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THE LEVY-KHINTCHINE REPRESENTATIONS AND FUNCTIONAL ALGEBRAS OF TEST FUNCTIONS

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Abstract: We discuss a new approach for the proof of the Levy-Khintchine formula for the V-infinitely divisible laws. Our proof is based on a description of the conditionally positive definite functions as positive functionals on semi-normed algebras of suitable test functions. In the framework of this approach we obtain integral representations of the common continuous positive definite functions and the logarithms of characteristic functions of the ordinary infinitely divisible and V-infinitely divisible distribution.

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1. Introduction

The Levy-Khintchine representation is important in probability theory, since it provides analytical descriptions of very important probability entities such as

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the infinitely divisible distributions and the infinitesimal operators of probability semi-groups. The classical proof of this formula is based on the so called approximation by means "the accompanying laws" (see, for example, [8], Part 5). Another approach is provided by Johansen [4]. He applies the Choquet's Theorem on the representation by means of the external points of a compact convex set (this result is also presented in the well known book of Y. Linnik and J. Ostrovsky [7], Part 11). Such a method could also be used for the proof of the classical Bochner's Theorem on the integral representation of the continuous positive definite function. A crucial component of Johansen's approach is the conception of the negative definite functions having properties analogous to the properties of the ordinary positive definite functions. It is necessary to note that a connection between the two mentioned notions has been discussed in [1]. A representation for infinitely divisible distributions on commutative local compact groups has been obtained with the help of "the shift-compact property of the divisors set" [11].

A generalization of the infinitely divisible property called V- infinitely divisible distributions has been offered in [13] and [15]. These distributions arise in problems of stochastic centering of sums of the real independent random variables and are related to the super-stable distributions of V. Zolotarev. Characterization of probability laws by means of stochastic properties of linear statistics can be considered as another source for such distributions.

The similarity of integral representations of ordinal characteristic function and logarithms of characteristic function of regular and V-infinitely divisible distributions leads to conclusion that all named representations can be interpreted and analyzed from some common general point of view. Apparently, it can be done using the Choquet's Theorem on the representation by means of the external points of a compact convex set. However such an approach may cause difficult analytical calculations, as it was demonstrated in [4]. A methodology to the study of the representations of the mentioned distribution from the viewpoint of suitable test functions algebras is discussed in the current paper. The key point of our research is a link from V-infinitely divisible distributions to the so named conditionally positive definite functions. Such functions have been introduced in the paper of Micchelli [9] and are widely utilized in the approximation theory. Integral representations of conditionally positive definite functions can be used for the Levy-Khintchine formulas' explorations and vice versa. A general description of the considered functions is provided by the integral representation of the positive functional produced on an appropriate semi-normed ring. The Bochner's Theorem, the Levy-Khintchine formula and a representation of characteristic functions of V-infinitely divisible distributions can be considered as partial cases of this general representation. As proved in [16], these representations can be obtained in the framework of our method by means of a generalization of the Bochner-Schwartz Theorem presented in [3].

The organization of this paper is as follows: in Section 2 we mention known and our results concerning V-infinitely divisible laws and their connection to the problems of characterization distributions by stochastic properties of linear statistics. In Section 3 we explore the relationship between the V-infinitely divisible laws and conditionally positive definite functions. Section 4 is devoted to the construction and research of semi-normed rings formed by suitable sets of test functions based on a new convolution-type operation. In Section 5 we derive an analogue of the Levy-Khintchine canonical representation for the characteristic functions of the V-infinitely divisible distributions. We conclude with examples of several known representations.

2. *V*-Infinitely Divisible Laws

Let us consider the set $B(\mathbb{R}^1)$ of all probability measures on $\mathbb{R}^1$ and denote by $\Phi(Q,t)$, $t \in \mathbb{R}^1$ the characteristic function of a probability measure $Q \epsilon B(\mathbb{R}^1)$.

Definition 1. A distribution $Q \in B(\mathbb{R}^1)$ is called V-infinitely divisible (V-i.d.) distribution if for each natural n there exists a distribution Q_n and a non-negative number a_n such that:

$$e^{-a_n|t|^2}\Phi(Q,t) = \Phi^n(Q_n,t), \tag{1}$$

for every $t \in \mathbb{R}^1$.

Let us define the subsets $V_s, s > 0$, of V-i.d. distributions in the relation to the asymptotic behavior of a sequence a_n in (1) by

$$a_n = O(1) * n^{\alpha}, \quad \alpha = 1 - \frac{1}{s+1}, \quad n \to \infty.$$

It is evident that:

1. V_0 is the set of regular i.d. distributions;
2. $V_s \subset V_{s+1}$ for every s;
3. $\Phi(Q,t) \neq 0$ for all t, if $Q \in V_s$ (it has been proved in [13] and [15]).

