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CHARACTERIZATIONS OF FUZZY PARACOMPACTNESS

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Abstract: In this paper new characterizations of fuzzy paracompactness are presented by mean of concepts of fuzzy star refinement and barycentric refinement

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1. Introduction

There are different kinds of paracompactness notions in fuzzy topological space. The first one (a-paracompactness, a*-paracompactness) introduced by Malghan and Benchalli [11] and may be not good extension, the second (S-paracompactness, S^*-paracompactness) presented by Luo [8] is good extension of ordinary paracompactness and the third (fuzzy paracompactness, *-fuzzy paracompactness) introduced by Abd-Elmonsef, Zeyada, El-deeb and Hanafy [1] which is implied by fuzzy compactness in the sense of Lowen [7] and also is good extension. Various matematicians studied these spaces [2, 3, 4, 8, 9].

In this paper we introduce new characterizations of fuzzy paracompactness in the sense of Luo by means of concepts of fuzzy star refinement and barycentric refinement. Then we prove that the Hausdorff fuzzy paracompactness in the sense of Malghan and Benchalli is collectionwise normal fuzzy.

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A fuzzy set μ is said to be quasi-coincident with the fuzzy set K if there exist an element $x \in X$ such that $\mu(x) + K(x) > 1$ and denoted by $\mu q K$.

Let $\mathcal{F} = \{ f_s : s \in S\}$ be a cover of X in the fuzzy topological space (X, T). The star of a fuzzy set μ with respect to cover $\mathcal{F}$ is

$$\text{st}\ (\mu, \mathcal{F}) = \vee\{f_s : \mu q f_s\}\,.$$

In particular the star of a fuzzy point x_λ with respect to $\mathcal{F}$ is denoted by $\text{st}\,(x_\lambda, \mathcal{F})$.

We say that a cover $\mathfrak{B} = \{b_t : t \in \ T\}$ of a set X is a star refinement of another cover $\mathcal{F} = \{f_s : s \in S\}$of the same set X if $\{\text{st}\,(\ b_t, \mathfrak{B}\ ) : t \in T\}$ is a refinement of $\mathcal{F}$ and denoted by $\text{st}\,(\ b_t, \mathfrak{B}\ ) \subset f_s$.

If $\{\text{st}\,(x_\lambda, \mathfrak{B}) : x_\lambda$ fuzzy point in $X\ \}$ is a refinement of $\mathcal{F}$ for every fuzzy point x_λ, we say that $\mathfrak{B}$ is a barycentric refinement of $\mathcal{F}$ and denoted by $\text{st}\,(x_\lambda, \mathfrak{B}) \subset f_s$.

2. Basic Definition

The following definitions have been used to obtain the results and properties developed in this paper.

Definition 1. (see [8]) Let $r \in (0, 1)$ and μ be a fuzzy set in a fuzzy topological space (X, T). The set μ is said to be r-paracompact (resp. r^*-paracompact) if for every $r-$open $Q-$cover of μ has an open refinement of it which is both locally finite (*-locally finite) in μ and an $r - Q-$cover of μ.

μ is called S-paracompact (S^*-paracompact) if for every $r \in (0, 1]$, μ is r-paracompact (resp. r^*-paracompact). We say that (X, T) is r-paracompact (resp. r^*-paracompact), S-paracompact (S^*-paracompact) if the set X is r-paracompact (resp. r*-paracompact), S-paracompact (S^*-paracompact).

Definition 2. (see [11]) Let $r \in [0, 1]$, a fuzzy topological space is said to be a-paracompact (resp a*-paracompact) if each a-shading (resp. a*-shading) of X by open fuzzy sets has a locally finite a-shading (resp. a*-shading) refinement by open fuzzy sets.

Definition 3. (see [6]) Let (X, T) be a fuzzy topological space. If A is a subset of X, then we denote χ_A its characteristic function. The set

$$[T] = \{A \subset X : \chi_A \in T\}$$

is called the original topology of X, and the crisp topological space $(X, [T])$ is called original topological space of (X, T).

Definition 4. (see [12]) A fuzzy topological space (X, T) is called a week induction of the topological space (X, T_0) if $[T] = T_0$ and each element of T is lower semi-continuous from (X, T_0) to $[0, 1]$.

