pi_1) \cap In(\Pi_2)$. Using the assertion of Exercise 9 we obtain $Z \in Ex(\Pi)$, which is in contradiction with (2).)

Step III. In the remainder of the proof we shall establish the relation $\Pi \subseteq \Pi_1 \cup \Pi_2$ by proving its contrapositive.

Exercise 12. Prove that $Ex(\Pi_1) \cap Ex(\Pi_2) \subseteq Ex(\Pi)$.

(*Hint.* Similar as in the previous exercise.)

The last exercise completes the proof of Theorem 2. □

5. The Game of Hex and its Connection with Theta-Curve Theorem

In what follows, we remind the reader to the article [3] about the game of Hex and to the connection between this game and the polygon case of theta-curve theorem.

Recall that the game of Hex was invented in 1942 by Piet Hein and later, in 1948, independently by John Nash. The game of Hex is played on a diamond-shaped board of tessellated hexagons, usually 11 on each edge, where by "tessellated" we mean fitted together like tiles to cover the board completely. An 7×7 Hex board is represented in Figure 7. Two opposite edges of the diamond are designated $\mathcal{O}$; the other two sides, $\mathcal{X}$ (the leftmost side of T (Figure 7) and the rightmost one of W are designated both $\mathcal{X}$ and $\mathcal{O}$). One player has a supply of X, the other of O counters. The players alternately place a piece on any vacant hexagon. The object of the game is for each player to complete an unbroken chain of his pieces between the sides labeled by his colour. The game cannot end in a draw because one player "must" win (sooner or later), which is claimed in the following (weak) Hex Theorem proved in [3]:

If every tile of the Hex board is marked either X or O, then there is either an X-path connecting edges of the diamond labeled by $\mathcal{X}$ or an O-path connecting edges of the diamond labeled by $\mathcal{O}$.

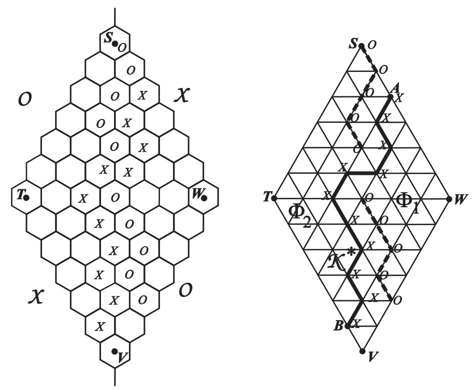

Figure 7: Hex board of size 7 and its dual graph

This theorem can be strengthened [3].

Theorem 3. *If every tile of the Hex board is marked either X or O, then there is either an X-path connecting edges of the diamond labeled by $\mathcal{X}$ or an O-path connecting edges of the diamond labeled by $\mathcal{O}$ but not both, i.e., it is impossible for both players to construct winning paths.*

Proof. To make it more clear, consider instead Hex board its dual graph (Figure 7) constructed in the following way. To each hexagon there is corresponding vertex of the graph and to each common edge of two adjacent hexagons there is a corresponding edge connecting the vertices corresponding to these hexagons. The above assertion says that if we have an X-path connecting the points A and B, belonging to opposite sides $[TV]$ and $[SW]$ of the parallelogram $STVW$, then it is impossible that an O-path connecting to vertices from the remaining two sides of the parallelogram exists.

The connection with our Theorem 1 is evident. Namely, the winning X-path represents a broken line entirely situated within the parallelogram $STVW$.

Consider first the case when this winning X-path is entirely situated within the interior of the parallelogram $STVW$, except its end-points A and B. Then, the polygon $STVW$ is separated into two polygons by this X-broken line (X-path). The sides $[ST]$ and $[VW]$ of the parallelogram are situated on the boundaries of different polygons obtained by X-broken line. Denote these polygons by Φ_1 and Φ_2, and this X-broken line by $\mathcal{K}^*$ (Figure 7). If there exists a winning O-broken line (O-path), then one of its end-points belongs to $[VW] \subseteq Bd(\Phi_1)\backslash\mathcal{K}^* \subseteq Ex(\Phi_2) \cap \Phi_1$ and the other one to $[TS] \subseteq Bd(\Phi_2)\backslash\mathcal{K}^* \subseteq Ex(\Phi_1) \cap \Phi_2$. Since all points of the O-broken line are from $\Phi_1 \cup \Phi_2$, there

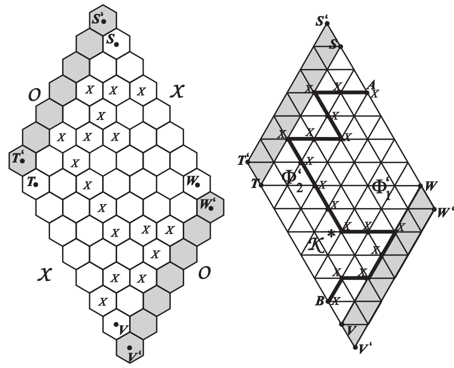

Figure 8: The widened hex board and its dual graph

exists a point on the O-broken line from intersection $Bd(\Phi_1) \cap Bd(\Phi_2) = \mathcal{K}^*$. So, O-broken line would have to cross the X-broken line. But it is possible only if the vertices of the dual graph and such vertex cannot be marked by both X and O simultaneously.

Consider now the possibility when this winning X-path contains some vertices belonging to the sides $[ST]$ and/or $[VW]$ which are neither A nor B (Figure 8). In this case, instead of the hex board, consider its *widened* hex board as it is shown in Figure 8. The *widened* parallelogram $S'T'V'W'$ is separated into two polygons (Φ_1' and Φ_2') by this X-broken line (X-path). The sides $[ST]$ and $[VW]$ of the parallelogram $STVW$ belong to different polygons because the parallelograms $SS'T'T$ and $VV'W'W$ are contained into different polygons. Consequently, a winning O-path would be crossed by the winning X-path again which is impossible. □

The reader interested in this topic can find a very elementary proof of the weak Hex Theorem (the state: "one of the players must win") and Nash's proof that there is a winning strategy for the player who starts on the homepage of the Dutch mathematician Jack van Rijswijck (www.cs.ualberta.ca/ javhar/hex).

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References

[1] O. Bodroža-Pantić, R. Doroslovački, K. Doroslovački, An elementary proof of a theorem concerning the division of region into two, *Rocky Mountain Journal of Mathematics*, In Print.

[2] A.F. Filippov, Élementarnoe dokazatelstvo teoremy Žordana (An elementary proof of the Jordan Curve Theorem), *Uspehi Mat. Nauk*, **5** (1950), 173-176.

[3] D. Gale, The game of Hex and the Brouwer Fixed-Point Theorem, *Amer. Math. Montly*, **86** (1979), 818-827.

[4] M.H.A. Newman, *Elements of the Topology of Plane Sets of Points*, Cambridge (1954).

[5] E.H. Spanier, *Algebraic Topology*, McGraw-Hill (1966).

[6] H. Tverberberg, *Bull. London Math. Soc.*, **12** (1980), 34-38.

[7] A.I. Volpert, Élementarnoe dokazatelstvo teoremy Žordana (An elementary proof of the Jordan Curve Theorem), *Uspehi Mat. Nauk*, **5** (1950), 168-172.