Examples of laws $Q \in V_s$ can be obtained by the Linnik's distribution having the characteristic function:

$$\Phi(Q,t) = \exp(-A\,|t|^2 - |t|^\gamma)\,,$$

where $\left[\frac{\gamma}{2}\right]$ is an even number such that γ is not an even number and A is a positive constant. It is easy to see that $Q\epsilon V_s$ for

$$s > \left(\frac{\gamma}{2} - 1\right).$$

Let us discuss how V-i.d. distributions arise in the characterizations of distributions by stochastic properties of linear statistics (see, for example [5], Part 2 and Part 5). We are particularly interested in characterizations by the properties of two linear sample statistics identically distributed. First let us state several results from [6] and ([5], Part 2). Let $\{x_j\}$, $j = 1, ..., m$ be a sequence of independent identically distributed random values in $\mathbb{R}^1$ that obey a distribution Q, and let $\{b_j\}$, $j = 1, ..., 2m$ be real numbers. We construct the forms :

$$L_1 = \sum_{j=1}^{m} b_j x_j\,, \quad L_2 = \sum_{j=1}^{m} b_{m+j} x_j$$

and consider the problem of the characterization probability laws by the properties of identical distribution of L_1 and L_2. This condition is equivalent to the following relation:

$$\prod_{j=1}^{m} \Phi(Q, b_j t) = \prod_{j=1}^{m} \Phi(Q, b_{m+j} t).$$

Since $\Phi(Q,t)$ is different from zero in some neighborhood of the zero point, we may consider, in this neighborhood, the function

$$\omega(t) = \log(\Phi(Q,t))$$

(here we choose the principal branch of the logarithmic function) and

$$\sum_{j=1}^{m} \omega(b_j t) = \sum_{j=1}^{m} \omega(b_{m+j} t) \tag{2}$$

in some neighborhood of the zero point. However, if

$$\max_{1 \le j \le m} |b_j| \neq \max_{1 \le j \le m} |b_{j+m}|\,, \tag{3}$$

then:

1. $\Phi(Q,t)\neq 0$ for all $t\in\mathbb{R}^1$, and the function $\omega(t)$ can be defined for all $t\in\mathbb{R}^1$;

2. $\omega(t)=O(t^\beta)$, for $t\to\infty$ and some $\beta>0$.

Polynomial expansion of the function $\omega(t)$ according to the roots of the function

$$\sigma(z)=\sum_{j=1,\ b_j\neq 0,\ b_{j+m}\neq 0}^{m}\left(|b_j|^z-|b_{m+j}|^z\right)$$

are the most important aspects in an investigation of the equality (2) and the general characterizations of the Gaussian law by means of two identical distributed linear statistics are given in terms of them. Such an expansion can be written when (3) holds, for each $\Lambda>0$ as

$$\int_0^t \log\left(\frac{t}{u}\right)u^{\Lambda-1}\omega(u)du=\Pi(t)+S(t),$$

where

$$\Pi(t)=t^\Lambda\sum_{k=0}^{k^*}t^{\gamma_k}P_{\gamma_k}(\log t)$$

and $\gamma_k,\ k=0,...,k^*$ are the real roots of $\sigma(z)$ in the interval $[0,\beta+1]$; P_{γ_k} $k=0,...,k^*$ are polynomials having degrees m_k bounded from above by a constant k^{**},

$$S(t)=t^\Lambda\sum_{k=0}^{\infty}\mathrm{Re}\left(\sum_{m=0}^{m_k}t^{\gamma_{m_k}}P^*_{\gamma_{m_k}}(\log t)\right),$$

γ_{m_k} are the not real roots of $\sigma(z)$, and $P^*_{\gamma_k}$ are certain polynomials. We restrict ourselves in this paper to the case where $S(t)\equiv 0$.

Theorem 1. *Let Q be a symmetric distribution satisfies (2) such that:*

1. $\sigma(2)=0$,

2. the relationship (3) is held,

3. $S(t)\equiv 0$.

Then $Q\in V_s$ for some $s>0$.

The theorem above can be proved in a manner very similar to those of Lemma 13 and Lemma 15, [6] and in the Lemma 2.4.2, [5]. For this reason we omit several technical details of the proof.