Definition 5. (see [13]) Let A be a subset of a fuzzy topological space (X, T). The collection

$$T_A = \{V_A : V_A = V \wedge A, A \in T\}$$

constitutes a fuzzy topology on A. Consequently, (A, T_A) is called a fuzzy subspace of (X, T) and V_A is an open subset of A in T_A.

Definition 6. (see [15]) A fuzzy topological space (X, T) is said to be separable iff there exists a countable sequence of fuzzy points $\{p_i\}_{i=1,2,..}$ such that for every member $\mu \neq 0$ of T there exist a p_i such that $p_i \in \mu$.

Definition 7. (see [6]) Let (X, τ) be a topological space and let $\omega(\tau)$ be the set of all semicontinuas function from (X, τ) to the unit interval equipped with the usual topology, then $(X, \omega(\tau))$ is called the induced fuzzy topological space by (X,τ)

Definition 8. (see [8]) Let $\mathcal{F}$ be a family of sets and μ be a set in a fuzzy topological space (X, T). We say that $\mathcal{F}$ is discrete in μ if for each point e in μ, there exist a $U \in Q(e)$ such that U is quasi-coincident with at most one member of $\mathcal{F}$and we say that $\mathcal{F}$ is δ-discrete in μ if $\mathcal{F}$ can be represented as a countable union of discrete subfamilies.

Definition 9. (see [14]) Let A and B be two fuzzy sets in X, the operation $A - B$ is defined as follows:

$$(A - B)(x) = \begin{cases} A(x)\,, & \text{if } A(x) > B(x)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Definition 10. A family $\{f_s\}_{s \in S}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be discrete if for every $x \in X$ there exists an open set f with $f(x) = 1$ such that $f_s < 1 - f$ for all but at most one $s \in S$.

3. Fuzzy Paracompactness

In this section we introduce new characterizations of fuzzy paracompactness in the sense of Luo by means of concepts of fuzzy star refinement and barycentric refinement. We need prove the following lemmas.

Lemma 3.1. *For every weekly induced fuzzy topological space (X,T), if an open$-Q-$cover of 1_α has a closed locally finite refinement then it also has an open fuzzy barycentric refinement.*

Proof. Let $U = \{u_t\}_{t\in T}$ be an open$-Q-$cover of $1_\alpha, \alpha \in (0,1)$. Take a locally finite closed refinement $\mathcal{F}$. Let $r = \min\{\alpha, 1-\alpha\}$ then $r \in (0,1)$. For every $t \in T$, let

$$\omega_t = \wedge\{\ f'\colon f\ \in \mathcal{F},\ \ f' \geq \chi_{(u_t)_{1-\alpha}}\},\ \text{where } f' \text{ is the complement of } f\,.$$

Now f' is open, then $\omega_t \in T$ and for each $f \in \mathcal{F}$ we have

$$\omega_t q\ f \Longleftrightarrow \chi_{(u_t)_{1-\alpha}} q\ f\,.$$

Since

$$\omega_t q\ f \Longleftarrow \chi_{(u_t)_{1-\alpha}} q\ f$$

and we can show that

$$\omega_t q\ f \Longrightarrow \chi_{(u_t)_{1-\alpha}} q\ f,$$

indeed let $\omega_t q\ f$ and $\chi_{(u_t)_{1-\alpha}}$ not quasi-coincident with f, then

$$\chi_{(u_t)_{1-\alpha}} \leq f' \Longrightarrow \omega_t \leq f' \Longrightarrow \omega_t \text{ not quasi-coincidente with } \ f,$$

then we get contradiction.

Take $u_t \in U$ such that $u_t \leq \omega_t$ and for every $t \in T$, let

$$v_t = \chi_{(\omega_t)_{1-r}} \wedge u_t\,.$$

By the weekly induced property of (X,T), $V = \{v_t\}_{t\in T}$ is an open refinement of U.