Proof. According to our assumption the function $\omega(t)$ can be written for all $t \in R^1$ in the form

$$\omega(t) = -Gt^2 - \sum_{k=0}^{k^*} t^{\gamma_k} P_{\gamma_k}(\log t), \tag{4}$$

where $G > 0$. Note, that if, in (4), all γ_k , $k = 0, ..., k^*$ are even integers and $m_k = 0$, $k = 0, ..., k^*$ then by the well-known theorem of Marcinkiewicz (see, for example [8], Part 7) Q is a normal distribution and the theorem is proved. Otherwise there exists a number γ_k which is not an even number or is not a simple zero point. Let us denote by γ^* the minimal such number. In order to avoid unnecessary considerations we suppose: $t^{\gamma_{k^*}} P_{\gamma_{k^*}}(\log t) \not\equiv 0$ and denote $\gamma_{k^*} = \gamma^{**}$. Obviously, the theorem is proved if we will show that for every natural n the function

$$f_n(t) = \exp\left(-B_n t^2 + \frac{\omega_1(t)}{n}\right)$$

can be a characteristic function of some distribution F_n for a sequence

$$B_n = O(1)n^{\delta} \ \ , \ n \to \infty$$

for some $\delta \leq 0$, where

$$\omega_1(t) = -\sum_{k=0}^{k^*} t^{\gamma_k} P_{\gamma_k}(\log t)\,.$$

Denote by $q(x)$ the density of Q then

$$q(x) = \frac{1}{2\pi}\mathrm{Re}(\int_0^\infty \exp(-itx + \omega(t))dt)\,.$$

Let us now investigate the asymptomatic behavior of Q. We are concerned about $x > Gx_0$, where x_0 will be chosen later. It is easy to show, by standard calculations, that

$$\int_0^\infty \exp(-itx + \omega(t))dt = \left(\int_{L_1} + \int_{L_2}\right) \exp(-itx + \omega(t))dt,$$

where L_1 is the interval in the complex plane $0 \le \mathrm{Im}(z) \le L$, $\mathrm{Re}(z) = 0$ and L_2 is the half-line $0 \le \mathrm{Re}(z)$, $\mathrm{Im}(z) = L$. If $L = x^{-1}\log^2(x)$ then

$$\int_{L_2} \exp(-itx + \omega(t))dt = O(\exp(-\log^2(x)))\,,\ x \to \infty\,.$$

For x_0 sufficiently large

$$\mathrm{Re}\left(\int_{L_1} \exp(-itx - \omega(t))dt\right) = \mathrm{Re}\left(C\int_0^\infty e^{-tx}t^{\gamma^*}(\log(t) + \frac{\pi}{2}i)^{m_{j^*}}\,dt\right) + O\left(\frac{G(\log(x)^{m_{j^*}})}{x^{\gamma^*+2}}\right),$$

where C is a constant and m_{j^*} is the degree of the polynomial $P_{\gamma^*}(\log t)$. Taking into account

$$\int_0^\infty e^{-tx}t^h * \log^g(t)dt = \frac{\Gamma(h+1)}{x^{h+1}}(-\log(x))^g + O\left(\frac{\log^{g-1}(x)}{x^{h+1}}\right)\,,\ x \to \infty\,,$$

for $h > 0$, $g \ge 0$, we obtain that

$$q(x) = C_0\frac{(\log(x))^{m_{j^*}}}{x^{\gamma^*+1}} + O\left(\frac{\log(x)^{m_{j^*}-1}}{x^{\gamma^*+1}}\right) \tag{5}$$

for some constant $C_0 > 0$ and $x > Gx_0$, where x_0 is chosen to be sufficiently large independently of G. Now, we introduce

$$q_n(x) = \frac{1}{\pi}\mathrm{Re}\,(\int_0^\infty \exp(-itx)f_n(t)dt),\ x \ge 0.$$

Substituting

$$y = \frac{x}{\sqrt{B_n}}\,,\ u = t\sqrt{B_n}\,.$$

We obtain

$$q_n(y\sqrt{B_n}) = \frac{1}{\pi\sqrt{B_n}}\mathrm{Re}\left(\int_0^\infty \exp\left(-iuy - u^2 + \omega_1\left(\frac{u}{\sqrt{B_n}}\right)\right)du\right)\,.$$

We can conclude that the functions $q_n(x)$ obey the asymptotic representation (5) but with a different value of C_0 for $y > y_0$, where y_0 is chosen to be sufficiently large independently of B_n. Let us fix x_0 and demonstrate that $q_n(x) > 0$ for $0 \le y < y_0$, where $B_n \ge 0$ is a sufficiently large. Denote $u_0 = \log(B_n)$ then:

$$q_n(x) = \frac{1}{\pi\sqrt{B_n}} \mathrm{Re} \left(\int_0^{u_0} \exp\left(-iuy - u^2 + \omega_1\left(\frac{u}{\sqrt{B_n}}\right)\right) du \right) + O\left(e^{-u_0^2}\right).$$

Choose

$$B_n = Cn^\alpha \log^{2m_{\gamma^{**}}}(n), \quad \alpha = -\frac{2}{\gamma^{**}},$$

where C is chosen independently of n so that

$$\left| \frac{1}{n} \sum_{k=0}^{k^*} \left(\frac{\log(B_n)}{\sqrt{B_n}}\right)^{\gamma_k} P_{\gamma_k}(\log\left(\frac{\log(B_n)}{\sqrt{B_n}}\right) \right| < 1.$$

Using the Taylor expansion we obtain:

$$q_n(y\sqrt{B_n}) = \frac{1}{\pi\sqrt{B_n}} \mathrm{Re} \int_0^\infty \exp\left(-iuy - \frac{u^2}{n}\right) dt + O\left(\left(\frac{\log^{m_{j^*}}(B_n)}{B_n}\right)^{-\frac{1+\gamma^*}{2}}\right),$$

that gives

$$q_n(y\sqrt{B_n}) = \frac{1}{2\pi} \exp(-\frac{y^2}{4}) + O\left(\left(\frac{\log^{m_{j^*}}(B_n)}{B_n}\right)^{-\frac{1+\gamma^*}{2}}\right).$$

Thus $q_n(x) > 0$ for $0 \le y < y_0$, and the theorem is proved. □

3. Conditionally Positive Definite Functions

In this section we take into consideration a connection between the notions of V-i.d. distributions and conditionally positive definite functions. We introduce the set D_0 of all test functions (i.e. infinitely differentiable complex functions having compact support) on the real line $\mathbb{R}^1$. D_k, $k > 1$ are subsets of D_0 such that:

$$\int x^j f(x)dx = 0,$$

for all $0 \leq j \leq k-1$. Note, whenever no integration limits are specified a lower limit of $-\infty$ and an upper limit of $+\infty$ is assumed. Denote by ∂ the differentiation operation on D_0, and by

$$I(f)(x) = \int_{-\infty}^{x} f(u)du$$

the integration operation on D_0. It can easily be verified that

$$\partial(D_k) = D_{k+1} \quad , \quad I(D_k) = D_{k-1}, \quad k > 0.$$

We will use the same notations for the conjugate operations in the dual space of generalized functions as well.

Definition 2. A continuos function h on $\mathbb{R}^1$ is called m-positive definite if

$$\int\int h(x-y)f(x)\overline{f(y)}dxdy \geq 0$$

for each $f \in D_m$, where $\overline{f(y)}$ is the conjugate of $f(y)$.

It is well known that a 0-positive definite function is a regular positive definite function. Due to the famous Bochner's Theorem the set of such functions coincides with the set of all normed characteristic functions. For a 1-positive definite h the function $(-h)$ is a negative definite function. Such a function arises (see [1]) as the logarithm of a characteristic function of an infinitely divisible distribution. As it was mentioned above, m-positive definite functions have been introduced in the approximation theory and are named conditionally positive definite functions (see [9]).

Theorem 2. *If $Q \in V_s$, then the function $f(t) = \log(\Phi(Q,t))$ is a $m-$positive definite function for $m > s+1$.*

Proof. Consider $Q \in V_s$ and take a test function $h \epsilon D_m$. As it was noted above

$$\int\int \Phi(Q_n, t_1 - t_2)h(t_1)\overline{h(t_2)}dt_1dt_2 \geq 0\,, \tag{6}$$

for the distribution Q_n defined by $Q \in V_s$ in the equation (1) for each natural n. Due to (1) and (6)

$$I_n = n \int\int \exp\left(\frac{f(t_1 - t_2)}{n} - c_n(t_1 - t_2)^2\right) h(t_1)\overline{h(t_2)} dt_1 dt_2 \geq 0\,, \qquad (7)$$

where

$$c_n = \frac{a_n}{n} = O(1) n^{-\frac{1}{s+1}}, \quad n \to \infty. \qquad (8)$$

It follows from (7) that:

$$I_n = \iint n\left(\exp\left(\frac{f(t_1 - t_2)}{n} - 1\right) * \exp\left(-c_n(t_1 - t_2)^2\right)\right) h(t_1)\overline{h(t_2)} dt_1 dt_2 + \int\int n \exp\left(-c_n(t_1 - t_2)^2\right) h(t_1)\overline{h(t_2)} dt_1 dt_2 \geq 0\,.$$

The Taylor-series expansion gives us

$$n \exp\left(-c_n(t_1 - t_2)^2\right) = \sum_{j=0}^{\infty} \frac{n\left(-c_n(t_1 - t_2)\right)^{2j}}{j!}\,.$$

Since $m > s + 1$ and using (8) we obtain for each value of $(t_1 - t_2)$

$$\lim_{n\to\infty} \sum_{j=m}^{\infty} \frac{n\left(-c_n(t_1 - t_2)\right)^{2j}}{j!} = 0\,.$$