Let x_λ be a fuzzy point in X such that $x_\lambda q v_t$

Now $v_t(x) \leq u_t(x)$ and then

$$\text{st}\ (x_\lambda, V) = \vee\{v_t\,(x) : \lambda + v_t\,(x) > 1\} \leq u_t\,,$$

then st $(x_\lambda, V) \leq u_t$ which prove that the covering V is a fuzzy barycentric refinement of U. □

Lemma 3.2. *For every weakly induced fuzzy topological space (X,T), if a $Q-$cover U of a fuzzy set X is fuzzy barycentric refinement of a $Q-$cover V, and a $Q-$cover V is a fuzzy barycentric refinement of $Q-$cover W, then U is a fuzzy star-refinement of W.*

Proof. Take $u \in U$ and take an x_λ satisfying $x_\lambda qu$. Since V is a barycentric refinement of W we can choose $w \in W$ with $\text{St}\,(x_\lambda, V) \leq w$. It will now be sufficient to show that $\text{St}\,(u, U) \leq w$. Let U be a $\alpha - Q-$cover of a fuzzy set A. Let $u \in U$ and for every fuzzy point x_λ then

$$\text{st}\ (x_\lambda, U) = \vee\{u : x_\lambda qu\}$$

and

$$u < \text{st}\left(x_{\lambda}, U\right) < v_t \,. \tag{1}$$

Thus we have

$$\begin{aligned} \text{st}\,(u,U) &= \bigvee_{x_\lambda \in u_s} \text{st}\ (x_\lambda, U) \\ &= \bigvee_{x_\lambda \in u} \{\vee\{u : x_\lambda q u_s\}\} = \bigvee_{x_\lambda \in u} \{u : \lambda + u_s\,(x) > 1\} \\ &\le \bigvee_{x_\lambda \in u} v_t, \quad \text{where } V = \{v_t\}_{t\in T}\,. \end{aligned} \tag{2}$$

Then

$$\text{st}\ (u,U) = \bigvee_{x_\lambda \in u} \text{st}\ (x_\lambda, U) < \bigvee_{x_\lambda \in u} v_t.$$

Now

$$\bigvee_{x_\lambda \in u} v_t = \vee\{v_t : x_\lambda \in u_s\} < \vee\{v_t : x_\lambda \in v_s\}\ .$$

From (1) we get

$$\bigvee_{x_\lambda \in u_s} v_t = \text{st}\,(x_\lambda, V)$$

and we have

$$\text{st}\,(u,U) < \text{st}\ \ (x_\lambda, V) < w_z. \qquad \square$$

Lemma 3.3. *Let (X,T) be a weakly induced fuzzy topological space, If every open $Q-$cover U of 1_α has an open star refinement then every open $Q-$cover of 1_α has also an open δ-discrete refinement.*

Proof. Let $U=\ \{u_t\}_{t\in T}$ be a $\alpha - Q-$cover of $1_\alpha, \alpha \in (1,0)$. Let $U_0 = U$ and let $\{U_1, U_2, \ldots\}$ be a sequence of open $Q-$covers of 1_α such that $U_{i+1}$is a star refinement of U_i, $i = 0,1,2,\ldots$. Let $U_x \in Q(x_\lambda)$ then the set:

$$U_{t,i} = \{U_x : x \in X \text{ such that st } (U_x, U_i) < U_t\}$$

is an open refinement of U for $i = 0,1,2,\ldots$, since $U_{t,i}$ is star refinement of U_i, then for every $U \in U_{t,i}$ there exist $V \in U_i$ such that:

$$\text{st}\ (U, U_{i+1}) \le V\,.$$

Then

$$V < \text{st}\ (x_\lambda, U_i) \le U_t$$

and $x_\lambda \in U$ which implies that

$$\text{st}\ (U, U_{i+1}) \le U_t.$$

Taking a well ordering relation $<$ on the set T and let:

$$V_{t_0,i} = U_{t_0,i} \setminus \bigvee_{t<t_0} \overline{U}_{t,i+1}$$

and

$$\mu_{V_{t_0,i}}(x) = \mu_{U_{t_0,i}}(x) - \sup_{t<t_0}\left(\overline{U}_{t,i+1}\right)(x), \tag{1}$$

we have either $t_1 < t_2$ or $t_2 < t_1$, from (1) we have either

$$\mu_{V_{t_2,i}} < 1 - \mu_{U_{t_1,i+1}}$$

or

$$\mu_{V_{t_1,i}} < 1 - \mu_{U_{t_2,i+1}}.$$

If $e_1 \in U_{t,i}$ and $e_2 \notin U_{t,i+1}$ then, there is no $U \in U_{i+1}$ such that $e_1, e_2 \in U$, thus for $t_1 \neq t_2$ if $e_1 \notin V_{t_1,i}$ and $e_2 \in V_{t_2,i}$, there is no $U \in U_{i+1}$ such that $e_1, e_2 \in U$. Now $V_{t,i}\ qU_{t,i}$ for at most one $t \in T$. Then the family of open sets $\{V_{t,i}\}_{t\in T}$ is discret. Finally let $x \in X$ and take $t(x)$ the smallest element in T such that