On the other hand $h \in D_m$, therefore

$$\int\int \left(\sum_{j=0}^{m-1} \frac{n\left(-c_n(t_1 - t_2)\right)^{2j}}{j!}\right) h(t_1)\overline{h(t_2)} dt_1 dt_2 = 0.$$

It follows:

$$\lim_{n\to\infty} I_n = \int\int f(t_1 - t_2) h(t_1)\overline{h(t_2)} dt_1 dt_2$$

and we conclude from (6)

$$\int\int f(t_1 - t_2) h(t_1)\overline{h(t_2)} dt_1 dt_2 \geq 0\,.$$

Thus, the theorem is proved. □

4. Integer Transform and Semi-Normed Ring

In this section we obtain an integral representation for the logarithm of a characteristic function of the $V-$i.d. distributions. For this purpose we endow the set D_n, for an even $n = 2m$, with a semi-normed ring structure using an appropriate integral transform having the kernel

$$K_n(x,t) = \frac{\exp(itx)}{(ix)^n}\,,$$

where $x \neq 0$, for $n > 0$. Note that the creation of a convolution-like operation, by this way, is similar to the approach for stochastic convolutions' construction described in [12] and [14]. Now, the integral transform is defined as

$$\Psi_n(h,x) = \int K_n(x,t)h(t)dt = \frac{1}{(ix)^n}\int \exp(itx)h(t)dt, \quad h \in D_n\,.$$

It is easy to see that each function $h \in D_n$ is uniquely reconstructed by its transform and the transform can be considered as a continuous function of $x \in \mathbb{R}^1$. Let us introduce a binary operation on D_n:

$$(h \circ g)(t) = I^{(n)}(h * g)\,,$$

where "$*$" denotes the regular convolution of the functions $h, g \in D_n$ and $I^{(n)}$ is the n-th iteration of the integral operator I on D_n described in the previous section.

Theorem 3. *If* $f = (h \circ g)$ *then:*
1. $f \in D_n$;
2. $\Psi_n(f,x) = \Psi_n(h,x)\Psi_n(g,x)$ *for all* $x \in \mathbb{R}^1$.

Proof. First, we show that $f \in D_n$. For this purpose we caluculate

$$T_j = \int t^j f(t)dt = \int t^j I^{(n)}(h*g)(t)dt, \quad j = 0, ..., n-1.$$

We obtain

$$T_j = \iint \left((u+v)^{j+n} + P_{n-1}(u+v)\right) h(u)g(v)dudv$$

for a polynomial P_{n-1} of a degree less than n and $T_j = 0$, $j = 0, ..., n-1$ since $h, g \in D_n$. In addition the following holds:

$$\Psi_n(h \circ g, x) = \frac{1}{(ix)^n} \int \exp(itx) I^{(n)}(h * g)(t)dt$$
$$= \frac{1}{(ix)^n} \int \frac{(\exp(itx) + P_{n-1}(tx))}{(ix)^n}(h * g)(t)dt,$$

where once more P_{n-1} is a polynomial of a degree less than n. Hence for all $x \in \mathbb{R}^1$

$$\Psi_n(h \circ g, x) = \frac{1}{(ix)^{2n}} \int \exp(itx)(h * g)(t)dt = \Psi_n(h, x)\Psi_n(g, x).$$

The theorem is proved. □

By the theorem above, the triplet $(D_n, +, \circ)$ yields a commutative ring. We can also define an involution operation $h^*(t) = \overline{h(-t)}$ and a family of semi-norms on this ring by:

$$\|h\|_j = \max_{|x| \le j} |\Psi_n(h, x)|, \quad j = 1, 2, 3, \dots.$$

It is easy to see that:

1. $\Psi_n(h^*, x) = \overline{\Psi_n(h, x)}$, for all x;
2. $\|h^*\|_j = \|h\|_j$, for all j;
3. $\|\Psi_n(h^* \circ h, x)\|_j = \|\Psi_n(h, x)\|_j^2$, for all j;
4. The topology produced by the system of semi-norms $\|\cdot\|_j$ is not weaker than "$*$"- the weak topology induced by the set of all continuous functions on $\mathbb{R}^1$.

Denote by R_n the closure of the symmetric algebra defined above, with the described topology. Let us build up a functional on R_n defined by a continuous function f:

$$F_n(f)(h) = \int (-1)^m f(t)h(t)dt$$

(here $n = 2m$).

Recall the following definition.

Definition 3. A linear functional F on R_n is called positive if

$$F_n(f)(h \circ h^*) \ge 0$$

for each $h \in D_n$.