$$x_\lambda \in U_{t(x),i}.$$

That means $U_{t(x),i}(x) > \lambda - 1$. Since $x_\lambda \notin U_{t,i}$ for $t < t(x)$ then

$$\text{st}\,(x_\lambda, U_{i+2}) < 1 - U_{t,i+1} = V_{t(x),i}.$$

We have $x_\lambda \in V_{t(x),i}$ and $V_{t(x),i}(x) > \lambda$. Then $\left\{V_{t(x),i}\right\}$ is a Q-cover of X.

Theorem 3.4. *Let (X,T) be a regular weakly induced fuzzy topological space and $\alpha \in (0,1)$, then the following conditions are equivalent:*

1. (X,T) is S-paracompact.

2. Every $\alpha-$ open $Q-$cover of X has an open barycentric refinement.

3. Every $\alpha-$ open $Q-$cover of X has an open star refinement.

4. Every $\alpha-$ open $Q-$cover of X has an open $\delta-$discrete refinement.

Proof. From Theorem 3.14, [8] [$iv \Rightarrow i$] and Lemma 3.1 we have $1 \Rightarrow 2$. By Lemma 3.2 $2 \Rightarrow 3$. By Lemma 3.3, $3 \Rightarrow 4$. From Theorem 3.14 we have $4 \Rightarrow 1$. □

4. Collectionwise Normal Fuzzy Topological Spase

In this section we give the definition of collectionwise normal fuzzy and prove that a Hausdorff fuzzy paracompactness in the sense of Malghan and Benchalli is collectionwise normal fuzzy.

Definition 4.1. A fuzzy topological space (X, T) is said to be collectionwise normal fuzzy if for every discrete family $\{f_s\}_{s\in S}$ of fuzzy closed subsets of X there exists a discrete family $\{V_s\}_{s\in S}$ of open subsets of X such that $f_s \leq V_s$ for every $s \in S$.

Theorem 4.2. *Every* 1^*-*paracompact Hausdorff fuzzy topological space* (X, T) *is collection wise normal fuzzy topological space.*

Proof. Let $\{f_s\}_{s\in S}$ be a discrete family of fuzzy closed subsets of X. For every $x \in X$, choose a neigborhood $g \in T$ such that $g(x) = 1$ and $f_s \leq 1 - \overline{g}$ for all but at most one set f_s. Now let

$$G = \vee \{g \in T : g(x) = 1\}$$

is 1*-shading of X. Let W be an open locally finite refinement of G.

Choose

$$\begin{aligned} V_s &= 1 - \vee\{\varpi : \omega \in W \text{ and } \varpi \leq 1 - \ f_s\} \\ &= 1 - \sup\{\varpi : \omega \in W \text{ and } \varpi \leq 1 - f_s\} \qquad (1) \end{aligned}$$

which leads to

$$V_s \geq 1 - (1 - f_s)$$

and then $f_s \leq V_s$.

To prove that $\{V_s\}_{s\in S}$ is discrete, we have $\overline{g} \leq 1 - f_s$ for all but at most one $s \in S$ and

$$\overline{\omega} \leq \overline{g} \leq 1 - f_s$$

for all but at most one $s \in S$. From (1)

$$V_s \leq 1 - \overline{\omega} \text{ iff } f_s \leq 1 - \overline{\omega}$$

for all but at most one $s \in S$ because:

(1) If $V_s \leq 1 - \overline{\omega}$ it is clear that $f_s \leq 1 - \overline{\omega}$.

(2) Suppose that $f_s \leq 1 - \omega$.

To prove $V_s \leq 1 - \overline{\omega}$,let $f_s \leq 1 - \overline{\omega}$ for all but at most one $s \in S$ and $V_s \nleq 1 - \overline{\omega}$ for all but at most $s \in S$, there for $V_s \leq 1 - \overline{\omega}$ is hold only for at most $s \in S$ and then $f_s \leq 1 - \overline{\omega}$ is hold also for at most one $s \in S$ and we get contradiction, then it must be $f_s \leq 1 - \overline{\omega}$.