Theorem 4. *If f is a m-positive definite function then the functional $F_n(f)(h)$ is a positive continuous functional.*

Proof. Let us calculate $F_n(f)(h \circ h^*)$, for some $h \in D_n$. According to the definition

$$F_n(f)(h \circ h^*) \\ = \int (-1)^m f(t)(h \circ h^*)dt = \iint (-1)^m I^{(n)}(f)(u-v)h(u)\overline{h(v)}dudv.$$

Choose $g \in D_m$, such that $\partial^{(m)}(g) = h$ (the m-th derivative of g equals h). We have:

$$\begin{aligned} F_n(f)(h \circ h^*) &= \iint (-1)^m I^{(n)}(f)(u-v)\partial^{(m)}(g)(u)\overline{\partial^{(m)}(g)(v)}dudv \\ &= \iint f(u-v)g(u)\overline{g(v)}dudv \end{aligned}$$

and $F_n(f)(h \circ h^*) \geq 0$ by Theorem 2. □

As a result of this theorem, representations of the positive functional on semi-normed rings can be exploited. Do-Shin has investigated such representations in [2]. We state here several relevant results of this paper (Theorems 5, 6 and 7) which we will use later.

Let H be a symmetric, complete and commutative ring with unit e.

Theorem 5. *If H is a ring with a countable set of semi-norms and a continuous involution then each positive functional on H is continuous.*

Recall that a linear functional ϕ on H is called real multiplicative if:
1. $\phi(h \circ g) = \phi(h)\phi(g)$ for all $h, g \in H$;
2. $\varphi(h^*) = \overline{\varphi(h)}$ for all $h \in H$.

We denote by $M(H)$ the topological space (with the weak topology) of all the real multiplicative functionals of H. A linear functional ϕ on H is called real multiplicative if:
1. $\phi(h \circ g) = \phi(h)\phi(g)$ for all $h, g \in H$;
2. $\varphi(h^*) = \overline{\varphi(h)}$ for all $h \in H$.

Theorem 6. *If H is a ring with a countable set of semi-norms Δ and*

$$\sup_{\alpha\in\Delta} \|x \circ x^*\|_\alpha = \sup_{\alpha\in\Delta} \|x\|_\alpha \sup_{\alpha\in\Delta} \|x^*\|_\alpha, \text{ for all } x \in H$$

then:
1. *$\|x \circ x^*\|_\alpha = \|x\|_\alpha \|x^*\|_\alpha$, for all $x \in H$;*
2. *H is a complete ring for each semi-norm;*
3. *H is completely isomorphic to the ring $C(M(H))$ of all continuous functions on $M(H)$ with it's intrinsic operations.*

Note that Theorem 6 is an analog of the famous Gelfand-Naimark Theorem on normed symmetric rings.

Theorem 7. *If H is a ring with a continuous involution and F is a positive functional such that $F(e) = 1$, then there exist a compact subset $M_F \subset M(H)$ and a normed measure μ on $\mathbb{R}^1$ so that for all $h \in H$:*

$$F(h) = \int\limits_{M_F} \phi(h) d\mu(\phi), \; h \in H. \tag{9}$$

It may be desirable to use the last theorem for our purpose, however the problem is that the built rings R_n do not contain a unit. Formally, a unit can be attached, but it would introduce the difficulty of a precise extension of a positive definite functional. We use the methodology of unit approximation (see, for example [10], Part 2.10).

Let us denote by $R_n^{'}$ the ring obtained by attaching the unit e to R_n and take a "cap functions" family:

$$\omega_\varepsilon(t) = \begin{cases} C_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |t|^2}\right), & |t| \leq \varepsilon\,, \\ 0, & \text{else}\,, \end{cases}$$

where C_ε is the normalizer so that:

$$\int \omega_\varepsilon(t) dt = 1. \tag{10}$$

It is well known that

$$\lim_{\varepsilon \to 0} \int f(t) \omega_\varepsilon(t) dt = f(0)$$

for each continuous function f. Let us introduce:

$$\omega_{\varepsilon,n}(t) = \partial^{(n)}\left(\omega_\varepsilon(t)\right).$$

Theorem 8. *The following statements hold:*
1. $\|\omega_{\varepsilon,n}(t)\|_j \leq 1$ for each semi-norm j;
2. $\lim\limits_{\varepsilon \to 0} \|h - h \circ \omega_{\varepsilon,n}(t)\| = 0$ for each $h \in R_n$ and each semi-norm j;
3. The sequence

$$c_n(f) = \int f(t)(\omega_{\varepsilon,n} \circ \omega^*_{\varepsilon,n})(t) dt$$

converges for each continuous function f, if $\varepsilon \to 0$.

Proof. Let us calculate:

$$\Psi_n(\omega_{\varepsilon,n},x)=\frac{1}{(ix)^n}\int\exp(itx)\omega_{\varepsilon,n}(t)dt=\int\exp(itx)\omega_{\varepsilon}(t)dt. \tag{11}$$

By (10) statement 1 of the theorem is true. Next

$$\begin{aligned}\|h-h\circ\omega_{\varepsilon,n}(t)\|_j &= \max_{|t|\le j}|\Psi_n(h,x)\,(1-\Psi_n(\omega_{\varepsilon,n},x))| \\ &\le \max_{|t|\le j}|\Psi_n(h,x)|*\max_{|t|\le j}|(1-\Psi_n(\omega_{\varepsilon,n},x))|\,.\end{aligned}$$

However, due to (11)

$$\lim_{\varepsilon\to 0}\Psi_n(\omega_{\varepsilon,n},x)=1.$$

Hence second assertion is also proved. Now we consider for $f=\exp(itx)$

$$\begin{aligned}c_n(f) &= \iint I^{(n)}(f)(u-v)\omega_{\varepsilon,n}(u)\overline{\omega_{\varepsilon,n}(v)}dudv \\ &= \frac{1}{(ix)^n}\iint\exp(i(u-v)x)\omega_{\varepsilon,n}(u)\overline{\omega_{\varepsilon,n}(v)}dudv \\ &= (ix)^n\left|\int\exp(iux)\omega_{\varepsilon}(u)du\right|^2\to(ix)^n\,,\quad\text{for }\ \varepsilon\to 0.\end{aligned}$$

As t,x can arbitrarily be chosen the proof is complite. □

Theorem 9. *For all m-positive definite functions f, the functional $F_n(f)$ can be extended as a positive continuous functional on $R_n^{'}$.*

Proof. Note, that due to the Cauchy-Bunyakovsky inequality

$$|F_n(f)(h\circ\omega_{\varepsilon,n})|^2\le F_n(f)(\omega_{\varepsilon,n}\circ\omega^*_{\varepsilon,n})*F_n(f)(h\circ h^*)\,,$$

where $h\in R_n$ and the function $\omega_{\varepsilon,n}$ is defined above. According to the previous theorem we conclude that

$$|F_n(f)(h)|^2\le C_n(f)F_n(f)(h\circ h^*)\,, \tag{12}$$

where

$$C_n(f)=\lim_{\varepsilon\to 0}F_n(f)(\omega_{\varepsilon,n}\circ\omega^*_{\varepsilon,n}).$$

Every element g of $R_n^{'}$ is represented in the form $g=\lambda e+h$, where λ is a complex number and $h\in R_n$. We extend the functional $F_n(f)$ on $R_n^{'}$ by:

$$F_n(f)(g)=\lambda C_n(f)+F_n(f)(h).$$

Let us check if this extension leads to a positive functional on $R_n^{'}$:

$$F_n(f)(g \circ g^*) = |\lambda|^2 F_n(f)(e) + \lambda F_n(f)(h^*) + \bar{\lambda} F_n(f)(h) + F_n(f)(h \circ h^*).$$

The relationship (12) implies $F_n(f)(g \circ g^*) \geq 0$ and, thus the theorem is proved. □

5. Integral Representations and the Levy-Khintchine Formulas

In this section we consider the integral representations of the m-positive definite functions based on the results obtained in the previous section. By Theorem 6 the ring $R_n^{'}$ is completely isomorphic to the ring $\mathcal{C}(M(R_n^{'}))$ of all continuous functions on $M(R_n^{'})$ with its intrinsic operations. Here $M(R_n^{'})$ is the topological space (with the weak topology) of all the real multiplicative functionals of H. Therefore, $M(R_n^{'})$ could be identified to $\mathbb{R}^1$ and from the Theorem 4 and Theorem 7 we conclude that for each m-positive definite function f:

$$F_{2m}(f)(h) = \int (-1)^m f(t)h(t)dt = \int \Psi_{2m}(h,x)\mu(dx)$$

$$= \iint K_{2m}(x,t)h(t)\mu(dx)dt = \iint \frac{\left(\exp(itx) - \sum_{j=0}^{2m-1} \frac{(itz)^j}{j!}\right)}{\left(ix\right)^{2m}} \mu(dx)h(t)dt,$$

for a finite measure μ on $\mathbb{R}^1$. Moreover

$$F_{2m}(f)(h) = \int \mu(dx) \int \frac{\left(\exp(itx) - \sum_{j=0}^{2m-1} \frac{(itz)^j}{j!}\right)}{\left(ix\right)^{2m}} h(t)dt$$

$$= \int h(t)dt \int \frac{\left(\exp(itx) - \sum_{j=0}^{2m-1} \frac{(itz)^j}{j!}\right)}{\left(ix\right)^{2m}} \mu(dx).$$

It implies that:

$$f(t) = P_{2m-1}(t) + \int \frac{\left(\exp(itx) - \sum_{j=0}^{2m-1} \frac{(itz)^j}{j!}\right)}{x^{2m}} \mu(dx), \tag{13}$$

where $P_{2m-1}(t)$ is a polynomial of degree $(2m-1)$. Result (13) can be summarized in the following form.