5. Other Results

In this section we prove some results corresponding to the same concepts in general topology.

Theorem 5.1. *If every fuzzy open set in a r-paracompact space is r-paracompact space then every subspace is r-paracompact space.*

Proof. Let A be a subset of fuzzy topological space (X, T), then

$$T_A = \{\upsilon_A : \upsilon_A = \upsilon \wedge A, \upsilon \in T\}$$

By hypothesis it is clear that $\vee\upsilon$ is open fuzzy set and so is r-paracompact, let$\{u_i\}_{i\in I}$ be an r-open Q-cover of $\vee\upsilon$ then $\{A \wedge u_i\}_{i\in I}$ is a r-open Q-cover of A.

Since $\vee\upsilon$ is r-paracompact then $\{u_i\}_{i\in I}$ has an open refinement locally finite $\{u_i^*\}_{i\in I}$, and this implies that $\{u_i^* \wedge A\}_{i\in I}$ is r-open $Q-$ cover refinement of $\{A \wedge u_i\}_{i\in I}$.

Then A is r-paracompact.

Theorem 5.2. *Let (X, T) be a weakly induced fuzzy topological space, then (X, T) is separable iff $(X, [T])$ is separable.*

Proof. let $\{p_i\}, i = 1, 2, 3, ...$be a countable sequence of fuzzy points such that for each $\mu \neq 0, \mu \in T$ contains some $p_i \in \mu$.

For all i$\in$ N,

$$\begin{aligned} p_i(x_i) &= \lambda_i,\ \lambda_i \in (0, 1], \\ p_i(x) &= 0 \text{ for } x \neq x_i. \end{aligned}$$

For each non empty $G \in [T]$ there exists some $i \in N$ such that $p_i \in \chi_G$ then $p_i(x_i) \leq \chi_G(x_i)$, $\chi_G(x_i)$ =1, thus $x_i \in G$ and $\{x_i\}_{i\in N}$ is dense in $(X, [T])$.

Conversely, if $\{x_i\}_{i\in N}$ is dense in $(X, [T])$, define

$$\begin{aligned} p_i(x_i) &= \lambda\,, \\ p_i(x) &= 0 \text{ for } x \neq x_i, \quad \text{where } \lambda \in (0, 1]. \end{aligned}$$

For each $\mu \in$T , μ^{-1}(λ,1]$\in$ [T] there exist $x_k \in \mu^{-1}$(λ,1] such that $\mu(x_k) > \lambda$, then p$_k \in \mu$

Theorem 5.3. *Let (X, T) be r-paracompact (resp. $r*-$paracompact) space, and let A, B a pair of fuzzy closed in X. If every fuzzy point $x_\lambda \leq B$ there exist fuzzy open sets μ_1, μ_2 such that $A \leq \mu_1$, $x_\lambda \leq \mu_2$ and $\mu_1 \wedge \mu_2 = 0$, then there exist open sets μ_3, μ_4 such that $A \leq \mu_3$, $B \leq \mu_4$ and $\mu_3 \wedge \mu_4 = 0$.*

Proof. Let $U = \{u_x : x \in B\}$ be a r-open Q-cover of B, then $U \vee \{B\'\}$ be an open Q-cover of 1_r. Since (X, T) is r-paracompact space, then there is a locally finite refinement

$$\omega = \{w_s : s \in S\}$$

such that it is $Q-$cover of 1_r. Let

$$S_0 = \{s \in S : w_s q B\} .$$

Then we have A is not quasi-coincident with $\overline{w}_s$ for every $s \in S$. By Theorem 2.1 from [14], $A \leq (\overline{w}_s)$ for every $s \in S_0$, thus $B \leq w_s$ and implies that

$$B \leq \bigvee_{s \in S_0} w_s .$$

By Corollary 2.13 from [8],

$$\overline{\bigvee_{s \in S_0} w_s} = \bigvee_{s \in S_0} \overline{w}_s .$$

Choose

$$\mu_3 = 1 - \bigvee_{s \in S_0} \overline{w}_s \text{ and } \mu_4 = \bigvee_{s \in S_0} w_s .$$

Then it is clear that $\mu_3 \wedge \mu_4 = 0$. □

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