Theorem 10. *Let f be a m-positive definite function then there exist a polynomial $P_{2m-1}(t)$, of degree $(2m-1)$, and a finite measure μ on $\mathbb{R}^1$ such that the representation (13) is held. The polynomial $P_{2m-1}(t)$ and the measure μ are uniquely defined by f.*

The uniqueness of (13) has been proved in [15]. We further discuss two meaningful partial cases of the representation (13).

Corollary 1. *A continuous function f on $\mathbb{R}^1$ satisfying $f(0) = 1$ is positive definite if and only if this function is a characteristic function of some distribution on $\mathbb{R}^1$.*

In this case f is a 0-positive definite function and due to (13) we obtain for $f(0) = 1$

$$f(t) = \int \exp(itx)\mu(dx),$$

where μ is a probability measure on $\mathbb{R}^1$. This is the Bochner representation for the positive definite functions.

Corollary 2. *A probability measure Q is infinitely divisible if and only if*

$$\log \Phi(Q,t) = ict + \int \frac{\left(\exp(itx) - 1 - itx\right)}{x^2} \mu(dx), \tag{14}$$

where c is a real constant and μ is a finite measure on $\mathbb{R}^1$.

This corollary provides a variant of the Levy-Khintchine formula for infinitely divisible distributions. As we mentioned earlier, if a distribution Q is infinitely divisible then $f(t)$=log$(\Phi(Q,t))$ is a 1-positive definite function. Taking into account that $f(0) = 0$, $f(-t) = \overline{f(t)}$ we obtain (14) from (13). In conclusion, we note that the equation (14) coincides, in the case $m > 1$, with the representation of $V-$ infinitely divisible distributions obtained in the papers [13] and [15].

References

[1] N. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publishing Co., New York (1965).

[2] S. Do Shin, On semi-normed rings with involution, *Izv. Akad. Nauk USSR*, **23**, No. 4 (1959), 509-528, In Russian.

[3] C. Jaeyoung, C. Soon-Yeong, K. Dohan, Bochner Schwartz type theorem for conditionally positive definite fourier hyperfunctions, *Positivity*, **7** (2003), 323-334.

[4] S. Johansen, An application of extreme-point methods to the representation of infinitely divisible distributions, *Z. Wahrscheinlichkeits-Theorie Verb. Geb.*, **5** (1966), 304-316.

[5] A. Kagan, J. Linnik, C.R. Rao, *Characterization Problems in Mathematical Statistics*, John Wiley and Sons (1973).

[6] Y. Linnik, Linear forms and statistical criteria, *Ukr. Math. J.*, **5**, No. 2-3 (1953), 207-243, 247-290, In Russian.

[7] Y. Linnik, J. Ostrovskii, *Decomposition of Random Variables and Vectors*, American Mathematical Society (1977).

[8] E. Lukacs, *Characteristic Functions*, Griffin, London (1970).

[9] C.A. Micchelli, Interpolation of scattered data: distance matrices and conditionally positive definite functions, *Constr. Approx.*, **2** (1986), 11-22.

[10] M.A. Naimark, *Normed Rings*, Nauka, Moscow (1968), In Russian.

[11] K.R. Partasarathy, R.R. Ranga, S.R.S. Valadhan, Probability distributions on locally compact abelian groups, *Illinois J. Math.*, **7** (1963), 337-369.

[12] V. Volkovich, Infinitely divisible distributions in algebras with stochastic convolution, *J. Soviet Math.*, **40**, No. 4 (1988), 459-467.

[13] V. Volkovich, Centering of probability distributions, *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk*, **2** (1990), 3-9, In Russian.

[14] V. Volkovich, On symmetric stochastic convolutions, *J. Theoret. Probab.*, **5**, No. 3 (1992), 417-430.

[15] V. Volkovich, On $V-$infinitely divisible distributions, *Theory Probab. Appl.*, **3** (1996), 503-517.

[16] V. Volkovich, On the Levy-Khintchine formulas from the generalized function point of view, In: *Transactions of XXIV International Seminar on Stability Problems for Stochastic Models, September 10-17, 2004*, Jurmala, Latvia (2004), 74-76.